Counting ~~sheaves on Calabi-Yau 4-folds~~ CY4 quiver representations (joint with Jeongseok Oh)

Happy Birthday Alastair!

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vRR

$\mathrm{Hilb}^d\mathbb{C}^4$

A length d subscheme $Z \subset \mathbb{C}^4$ is described by

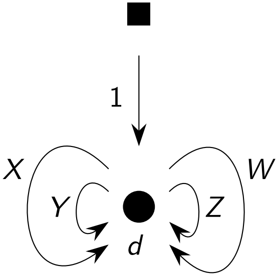
- a d-dimensional vector space $\Gamma(\mathcal{O}_Z) \cong \mathbb{C}^d$,
- with 4 commuting operators $X, Y, Z, W \colon \mathbb{C}^d \circlearrowleft$ and
- a framing $1 \colon \mathbb{C} \to \mathbb{C}^d$ whose image generates $\mathbb{C}^d$ under the action of $\mathbb{C}[X, Y, Z, W]$

up to the action of $GL(d, \mathbb{C})$.

Last condition is a King stability condition for this GIT data.

Quiver description

$$\underline{\mathbf{x}} := (X, Y, Z, W) \in (\mathbb{C}^4)^* \otimes \mathrm{End}(\mathbb{C}^d)$$



with relations $s = 0 \in E$, where

$$s := \underline{\mathbf{x}} \wedge \underline{\mathbf{x}} = \Big([X, Y], [X, Z], \dots\Big) \in \Lambda^2(\mathbb{C}^4)^* \otimes \mathrm{End}(\mathbb{C}^d) =: E$$

Pairing and superpotential

Notice $E = \Lambda^2(\mathbb{C}^4)^* \otimes \mathrm{End}(\mathbb{C}^d)$ has an obvious symmetric nondegenerate pairing

$$q\colon E \otimes E \xrightarrow{\wedge \otimes \mathrm{tr}} \Lambda^4(\mathbb{C}^4)^* \cong \mathbb{C}$$

with respect to which $s = \underline{\mathbf{x}} \wedge \underline{\mathbf{x}}$ is isotropic: $q(s, s) = 0$.
(For any $\underline{\mathbf{x}}$! Not just those which satisfy the relations $s = \underline{\mathbf{x}} \wedge \underline{\mathbf{x}} = 0$.)

Can encode in 6 extra degree -1 edges $X \wedge Y$, $X \wedge Z, \ldots$ labelled by a basis of $\Lambda^2(\mathbb{C}^4)^*$ and superpotential (sum of degree -1 cycles)

$$\sum_{i<j} (X_i \wedge X_j) \circ [X_k, X_\ell],$$

with k, ℓ chosen so that $ijk\ell$ is an even permutation of 1234.

(Setting the derivative w.r.t. the degree -1 edge $X_i \wedge X_j$ equal to zero recovers the relation $[X_k, X_\ell] = 0$.)

CY4 quivers

More generally there is a notion of CY4 quiver (Lam, building on work of Ginzburg, van den Bergh, physicists....):

- A finite quiver Q (vertices, degree 0 edges),
- An extra set E of degree -1 edges between the same vertices,
- A symmetric pairing $q\colon E \times E \to \mathbb{C}$ satisfying
  - $q(e, e') = 0$ unless $e \circ e'$ forms a cycle,
  - nondegeneracy $\det\big(q(e_i, e_j)\big)_{e_i, e_j \in E} \neq 0$,
- A superpotential s (a $\mathbb{C}$-linear combination of degree -1 cycles) such that
- $q(s, s) \in [\mathbb{C}Q, \mathbb{C}Q]$ ($\implies$ trace zero in any representation)

When we impose the relations $\{\partial_e s\}_{e \in E}$ we find the category of representations is CY4.

Counting representations

Picking dimension vectors we find the representations satisfying the relations are cut out of the smooth space of all (King stable) representations by an

isotropic section s of an orthogonal bundle (E, q), $q(s, s) = 0$.

Too many equations in too few unknowns. Eg 6 equations $[X, Y] = 0 = \dots$ in 4 unknowns X, Y, Z, W (modulo gauge).

