Counting Sheaves on CY4

Joint with Jeongseok Oh (KIAS).

1. Gauge theory motivation / Borisov-Joyce virtual cycle
2. Local model (Darboux thm of Brav-Bussi-Joyce, Bouaziz-Grojnowski)
3. Edidin-Graham $\sqrt{\text{Euler}}$ class of $SO(N,\mathbb{C})$ bdls
4. EG class localised w.r.t. a) isotropic section b) isotropic cone
5. Algebraic virtual cycle

$$[M]^{vir}_{BJ} \in H_{2*}(M,\mathbb{Z}) \longrightarrow H_{2*}(M,\mathbb{Z}[\tfrac{1}{2}])$$

$$[M]^{vir}_{OT} \in A_*(M,\mathbb{Z}[\tfrac{1}{2}]) \longrightarrow H_{2*}(M,\mathbb{Z}[\tfrac{1}{2}])$$

6. Torus localisation formula Justify calcs of Cao, Kool, Leung, Maulik, Monavari, Nekrasov, Piazzalunga, Toda modulo choice of orientation.

§1. Gauge theory

X smooth projective variety $/\mathbb{C}$

$\mathcal{M}$ moduli space of <u>stable</u> holomorphic bundles $/X$ of fixed topological type

simple

sheaves

(spherical twist trick on CYs)

$\updownarrow$

Integrable $\bar\partial$ operators $\bar\partial_A : \Omega^0(F) \to \Omega^{0,1}(F)$ on fixed C^∞ bdl $F \to X$

(fixed $\bar\partial_{\det A}$ on $\det F$)

$$\boxed{F_A^{0,2} = \bar\partial_A^2 = 0} \in \Omega^{0,2}(\mathrm{End}_0 F)$$

modulo gauge $\Omega^0(\mathrm{Aut}\, E)$

Linearisation fits into elliptic complex

$$\underbrace{\Omega^0(\mathrm{End}_0 F) \xrightarrow{\bar\partial_A} \Omega^{0,1}(\mathrm{End}_0 F) \xrightarrow{\bar\partial_A} \Omega^{0,2}(\mathrm{End}_0 F)}\xrightarrow{\bar\partial_A} \Omega^{0,3}(\mathrm{End}_0 F) \xrightarrow{\bar\partial_A} \Omega^{0,4}(\mathrm{End}_0 F) \longrightarrow \cdots$$

- Not Fredholm if just take this bit in dims >2. Overdetermined: more eqns than unknowns $vd = -\infty$

- Can try to truncate (replace $\Omega^{0,2}$ by $\ker \bar\partial_A : \Omega^{0,2} \to \Omega^{0,3}$) but this ker jumps in general as vary A.

(Works in dim 3 when $K_X \leq 0 \Rightarrow$ virtual cycle and invts)

- Cohomology groups
  - $H^0(\mathrm{End}_0 F)$ infinitesimal autos
  - $H^1(\mathrm{End}_0 F)$ deformations
  - $H^2(\mathrm{End}_0 F)$ obstructions
  - $H^3(\mathrm{End}_0 F) \neq 0$ in general $\Rightarrow$ no "perfect obstruction theory"

(X,Ω) CY^4 case: the complex is self-dual

+ metrics $\begin{cases} \text{Ricci-flat on } X \\ \text{hermitian on } F \end{cases}$

suggests halving it

(cf. asd eqⁿˢ on 4-mflds)

Complex anti-linear Hodge star-like operator

$$\tilde{*} : \Lambda^{0,2}(\mathrm{End}_0 F) \xrightarrow{\wedge\Omega} \Lambda^{4,2}(\mathrm{End}_0 F) \xrightarrow{\bar{*}} \Lambda^{0,2}(\mathrm{End}_0 F)$$

$\tilde{*}^2 = \mathrm{id} \Rightarrow \Lambda^{0,2} = \Lambda^{0,+} \oplus \Lambda^{0,-}$ ±1 eigen spaces

real subbdls of $\Lambda^{0,2}(\mathrm{End}_0 F)$

"Half" of $F_A^{0,2} = 0$ eqⁿ is "complex asd eqⁿ"

$\boxed{F_A^{0,+} = 0}$ $\in \Omega^{0,+}(\mathrm{End}_0 F)$ Elliptic when coupled with HYM eqⁿ $\omega \cdot F_A^{1,1} = 0$

Special case of $SU(4)/\mathrm{Spin}(7)$ instanton eqⁿs.

