Counting Sheaves on CY4
Joint with Jeungsuk Oh (KIAS).

1. Gauge theory motivation/Borisov-Joyce virtual cycle
2. Local model (Darboux thm of Brouz-Bussi-Joyce, Bonaziz-Grojnowski)
3. Edelin-Graham $\widetilde{Euler}$ class of $SO(n,c)$ balls
4. EG class localised w.r.t. a) isotropic section b) isotropic cone
5. Algebrave virtual cycle $[M]^v \in H^{2*}(\mu, \mathbb{Z})$ $\rightarrow$ $H^{2*}(\mu, \mathbb{Z}[\frac{1}{2}])$
6. Torus localization formula
   Justify cals of Cao, Kool, Lerung, Manlik, Monavari, Nekrasov, Piazzalunga, Toda modulo choice of orientation.
§1. Gauge theory

* X smooth projective variety
* $\cal M$ moduli space of stable holomorphic bundles of fixed topological type (spheral twist trick on CYs)

Integrable operators $\bar{\partial}_A : \Omega^0(F) \rightarrow \Omega^{0,1}(F)$ on fixed $C^\infty$ ball $F \rightarrow X$

modulo gauge $\Omega^0(\text{Aut} F)$
Linearisation fits into elliptic complex

\[ \Omega^0(\text{End}_0F) \xrightarrow{\delta_0} \Omega^1(\text{End}_0F) \xrightarrow{\delta_0^2} \Omega^2(\text{End}_0F) \xrightarrow{\delta_0^3} \Omega^3(\text{End}_0F) \xrightarrow{\delta_0^4} \Omega^4(\text{End}_0F) \rightarrow \cdots \]

- Not Fredholm if just take this bit in dims > 2. Overdetermined: more eqs than unknowns, \( r_d = -\infty \)

- Can try to truncate (replace \( \Omega^2 \) by \( \delta_0^2 = \delta_0 \)\( \Omega^3 \rightarrow \Omega^0 \))
  - but this bar jumps in general as vary A.
  - (Works in dim 3 when \( k_x \leq 0 \) \( \Rightarrow \) virtual and inverts cycle)

- Cohomology groups
  - \( H^0(\text{End}_0F) \) infinitesimal automorphisms
  - \( H^1(\text{End}_0F) \) deformations
  - \( H^2(\text{End}_0F) \) obstructions
  - \( H^3(\text{End}_0F) \neq 0 \) in general \( \Rightarrow \) no "perfect obstruction theory"
C4 case: the complex is self-dual
suggests halving it

(cf. and eqs. on 4-mfds)

Complex anti-linear Hodge star-like operator

\[ \tilde{\star} : \Lambda^{0,2}(\text{End}_0 F) \stackrel{\tilde{\star}}{\longrightarrow} \Lambda^{4,2}(\text{End}_0 F) \stackrel{\star}{\longrightarrow} \Lambda^{0,2}(\text{End}_0 F) \]

\[ \tilde{\star}^2 = \text{id} \Rightarrow \Lambda^{0,2} = \Lambda^{0,2}_+ \oplus \Lambda^{0,2}_- \text{ eigen spaces} \]

real subbds of \( \Lambda^{0,2}(\text{End}_0 F) \)

"Half" of \( F_{A}^{0,2} = 0 \) eqs. is "complex and eqn."

\[ F_{A}^{0,1} = 0 \] (SL\( ^c \)) (End_0 F)

Elliptic when coupled with HYM eqs. w. \( F_{a}^{1,1} = 0 \)

Special case of \( SU(4)/Spin(7) \) instanton eqs.
Half holomorphic = holomorphic?

$F_A^{0,2} = 0 \iff F_A^{0,+} = 0$ and $\int_X \Omega = 0$ (cf. Donaldson theory
A flat $\iff$ and $\Pi = 0$)

$-4\pi^2 \int_X \Omega = \int_X \text{tr} (F_A^{0,2} \wedge \Omega) = \int_X \text{tr} (F_A^{0,+} \wedge F_A^{0,+}) \wedge \Omega + \int_X \text{tr} (F_A^{0,-} \wedge F_A^{0,-}) \wedge \Omega$

$= \int_X \text{tr} (F_A^{0,+} \wedge F_A^{0,+}) \wedge \Omega - \int_X \text{tr} (F_A^{0,-} \wedge F_A^{0,-}) \wedge \Omega$

$= \int_X \text{tr} (F_A^{0,+} \wedge F_A^{0,+}) - \int_X \text{tr} (F_A^{0,-} \wedge F_A^{0,-})$

$= 11 F_A^{0,+} \| \Omega^2 - 11 F_A^{0,-} \| ^2$

Therefore $\| F_A^{0,2} \|^2 = \| F_A^{0,+} \|^2 + \| F_A^{0,-} \|^2 = 2 \| F_A^{0,+} \|^2 + 4 \pi^2 \int_X \Omega$

So when $\Pi = 0$ (only) we get natural way to compactify
moduli space of SU(4) instantons using algebraic geometry.
So can consider $M$ as being cut out (set theoretically!) by $f_A^{0+} = 0$ instead of $f_A^{0,12} = 0$ to give it a Kuranishi structure.