Real equations

(Following Acharya-O'Loughlin-Spence, Baulieu-Kanno-Singer, Borisov-Joyce, Cao-Leung, DT, etc) Nekrasov and Nekrasov-Piazzalunga instead "halved" the equations to $s_+ = 0$, where

$$E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}, \quad s = (s_+, s_-)$$

$(0 = q(s,s) = |s_+|^2 - |s_-|^2 \implies |s|^2 = 2|s_+|^2$, so $s_+ = 0 \iff s = 0)$

by splitting $\Lambda^2(\mathbb{C}^4)^* \cong \Lambda^+(\mathbb{C}^4)^* \oplus \Lambda^-(\mathbb{C}^4)^*$ into real subspaces; the ± 1 eigenspaces of $\overline{*}\colon \Lambda^{2,0} \to \Lambda^{2,4} \cong \Lambda^{2,0}$.

So they consider the real equations

$$[X, Y]_+ = 0 = [X, Z]_+ = \dots,$$

conjecture a localisation formula and deduce a good counting theory, nice answers (better than MacMahon!).

Holomorphic equations

We will work within algebraic geometry with maximal isotropic subbundles $\Lambda \subset E$ instead of maximal positive definite subbundles $E_{\mathbb{R}} \subset E$.

E.g split $\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C} := \langle X, Y, Z \rangle \oplus \langle W \rangle$ and so

$$\begin{aligned}
\Lambda^2(\mathbb{C}^4)^* &\cong \Lambda^2(\mathbb{C}^3)^* \oplus (\mathbb{C}^3)^* \\
&\cong \langle X \wedge Y, X \wedge Z, Y \wedge Z \rangle \oplus \langle X \wedge W, Y \wedge W, Z \wedge W \rangle.
\end{aligned}$$

This half of the equations now have bigger zero locus (dependent on the choice of $\Lambda \subset E$).

The rest of the talk shows how to fix this.

Kool-Rennemo re-express Nekrasov-Piazzalunga's work within our framework, then use our torus localisation formula (below) to prove Nekrasov-Piazzalunga's formula.
[Cf. work of Bojko in compact CY4 case.]

Quiver trees

Jeongseok Oh, Namibia, 10 Jan 2023.

Orthogonal bundles

Work over a fixed complex quasi-projective scheme Y.

- $E \to Y$ holomorphic rank r vector bundle $\implies$ Zariski locally trivial.
- $q\colon E \otimes E \to \mathcal{O}_Y$ non-degenerate quadratic form.
- Gram-Schmidt process (uses square roots!) $\implies$ étale locally trivial. E corresponds to an étale locally trivial principal $O(r, \mathbb{C})$ bundle.

Special orthogonal bundles

q gives an isomorphism $E \cong E^* \implies \det E \cong \det E^*$

$$\implies (\det E)^{\otimes 2} \cong \mathcal{O}_Y. \qquad (*)$$

Definition An orientation on E is a trivialisation

$$o\colon \mathcal{O}_Y \xrightarrow{\sim} \det E$$

such that $(-1)^{r(r-1)/2} o^{\otimes 2}$ is the inverse of $(*)$.
(Sign due to convention $(e_1 \wedge \cdots \wedge e_r)^* = f_r \wedge \cdots \wedge f_1$.)

A $\mathbb{Z}/2$ choice, when one exists.

Oriented orthogonal bundles (E, q, o) correspond to étale locally trivial principal $SO(r, \mathbb{C})$ bundles.

Maximal real positive definite subbundles

- Orthogonal bundles admit maximal real positive definite subbundles $E_{\mathbb{R}} \subset E$ (unique up to homotopy) on which $q|_{E_{\mathbb{R}}}$ is a real positive definite quadratic form (i.e. inner product, metric). (E.g. $\mathbb{R}^r_{x_1,\ldots,x_r} \subset \mathbb{C}^r_{z_1,\ldots,z_r}$ maximal positive definite real for $q = \sum z_j^2 = \sum (x_j^2 - y_j^2) + 2i \sum x_j y_j$.)
- Conversely a real bundle $E_{\mathbb{R}}$ with a real inner product $q_{\mathbb{R}}$ gives a complex bundle $E := E_{\mathbb{R}} \otimes \mathbb{C} = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}$ with quadratic form $q = q_{\mathbb{R}} \otimes \mathbb{C}$.
- Homotopy equivalence $O(r,\mathbb{R}) \overset{\sim}{\hookrightarrow} O(r,\mathbb{C})$.
- Homotopy equivalence $SO(r,\mathbb{R}) \overset{\sim}{\hookrightarrow} SO(r,\mathbb{C}) \implies$ orientations o on E (in the above sense) $\longleftrightarrow$ orientations $o_{\mathbb{R}}$ on $E_{\mathbb{R}}$ (in the usual sense).
- If $e_1, \ldots, e_r$ are local orthonormal sections of $E_{\mathbb{R}}$ then they also form a local $\mathbb{C}$-basis of E, and $o = e_1 \wedge \cdots \wedge e_r$ is a local orientation of E corresponding to the local orientation $o_{\mathbb{R}} = e_1 \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} e_r$ of $E_{\mathbb{R}}$.