Half holomorphic = holomorphic?

$$\boxed{F_A^{0,2}=0 \iff F_A^{0,+}=0 \ \text{and} \ \int_X p_1(F)\wedge\Omega=0}$$

(cf. Donaldson theory
A flat $\iff$ asd and $p_1=0$)

$$-4\pi^2\int_X p_1(F)\wedge\Omega = \int_X tr\left(F_A^{0,2}\wedge F_A^{0,2}\right)\wedge\Omega = \int_X tr\left(F_A^{0,+}\wedge F_A^{0,+}\right)\wedge\Omega + \int_X tr\left(F_A^{0,-}\wedge F_A^{0,-}\right)\wedge\Omega$$

$$= \int_X tr\left(F_A^{0,+}\wedge \tilde{*} F_A^{0,+}\right)\wedge\Omega - \int_X tr\left(F_A^{0,-}\wedge \tilde{*} F_A^{0,-}\right)\wedge\Omega$$

$$= \int_X tr\left(F_A^{0,+}\wedge \tilde{*} F_A^{0,+}\right) - \int_X tr\left(F_A^{0,-}\wedge \tilde{*} F_A^{0,-}\right)$$

$$= \|F_A^{0,+}\|^2 - \|F_A^{0,-}\|^2$$

Therefore $\|F_A^{0,2}\|^2 = \|F_A^{0,+}\|^2 + \|F_A^{0,-}\|^2 = 2\|F_A^{0,+}\|^2 + 4\pi^2\int_X p_1(F)\wedge\Omega$

So when $p_1 \cup [\Omega]=0$ (only) we get natural way to compactify moduli space of $SU(4)$ instantons using algebraic geometry.

So can consider $\mathcal{M}$ as being cut out (set theoretically!) by $F_A^{0,+} = 0$ instead of $F_A^{0,2} = 0$ to give it a Kuranishi structure.

This is essentially (not really) what [BJ] do to define $[\mathcal{M}]^{\mathrm{vir}} \in H_{\mathrm{vd}_{\mathbb{R}}}(\mathcal{M}, \mathbb{Z})$.

- $\mathrm{vd}_{\mathbb{R}}$ can be odd
- hard to compute $[\mathcal{M}]^{\mathrm{vir}}$
- can only handle $p_1 \cup [S] = 0$ case at present so that $F_A^{0,+} = 0 \iff F_A^{0,2} = 0 \Rightarrow$ can use algebraic geometry to compactify $\mathcal{M}$.
- would like a construction in algebraic geometry (implies $[\mathcal{M}]^{\mathrm{vir}} = 0$ if $\mathrm{vd}_{\mathbb{R}}$ odd)

Note $\Omega^{0,2}(\mathrm{End}_0 F)$ has a complex quadratic form $Q(a,b) := \int \mathrm{tr}(a \wedge b) \wedge \Omega$ on it w.r.t. which $F_A^{0,2}$ is <u>isotropic</u> $Q(F_A^{0,2}, F_A^{0,2}) = 0$ iff $p_1 \cup [\Omega] = 0$

§2 <u>Local model</u> (Darboux theorem of [BBJ, BG])

About any $F \in \mathcal{M}$ is Zariski locally

$$\boxed{Z(s) \subset \underset{H^1(\mathrm{End}_0 F)}{\mathrm{Ext}^1(F,F)_0}}$$

where s is an <u>isotropic</u> section $Q(s,s) = 0$ of a quadratic bundle $(E, Q) \to \mathrm{Ext}^1(F,F)_0$.

E modelled on $\mathrm{Ext}^2(F,F)_0$ with Serre duality quadratic structure Q at F

(but Q is only étale/analytically locally trivial)

$SO(n,\mathbb{C}) \underset{\text{equiv}}{\overset{\text{htpy}}{\sim}} SO(n,\mathbb{R})$

So $SO(n,\mathbb{C})$ bundles E can be written $E = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E_{\mathbb{R}} \oplus i E_{\mathbb{R}} = E^+ \oplus E^-$

orientation dealt with by [Cao-Gross-Joyce]

$SO(n,\mathbb{R})$ bdl, metric

negative definite

So admit a characteristic class $e(E_{\mathbb{R}})$ s.t. $e(E_{\mathbb{R}})^2 = e(E) = 0_E^!(0_E)$

Call this $\sqrt{e}(E)$.