This is essentially what [BEJ] do to define $[M]^{\tilde{\omega}} \in H^*_\mathbb{Z}(M, \mathbb{Z})$.

- $\nu_{\mathbb{R}}$ can be odd
- Hard to compute $[M]^{\tilde{\omega}}$
- Can only handle $p = 5$ case at present

So that $f_A^{0+} = 0 \iff f_A^{0,12} = 0 \Rightarrow$ can use algebraic geometry to compactify $M$.

Would like a construction in algebraic geometry (implies $[M]^{\tilde{\omega}} = 0$ if $\nu_{\mathbb{R}}$ odd)
Note $\mathcal{O}^{0,2}(\text{End}_0 F)$ has a complex quadratic form $\Omega(a,b) = \text{str}(anb)$ on it \textit{w.r.t.} which $F^{0,2}_a$ is \underline{isotropic} $\Omega(F^{0,2}_a, F^{0,2}_a) = 0$ if $P_{\nu}[\Omega] = 0$.

\[ \begin{align*}
\text{Local model} & \quad \text{(Darboux theorem of [BjJ, BG])} \\
\text{About any } F \in M & \text{ is Zariski locally where } S \text{ is an \underline{isotropic} section} \\
\Omega(s, s) = 0 & \text{ of a quadratic bundle } (E, \Omega) \to \text{Ext}'(F, F). \\
\end{align*} \]

$E$ modelled on $\text{Ext}^2(F, F)$ with \underline{some} duality quadratic structure $\Omega$ at $F$

\[ \begin{align*}
\mathbb{Z}(S) \subset \text{Ext}'(F, F) \\
\mathbb{F}^{0,2}(\text{End}_0 F) \\
\end{align*} \]
$\text{So}(n, \mathbb{C}) \cong \text{SO}(n, \mathbb{R})$

So $\text{SO}(n, \mathbb{C})$ bundles $E$ can be written $E = E_\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} = E^+ \oplus E^-$ negative definite $\text{SO}(n, \mathbb{R})$ ball, metric

So admit a characteristic class $c(E_\mathbb{R})$ s.t. $c(E_\mathbb{R})^2 = c(E) = 0^1_\mathbb{E}^1_\mathbb{E}$

Call this $\mathcal{E}(E)$.

Morally, [BJ] are seeing $M$ as $\mathcal{E}(E)$ cut out by projection $S_+$ of isotropic section $S$ of $E$.

Makes sense because

\[ \alpha(s, s) = 0 = \alpha(s_+, s_+) + \alpha(s_-, s_-) \iff |s|^2 = 2|s_+|^2 \]

$s = 0$ iff $s_+ = 0$
§3 Ededin–Graham

But an algebraic way of seeing $\tilde{\text{Se}}(E)$ using maximal isotropic subbundles

$0 \to \Lambda \to E \to \Lambda^* \to 0$

of $E$ instead of real subbundles.

Note if $\exists \Lambda$ then $e(E) = e(\Lambda)e(\Lambda^*)$

$= (-1)^n e(\Lambda)^2$ so $\boxed{e(\Lambda) = (-1)^n \tilde{\text{Se}}(E)}$

And $\Lambda \cap E^\perp = \{0\}$ so projection $\Lambda \to E^+$ is isomorphism.

(Relevance suggested by Co-Leung)

$\text{So}(m, \mathbb{C})$ split; take $m = 2n$ for simplicity in this talk

(if $m = 2n + 1$ extended by 2-torsion line bundle $\Rightarrow e = 0$)
Work on a cover $\tilde{Y} \rightarrow Y$ on which $\pi^*E$ has a canonical maximal isotropic $0 \rightarrow \Lambda \rightarrow \pi^*E \rightarrow \Lambda^* \rightarrow 0$

(cf. splitting principle)

$\tilde{Y}$: perverse isotropic flag variety of $E$

$\{ \Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_{n-1} \leq E : \dim \Lambda_i = i \}$

$\alpha |_{\Lambda_i} \equiv 0$

$\pi^*: A^*(Y) \rightarrow A^*(\tilde{Y})$ injective so any class $\xi(e)$ s.t. $\pi^* [e] \in c_n(\Lambda)$ is unique.
Define $\overline{e}(E) = \pi_*(c_n(A) \cup h)$

where $h$ is a canonical class s.t. $\pi_* h = 1$.

This requires $\mathbb{Z}[\frac{1}{2}]$ coefficients.

$[\overline{e}(E) \in A^n(Y, \mathbb{Z}[\frac{1}{2}]) \Rightarrow H^{2n}(Y, \mathbb{Z}[\frac{1}{2}])]$

If $E$ is a fibration, $A' \leq E$ then $\overline{e}(E) = c_n(A')$. 

If $E$ has isotropic $A' \leq E$ then $\overline{e}(E) = c_n(A')$. 

same image
§4 Localising EG class.