Square root Euler class

From now on we fix $r = 2n$ and work with $SO(2n, \mathbb{C})$ bundles (E, q, o) over Y.

- The Euler class $e(E_{\mathbb{R}}) \in H^{2n}(Y, \mathbb{Z})$ is a characteristic class of (E, q, o).
- Satisfies $(-1)^n e(E_{\mathbb{R}})^2 = e(E) = c_{2n}(E) \in H^{4n}(Y, \mathbb{Z})$ so we call it a square root Euler class $\sqrt{e}(E) \in H^{2n}(Y, \mathbb{Z})$ of E.
- Field and Totaro showed it does not lift to $A^n(Y, \mathbb{Z})$.
- Edidin-Graham showed $\pm\sqrt{e}(E)$ does lift to $A^n\left(Y, \mathbb{Z}\left[\frac{1}{2}\right]\right)$.

In place of maximal positive definite real subbundles they used maximal isotropic holomorphic subbundles.

Maximal isotropic subbundles

We call a holomorphic subbundle $\Lambda \subset E$ isotropic if $q|_\Lambda \equiv 0$.
We call it maximal isotropic if in addition it has rank n.

E.g. in $(\mathbb{C}^2, z_1^2 + z_2^2)$, $\binom{1}{0}$ and $\binom{0}{1}$ span a real positive definite subspace, while the vectors $\binom{1}{i}$ and $\binom{1}{-i}$ are isotropic.

$\{y_1 = 0 = y_2\} = \mathbb{R}^2_{x_1,x_2} \subset \mathbb{C}^2$ is a maximal positive definite real subspace; $\mathbb{C}$-lines $z_1 + iz_2 = 0$ and $z_1 - iz_2 = 0$ are (the only) maximal isotropics.

We get an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow E \xrightarrow{q} \Lambda^* \longrightarrow 0$$

and so $(-1)^n c_n(\Lambda)^2 = c_{2n}(E)$, that is $(-1)^n e(\Lambda)^2 = e(E)$.
(And notice the composition $\Lambda \hookrightarrow E \twoheadrightarrow E_\mathbb{R}$ is an isomorphism.)

(Writing $E = E_\mathbb{R} \oplus iE_\mathbb{R}$ corresponds to writing $q = z_1^2 + z_2^2 + \dots$. Locally writing $E = \Lambda \oplus \Lambda^*$ corresponds to writing $q = z_1 z_{n+1} + z_2 z_{n+2} + \dots$.)

Edidin-Graham class, I

So if E admits a maximal isotropic subbundle $\Lambda \subset E$, Edidin-Graham lift $\sqrt{e}(E) = e(E_{\mathbb{R}}) \in H^{2n}(Y, \mathbb{Z})$ to

$$\pm c_n(\Lambda) \ \in \ A^n(Y, \mathbb{Z}).$$

They show this is independent of Λ up to sign.
(E.g. replacing Λ by Λ^* in $E = \Lambda \oplus \Lambda^*$ changes $c_n(\Lambda)$ by $(-1)^n$.)

