Manifolds with Integral Ricci Curvature Lower Bounds

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Simons Collaboration Conference on Structure of Collapsed Special Holonomy Spaces, Durham, April 9, 2018

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  Structure of spaces with two-sided Ricci bound: L^2 bound for curvature tensor—Cheeger-Naber (2015) and Jiang-Naber

Integral Curvature Bounds

Commonly used integral curvature

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Generalizes two-sided bounds

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Scale invariant curvature quantity (with $H = 0$):

$$k(x,p,R) = R^2\left(\fint_{B_R(x)} \rho_-^p\right)^{\frac{1}{p}}, \qquad k(p,R) = \sup_{x\in M} k(x,p,R).$$

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Answer: Yes and No

Earlier Work

- Gallot (1988), D. Yang (1988-92),
  Petersen-Wei (1997, 2001), Petersen-Sprouse (1998),
  Dai-Petersen-Wei (2000), Dai-Wei (2004),
  Aubry (2007, 2009), Tian-Z. Zhang (2016)

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- Anderson (1989), Anderson-Cheeger (1991), Cheeger (2003), Cheeger-Tian (2006) Streets (2013)

No: Examples

Example (S. Gallot, D. Yang) $\exists$ sequence of Riemannian manifolds with

$$\begin{aligned} \operatorname{diam} M &\le D \\ \operatorname{vol} M &\ge v \\ \int_M |\sec|^p \, d\operatorname{vol} &\le \Lambda \end{aligned}$$

but with arbitrarily short closed geodesics.

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No Volume Doubling

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M^n, $\mathrm{Ric}_M \geq (n-1)H$, then $\Delta r \leq \Delta_H r$, where r is the distance function.

For simplicity, take $H = 0$, we have

$$\mathrm{Ric}_- \equiv 0 \Longrightarrow \psi \equiv 0,$$

where $\psi = \left(\Delta r - \frac{n-1}{r}\right)_+$ — amount Laplacian comparison failed.

Laplacian Estimate for Integral Curvature

Theorem (Petersen-Wei,1997,GAFA)

Given M^n, *for* $p > \frac{n}{2}$, $r > 0$,

$$\|\psi\|_{2p.B(x,r)} \leq C(n,p)\|\mathrm{Ric}_-\|^{\frac{1}{2}}_{p,B(x,r)}.$$

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Lemma

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = \Delta r.$$

Volume Estimate for Integral Curvature

Theorem (Petersen-Wei,1997,GAFA)

Given M^n, *for* $p > \frac{n}{2}$, $R \geq r > 0$,

$$\left(\frac{vol(B_R(x))}{R^n}\right)^{\frac{1}{2p}} - \left(\frac{vol(B_r(x))}{r^n}\right)^{\frac{1}{2p}} \leq C(n,p) R^{1-\frac{n}{2p}} \left(\|Ric_-\|_{p,B_R(x)}\right)^{\frac{1}{2}}.$$

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In other words,

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In fact true for volume of geodesic sphere.

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- With volume growth condition $\mathrm{vol}(B_R(x)) \ge cR^n$, $k(x,p,R) \le c^{-\frac{1}{p}} R^{2-\frac{n}{p}} \|\mathrm{Ric}_-\|_{p,B_R(x)} \le \epsilon_0$ when $\|\mathrm{Ric}_-\|_p$ is bounded and $R \le R_0$.

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- When $k(p,r)$ is small for some r, it gives control on $k(p,r)$ for all r.

Isoperimetric constant estimate for integral curvature

Theorem (D. Yang1992)

Given $p > n/2$ and $v > 0$, there is an $\varepsilon(n, p, v) > 0$ such that if $vol(B_1(x)) \geq v$ and $k(p, 1) \leq \varepsilon$, then $Is(B_{\frac{1}{2}}(x)) \leq C(n, p, v)$, where $Is(B_{\frac{1}{2}}(x)) = \sup\{\frac{vol(\Omega)^{1-\frac{1}{n}}}{vol(\partial\Omega)} : \Omega \subset B_{\frac{1}{2}}(x)\}$.