Key technical tool: costruction localization (Kiem-Li)

Given $C \subset E \overset{s}{\to} O_Y$ (set theoretically) such that $C \subset \ker(s)$ and $s_C$ (set theoretically) then can localise the intersection class $O_E^{!} C$

to $Z(s) \cap C \cap O_E$ (ie get a cycle here whose pushforward to $O_E \cong Y$ is $O_E^{!} C$).
Linearise intersection of $\Gamma_s \subset E$ and $\Lambda \subset E$

$$N_{\Gamma_s \cap \Lambda/E} \subset N_{\Lambda/E} = \gamma^* \Lambda^* \xrightarrow{\tau_{\Lambda, \text{tangent}}} 0^\text{Tot}(\Lambda)$$

"normal cone" to $\Gamma_s \cap \Lambda$ inside $E$

**Fact:** $s$ isotropic $\Rightarrow N_{\Gamma_s \cap \Lambda/E}$ lies in $ker(\tau_{\Lambda, \text{tangent}})$

So coaction localisation gives class in

$$Z(\tau^\text{tangent}) \cap \Lambda \cap \Gamma_s = \mathbb{Z}(s) \subset \gamma$$

as required.

PUSH forward to $\gamma$ is $\ell(\Lambda^*) = (-1)^{r(E)}$. 

\[ T_{\gamma}(\Lambda) \xrightarrow{1} \gamma \]
§4(b) Localizing [EG] class by isotropic cone

\[ C \subset E \]

\[ p \rightarrow y \]

Isotropic means \( \mathcal{O}_C \rightarrow p^* E \) is an isotropic section.

\[ \Rightarrow \text{can localize as in (a) to its zeros} \]

\[ \Rightarrow \text{Given } \sqrt{D} \in C \text{ class in } \]

\[ \mathcal{Z}(t^+_E \text{tangent}) \cap C = \mathcal{O}_E \cap C = \text{supp}_y (C) \]

with correct properties.
$\mathcal{D}T^4$ theory

$\exists \xrightarrow{(\text{twisted})} \text{universal sheaf}$

$X \times M \xrightarrow{\pi} M$

$R_{\pi} \text{Hom}(\mathcal{E}, \mathcal{E}) \big|_{\mathcal{M}} \xrightarrow{\mathcal{A}_{t}} \mathcal{L}_{\mathcal{M}}$

$\text{II} \text{I} \text{Ie} \text{rr} \text{d} \text{u} \text{i} \text{ty} \text{ d} \text{o} \text{m} \pi$

$R_{\pi} \text{Hom}(\mathcal{E}, \mathcal{E}) \big|_{\mathcal{M}} \xrightarrow{[-1]}$

$\text{obstruction}\text{ theory}$

$\text{imperfect: } n \text{ degs } -2, -1, 0$
Some homological algebra can represent thin by $E_0 \xrightarrow{a} E_1 \xrightarrow{a} E_0 \rightarrow \mathbb{I}_M$

where $E_1$ is an $SO(n,\mathbb{C})$ bundle.

Locally $M$ cut out of smooth ambient space modelled on $E_0$ by section $s$ of $E_0 \cong E_1$.

Beltrami-Farkas: Cone $C := \lim_{\lambda \to 0} \Gamma_{\lambda s} C_{E_1}$ glues globally.

Darboux local model $\Rightarrow$ S isotropic $\Rightarrow$ Cone $C$ isotropic

$E$ localized $\sqrt{O_{E_1}} C = \{ M \}^{\mathbb{C}} \in A_\ast (M, \mathbb{Z}[\frac{1}{2}])$. 
$[M]^\nu$ well defined, independent of choices.

Surely equal (in $H^\nu_0 (\mathcal{M} , \mathbb{Z}(\frac{\nu}{2}))$) to $[M]^\nu_{BJ}$

Local holomorphic Kuranishi models glue in $C^0$ sense to global chart; need

\[
\left[ \lim_{\lambda \to 0} \Gamma_{\lambda s} \right] = C
\]

then homotop the cosection intersection with $\mathcal{L} cE_1$ with BJ's intersection with $E cE_1$. 

§5 Tors localization

\[ T \mathcal{N}(x, \mathbb{L}) \text{ preserving } \mathbb{L} \]

\[ \Rightarrow T \mathbb{M} \text{ preserving self-duality of } \mathbb{R}^3, \mathbb{R}^3, (3, 3) \]

\[ \text{work} \quad \text{on } M^T \rightarrow M, \text{ splits into fixed + moving parts} \]

\[ (E_0^*)^* \rightarrow E_1^\text{fix} \rightarrow E_0^\text{fix} \oplus (E_0^*)^* \rightarrow E_1^\text{mov} \rightarrow E_0^\text{mov} \]

\[ \downarrow \quad \text{gives } [M^T]^w \text{ as before} \]

\[ \text{This is } N_t. \text{ We would like to take } \frac{1}{2} \text{ of this.} \]
The formula

Define \( \sqrt{E_T(N^{\text{min}})} := e_T(E_0^{\text{min}})^* \sqrt{e_T(E_1^{\text{min}})} \)

Then

\[
[M]^{\text{min}} = I \times \frac{[M^T]^{\text{min}}}{\sqrt{E_T(N^{\text{min}})}}
\]