Define $(-1)^{|\Lambda|}$ as the image of $i^n o$ under

$$\det E \ \cong \ \det(\Lambda) \otimes \det(\Lambda^*) \ \cong \ \mathcal{O}_Y$$

(orient Λ by its complex structure, $E_{\mathbb{R}}$ by $o_{\mathbb{R}}$, then $\Lambda \hookrightarrow E \twoheadrightarrow E_{\mathbb{R}}$ preserves orientation iff $(-1)^{|\Lambda|} = +1$) and define

$$\sqrt{e}(E) \ := \ (-1)^{|\Lambda|} c_n(\Lambda) \ \in \ A^n(Y, \mathbb{Z}).$$

Edidin-Graham class, II

In general $\Lambda \subset E$ does not exist so we pass to a tautological bundle $\rho\colon \widetilde{Y} \to Y$ on which it does (cf. splitting principle). E.g. can set

$$\widetilde{Y} \;:=\; \mathrm{OGr}^{+}(E) \;=\; \big\{(y, \Lambda_y) \,:\, y \in Y,\ \Lambda_y \subset E_y \text{ positive maximal isotropic}\big\}$$

with tautological positive maximal isotropic $\Lambda_\rho \subset \rho^* E$.

Edidin-Graham prove that $c_n(\Lambda_\rho)$ descends to Y if we invert 2: there's a distinguished class with degree 2^{n-1} over Y,

$$h \;\in\; A^{n(n-1)}\big(\widetilde{Y}, \mathbb{Z}\big) \;\text{ with }\; \rho_* h \;=\; 2^{n-1}$$

so we may define

$$\sqrt{e}(E) \;:=\; \frac{1}{2^{n-1}}\,\rho_*\big(h \cup c_n(\Lambda_\rho)\big) \;\in\; A^n\big(Y, \mathbb{Z}[\tfrac{1}{2}]\big).$$

Edidin-Graham class, III

This $\sqrt{e}(E)$ has all the properties we'd like of it; for instance $\sqrt{e}(E)^2 = (-1)^n e(E)$, and it is the unique class with the property

$$\rho^* \sqrt{e}(E) \;=\; c_n(\Lambda_\rho).$$

More generally, if E (rather than $\rho^* E$) admits a maximal isotropic $\Lambda \subset E$ then

$$c_n(\Lambda) \;=\; (-1)^{|\Lambda|} \sqrt{e}(E).$$

There's a Whitney sum formula $\sqrt{e}(E_1 \oplus E_2) = \sqrt{e}(E_1)\sqrt{e}(E_2)$ and

$$\sqrt{e}(E) \;=\; e(E_{\mathbb{R}}) \text{ in } H^{2n}\left(Y, \mathbb{Z}\left[\tfrac{1}{2}\right]\right).$$

Localised Edidin-Graham class

Given an $SO(2n,\mathbb{C})$ bundle (E,q,o) over Y and an isotropic section $s \in \Gamma(E)$, $q(s,s)=0$, we construct a localised operator

$$\begin{array}{ccc} A_*\left(Y,\mathbb{Z}\left[\tfrac{1}{2}\right]\right) & \xrightarrow{\sqrt{e}(E,s)} & A_{*-n}\left(Z,\mathbb{Z}\left[\tfrac{1}{2}\right]\right) \\ & \searrow^{\sqrt{e}(E)\cap(\,\cdot\,)} & \downarrow \\ & & A_{*-n}\left(Y,\mathbb{Z}\left[\tfrac{1}{2}\right]\right), \end{array}$$

where $Z := Z(s)$. Using $\mathbb{Z}\left[\frac{1}{2}\right]$ coefficients (passing to the cover $\rho\colon \widetilde{Y} \to Y$ then later pushing down by $\frac{1}{2^{n-1}}\rho_*\left(h \cap (\,\cdot\,)\right)$) we can assume we have a positive maximal isotropic $\Lambda \subset E$,

$$0 \longrightarrow \Lambda \longrightarrow E \xrightarrow{p} \Lambda^* \longrightarrow 0.$$

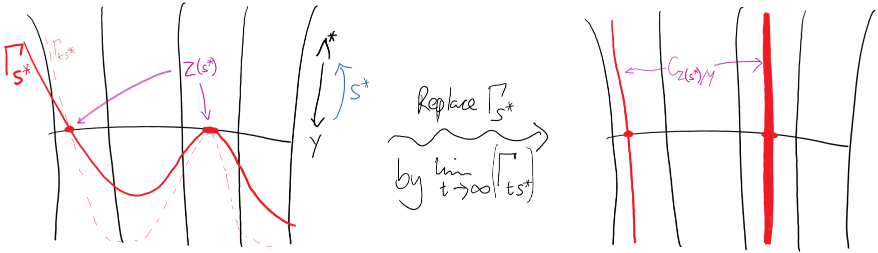
Since $\sqrt{e}(E) = e(\Lambda) = (-1)^n e(\Lambda^*)$ we first project $s^* := p(s)$ and intersect $\Gamma_{s^*} \subset \Lambda^*$ with 0_{Λ^*}.