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Theorem (Gallot1988, Petersen-Sprouse1998)

Given M^n closed manifold with diameter D, for $p > n/2$, there is an $\varepsilon(n, p) > 0$ such that if $k(p, D) \leq \varepsilon$, then $Is(M) \leq C(n, p, D)/vol(M)^{\frac{1}{n}}$, where $Is(M) = \sup\{\frac{vol(\Omega)\}^{1-\frac{1}{n}}}{vol(\partial\Omega)} : \Omega \subset M, vol(\Omega) \leq \frac{1}{2} vol(M)\}$.

Further prior results

Theorem (Dai-Petersen-Wei 2000)

Given $M^n, p > \frac{n}{2}, D > 0$, *there exists* $\epsilon(n, p, D)$ *such that if* $diam_M \leq D, \left(⨍ |Rm|^p\right)^{\frac{1}{p}} \leq \epsilon$, *then* M^n *is diffeomorphic to an infra-nilmanifold.*

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Integral version of Synge theorem is not true.

Main Result

Theorem (Dai-Wei-Z.Zhang2018, AIM)

Given M^n, $p > \frac{n}{2}$, there exists $\varepsilon = \varepsilon(p, n)$ such that if $k(p, 1) \leq \varepsilon$, then for any $x \in M$, $r \leq 1$ with $\partial B_r(x) \neq \emptyset$,

$$Is(B_r(x)) \leq C(n)r/(vol(B_r(x)))^{\frac{1}{n}}.$$

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$$Is(B_r(x)) \le C(n) r / (vol(B_r(x)))^{\frac{1}{n}}.$$

Corollary

Under the same assumption as above, we have the Sobolev inequality

$$\left(\fint_{B_r(x)} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \le C(n) r \fint_{B_r(x)} |\nabla f|,$$

for all $f \in C_0^\infty(B_r(x))$ *where* $r \le 1$.

Applications

Without Noncollapsing Assumption

- Harnack inequality, Heat kernel bounds

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Theorem (Gradient Estimate)

Given M^n, $p > n/2$, there is an $\varepsilon(n,p) > 0$ and $C(n,p) > 1$ such that for $R \le 1$ if $k(p,1) \le \varepsilon$ and u is a function on $B_1(x)$ satisfying

$$\Delta u = f,$$

then

$$\sup_{B_{\frac{R}{2}}(x)} |\nabla u|^2 \le C(n,p) R^{-2} \left[\left(\|u\|^*_{2,B_R(x)}\right)^2 + \left(\|f\|^*_{2p,B_R(x)}\right)^2 \right].$$

Idea of Proof — Step 1

- When $k(p,1)$ is small, small ball have small volume relative to big ball
(measure can't concentrate on small ball)

Theorem

Given (M^n, g), $p > \frac{n}{2}$, *there exist* $\varepsilon = \varepsilon(n,p) > 0$ *and* $r_0 = r_0(n) > 0$ *such that if* $k(p,1) \le \varepsilon$, *then*

$$\frac{vol(B_{r_0}(x))}{vol(B_1(x))} \le \frac{1}{2}, \ \forall x \in M.$$

Idea of Proof — Step 2

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Theorem

Let H be any hypersurface dividing M into two parts. For any ball $B = B_r(x)$ we have

$$\begin{aligned} & \min\left(vol(B \cap M_1), vol(B \cap M_2)\right) \\ \leq \quad & 2^{n+1} r \left[vol(H \cap B_{2r}(x)) + vol(B_{2r}(x)) \| Ric_- \|^{*\frac{1}{2}}_{p, B_{2r}(x)} \right]. \end{aligned}$$

Combining above we control the error term and obtain

Proposition (Key Estimate)

Given a hypersurface H dividing M^n into two parts, there exists $\varepsilon = \varepsilon(p, n)$ such that if $k(x, p, 1) \leq \varepsilon$, then for a metric ball $B = B_r(x)$, $r \leq \frac{1}{2}$, which is divided equally by H, we have

$$vol(B_r(x)) \leq 2^{n+3} r \, vol(H \cap B_{2r}(x)).$$

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- Vitali's covering lemma

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For any $x \in \Omega$, let r_x be the smallest radius such that

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The domain Ω has a covering

$$\Omega \subset \bigcup_{x \in \Omega} B_{2r_x}(x).$$

Thank You