Degeneration

First linearise $\Gamma_{s^*} \subset \Lambda^*$ about $Z^* := Z(s^*)$ by deforming it to the normal cone

$$\Lambda^* \supset \Gamma_{ts^*} \overset{t\to\infty}{\rightsquigarrow} C_{Z^*/Y} \subset \Lambda^*|_{Z^*}.$$

(Cone means $\mathbb{C}^*$-invariant; in lci case $C_{Z^*/Y}$ is $N_{Z^*/Y}$.)



So can replace $\Gamma_{s^*} \cap 0_{\Lambda^*}$ (in Y) by $C_{Z^*/Y} \cap 0_{\Lambda^*|_{Z^*}}$ (in Z^*).

Localisation to Z^*

So we can localise $e(\Lambda^*)$ to Z^* by

$$0^!_{\Lambda^*|_{Z^*}} C_{Z^*/Y} \ \in \ A_{\dim Y - n}(Z^*), \qquad (*)$$

where the Gysin map $0^!$ is the inverse of the Thom isomorphism

$$A_{\dim Y-n}(Z^*) \underset{0^!_{\Lambda^*|_{Z^*}}}{\overset{\pi^*}{\rightleftarrows}} A_{\dim Y}\big(\Lambda^*|_{Z^*}\big).$$

(Push forward of $(*)$ via $Z^* \hookrightarrow Y$ gives $e(\Lambda^*) = (-1)^n \sqrt{e}\,(E)$.)

But we would like to localise further to $Z = Z(s) \subset Z^*$ by using the "<u>other half</u>" of the section s. That is, by

$$s|_{Z^*} \ \in \ \Gamma\big(\Lambda|_{Z^*}\big) \ \subset \ \Gamma\big(E|_{Z^*}\big).$$

Cosection localisation

$s|_{Z^*} \in \Gamma\big(\Lambda|_{Z^*}\big)$ defines a cosection

$$\widetilde{s} \,:\, \Lambda^*\big|_{Z^*} \longrightarrow \mathcal{O}_{Z^*}$$

and the s isotropic condition gives $C_{Z^*/Y} \;\subset\; \ker \widetilde{s}$.

So where $\widetilde{s}$ is nonzero there's a trivial normal direction to $C_{Z^*/Y}$ in $\Lambda^*|_{Z^*}$; intersection can be made zero here.

By the cosection localisation of Kiem-Li can localise $0^!_{\Lambda^*|_{Z^*}}[C_{Z^*/Y}]$ to $Z(\widetilde{s}) = Z$ and define

$$\sqrt{e}\,(E,s) \cap [Y] \;:=\; (-1)^n\, 0^{!,\,\mathrm{loc}}_{\Lambda^*|_{Z^*},\,\widetilde{s}} \;\in\; A_{\dim Y - n}(Z).$$

Its push forward via $Z \hookrightarrow Y$ gives $\sqrt{e}\,(E)$.

Square root Gysin operator

Similarly for any isotropic cone $C \subset (E, q, o)$ can define

$$\sqrt{0^!_E} \;:\; A_*\big(C, \mathbb{Z}[\tfrac{1}{2}]\big) \longrightarrow A_{*-n}\big(Y, \mathbb{Z}[\tfrac{1}{2}]\big)$$

such that, if C factors through a maximal isotropic $\Lambda \subset E$,

$$\sqrt{0^!_E} \;=\; (-1)^{|\Lambda|}\, 0^!_\Lambda.$$

Let $\pi\colon C \to Y$. Then π^*E has a tautological section τ_E which is isotropic and has zero locus $Y \subset C$ (the 0-section of C) so we define $\sqrt{0^!_E}$ to be the operator $\sqrt{e}\,(\pi^*E, \tau_E)$.

Has good properties: Whitney sum formula, compatibility with $\sqrt{e}\,(E, s)$ under $\Gamma_{ts} \rightsquigarrow C_{Z(s)/Y}$, etc.

Moduli of sheaves on Calabi-Yau 4-folds

Fix a smooth projective fourfold $(X, \mathcal{O}_X(1))$ with $K_X \cong \mathcal{O}_X$, and a Chern character $c \in H^*(X, \mathbb{Q})$ such that Gieseker semistable sheaves of class c are all stable.

The moduli space $M = M(X, c)$ of stable sheaves of Chern character c is projective. Deformation theory at a point $F \in M$:

- Automorphisms $\mathrm{Hom}_X(F, F) = \mathbb{C} \cdot \mathrm{id}_F \implies$ can ignore
- Deformations $\mathrm{Ext}^1_X(F, F)$
- Obstructions $\mathrm{Ext}^2_X(F, F) \cong \mathrm{Ext}^2_X(F, F)^*$ by Serre duality
- Higher obstructions $\mathrm{Ext}^3_X(F, F) \cong \mathrm{Ext}^1_X(F, F)^*$
  ($\implies$ no Li-Tian/Behrend-Fantechi virtual cycle)
- Higher higher obstructions $\mathrm{Ext}^4_X(F, F) = \mathbb{C} \implies$ can ignore

So the obstruction theory $R\pi_* R\,\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ is 3-term: a self-dual complex of bundles $E_0 \xrightarrow{a} E_1 \xrightarrow{a^*} E_0^*$ over M, with $E_1 \cong E_1^*$ an orthogonal bundle.

Model

Morally, M can be described as the zero locus of an isotropic section of an orthogonal bundle over a smooth ambient space

$$\begin{array}{ccc} & & (E,q) \\ & & \downarrow \uparrow s \\ M = s^{-1}(0) & \subset & \mathcal{A}, \end{array} \qquad q(s,s)=0,$$

such that $E_0 \xrightarrow{a} E_1 \xrightarrow{a^*} E_0^*$ is $T_{\mathcal{A}}|_M \xrightarrow{ds} E|_M \xrightarrow{ds^*} \Omega_{\mathcal{A}}|_M$.

Globally true with $\mathcal{A}$, E infinite dimensional via gauge theory.

Locally true with everything algebraic [BG, BBJ, BBBJ, PTVV], taking $\mathcal{A}$ to be open in $\mathrm{Ext}^1(F,F)$ and E to be trivial bundle with fibre $\mathrm{Ext}^2(F,F)$.

Can also orient (E,q) by work of [CGJ].

Virtual cycle

Can't patch local models, but using the obstruction theory we can patch linearised version $\lim_{t\to\infty}\Gamma_{ts}\subset E$ to give an isotropic cone

$$C \;\subset\; E|_M \;=\; E_1.$$

Then we can define $[M]^{\mathrm{vir}} := \sqrt{0^!_{E_1}}\,[C]$.

[LT/BF] virtual cycle $0^!_E[C]$ wrong here $\longleftrightarrow$ stupid truncation
$T_{\mathcal{A}}|_M \xrightarrow{ds} E|_M$ of the self-dual deformation-obstruction complex
$T_{\mathcal{A}}|_M \xrightarrow{ds} E|_M \xrightarrow{ds^*} T^*_{\mathcal{A}}|_M$. Instead we "halve" it with $T_{\mathcal{A}}|_M \to \Lambda$.

Borisov-Joyce instead intersect $[C]$ with $E_{\mathbb{R}}\subset E$ (ish)
(taking the half $T^*_{\mathcal{A}}|_M \xrightarrow{ds_+} E_{\mathbb{R}}$ of the obstruction complex)
We show the result is the same. In particular [BJ]'s class is zero when it is odd-dimensional.

K-theoretic Euler class

K-theory is an oriented (generalised) cohomology theory: it has a theory of Chern classes.

The K-theoretic Euler class of a bundle $E \to Y$ is

$$\mathfrak{e}(E) \;=\; 0_E^*[\mathcal{O}_{0_E}] \;\in\; K^0(Y),$$

where $0_E \subset E$ is the zero-section and 0_E^* is the (derived) pullback in K-theory. Resolving the structure sheaf of $0_E \subset E$ by its Koszul resolution $\Lambda^\bullet \pi^* E^*$ on $\pi\colon E \to Y$ shows this is

$$\mathfrak{e}(E) \;=\; \Lambda^\bullet E^* \;\in\; K^0(Y).$$

When E admits a transverse section s this is just $[\mathcal{O}_{Z(s)}]$.

Square-root K-theoretic Euler class

For an $SO(2n, \mathbb{C})$ bundle (E, q, o) with a maximal isotropic $\Lambda \subset E$ we define

$$\sqrt{\mathfrak{e}}(E) \;:=\; (-1)^{|\Lambda|}\,\Lambda^\bullet(\Lambda^*) \otimes \sqrt{\det\Lambda} \;\in\; K^0\big(Y, \mathbb{Z}\big[\tfrac{1}{2}\big]\big).$$

$\sqrt{L} := 1 + \frac{1}{2}(L-1) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}(L-1)^2 + \cdots$ uniquely defined over $\mathbb{Z}\big[\frac{1}{2}\big]$.

More generally work on the cover $\rho\colon \widetilde{Y} \to Y$ with $\Lambda_\rho \subset \rho^* E$:

$$\sqrt{\mathfrak{e}}(E) \;:=\; \rho_*\big[\Lambda^\bullet(\Lambda_\rho^*) \otimes \sqrt{\det\Lambda_\rho}\,\big] \;\in\; K^0\big(Y, \mathbb{Z}\big[\tfrac{1}{2}\big]\big).$$

Building on work of Anderson we show these are compatible.
They satisfy $\sqrt{\mathfrak{e}}(E)^2 = (-1)^n \mathfrak{e}(E)$.

Virtual structure sheaf

Using Kiem-Li's recent K-theoretic cosection localisation we define a class $\mathfrak{e}(E, s)$ localised to the zeros of an isotropic section, and a square-root K-theoretic Gysin map

$$\sqrt{0_E^*} \,:\, K_0\big(C, \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \longrightarrow K_0\big(Y, \mathbb{Z}\big[\tfrac{1}{2}\big]\big)$$

for $C \subset E$ an isotropic cone.

In this way we can define a K-theoretic virtual cycle on moduli spaces M of stable sheaves on Calabi-Yau 4-folds.

Let $T \to E \to T^*$ be a self-dual representative of its deformation-obstruction complex $T_M^{\mathrm{vir}} := R\pi_* R\mathscr{H}om(\mathcal{F}, \mathcal{F})[1]$, and set

$$\widehat{\mathcal{O}}_M^{\mathrm{vir}} \;:=\; \sqrt{0_E^*}\,[\mathcal{O}_C] \cdot \sqrt{\det T^*} \;\in\; K_0(M).$$

Well-defined, independent of choices.

Virtual Riemann-Roch and torus localisation

The two classes are related by a virtual Riemann-Roch formula

$$\chi(\widehat{\mathcal{O}}_M^{\mathsf{vir}}) \;=\; \int_{[M]^{\mathsf{vir}}} \sqrt{\mathrm{td}}(T_M^{\mathsf{vir}}).$$

The advantage of algebraic classes is that they are more computable, for instance by torus localisation.

Suppose $\mathsf{T} := \mathbb{C}^*$ acts on a quasi-projective Calabi-Yau 4-fold X preserving the holomorphic 4-form. Let $\iota\colon M^{\mathsf{T}} \hookrightarrow M$ denote the fixed locus of the induced T action on M. Then

$$[M]^{\mathsf{vir}} \;=\; \iota_* \frac{\left[M^{\mathsf{T}}\right]^{\mathsf{vir}}}{\sqrt{e_{\mathsf{T}}}(N^{\mathsf{vir}})} \;\in\; A^{\mathsf{T}}_{\frac{1}{2}\mathsf{vd}}(M,\mathbb{Q})\left[t^{-1}\right],$$

$$\widehat{\mathcal{O}}_M^{\mathsf{vir}} \;=\; \iota_* \frac{\widehat{\mathcal{O}}_{M^{\mathsf{T}}}^{\mathsf{vir}}}{\sqrt{\mathfrak{e}_{\mathsf{T}}}(N^{\mathsf{vir}})} \;\in\; K_0^{\mathsf{T}}(M) \otimes_{\mathbb{Z}[\mathfrak{t},\mathfrak{t}^{-1}]} \mathbb{Q}(\mathfrak{t}).$$