Morally, [BJ] are seeing M as $\sqrt{e}(E)$ cut out by projection s_+ of isotropic section s of E.

$s = (s_+, s_-)$

Makes sense because

$$Q(s,s) = 0 = Q(s_+,s_+) + Q(s_-,s_-) = |s_+|^2 - |s_-|^2 \iff |s|^2 = 2|s_+|^2$$

$s = 0$ iff $s_+ = 0$

§3 Ededin–Graham

(Relevance suggested by Cao–Leung)

But $\exists$ algebraic way of seeing $\sqrt{e}(E)$ using maximal isotropic subbundles

SO(m, $\mathbb{C}$) bdl; take $m=2n$ for simplicity in this talk

$$0 \to \Lambda \to E \to \Lambda^* \to 0$$

of E instead of real subbundles.

(if $m=2n+1$ Λ^* extended by 2-torsion line bdl $\Rightarrow e=0$)

Note if $\exists \Lambda$ then $e(E) = e(\Lambda)e(\Lambda^*)$ $= (-1)^n e(\Lambda)^2$ so $\boxed{e(\Lambda) = (-1)^n \sqrt{e}(E)}$

And $\Lambda \cap E^{\pm} = \{0\}$ so projection $\Lambda \to E^+$ is isomorphism.

Work on a cover $\tilde{Y} \xrightarrow{\pi} Y$ on which π^*E has a canonical maximal isotropic $0 \to \Lambda \to \pi^*E \to \Lambda^* \to 0$

(cf. splitting principle)

$\tilde{Y}$ = fibrewise isotropic flag variety of E

$$= \left\{ \Lambda_1 \subseteq \Lambda_2 \subseteq \dots \subseteq \Lambda_{n-1} \subseteq E \; : \; \begin{array}{l} \dim \Lambda_i = i \\ q|_{\Lambda_i} \equiv 0 \end{array} \right\}$$

$\pi^* : A^*(Y) \longrightarrow A^*(\tilde{Y})$ injective so any class $\sqrt{e}(E)$ s.t. $\pi^* \sqrt{e}(E) = c_n(\Lambda)$ is <u>unique</u>.

[EG] define $\sqrt{e}(E) := \pi_*(c_n(\Lambda) \cup h)$

where h is a canonical class s.t. $\pi_* h = 1$.

requires $\mathbb{Z}[\frac{1}{2}]$ coefficients.

$$\left[\begin{array}{c} \sqrt{e}(E) \in A^n(Y, \mathbb{Z}[\tfrac{1}{2}]) \searrow \\ \qquad\qquad H^{2n}(Y, \mathbb{Z}[\tfrac{1}{2}]) \\ e(E_{\mathbb{R}}) \in H^{2n}(Y, \mathbb{Z}) \nearrow \end{array}\right]$$ same image

If $\exists$ max isotropic $\Lambda' \subset E$ then $\sqrt{e}(E) = c_n(\Lambda')$.

§4 Localising EG class.

Key technical tool: **Cosection localisation** (Kiem-Li)

Given $C \subset E \xrightarrow{s} \mathcal{O}_Y$ $\quad s \in \Gamma(E^*)$ cosection

cycle (eg cone) $\qquad$ vector bundle/Y

Such that $C \subset \mathrm{kernel}(s)$ (set theoretically)

then can localise the intersection class $0_E^! C$

to $\underline{Z(s)} \cap C \cap 0_E$

(ie get a cycle here whose pushforward to $0_E \cong Y$ is $0_E^! C$.)

Linearise intersection of $\Gamma_s \subset E$ and $\Lambda \subset E$

$$\underbrace{N_{\Gamma_s \cap \Lambda / E}}_{\text{"normal cone" to } \Gamma_s \cap \Lambda \text{ inside } E} \subset N_{\Lambda/E} = q^* \Lambda^* \xrightarrow{\tau_\Lambda^{taut}} \mathcal{O}_{\mathrm{Tot}(\Lambda)}$$

tautological section of Λ on $\mathrm{Tot}(\Lambda)$

$$\left[\begin{array}{c} \mathrm{Tot}(\Lambda) \\ \downarrow q \\ Y \end{array}\right]$$

FACT: s isotropic $\Rightarrow N_{\Gamma_s \cap \Lambda / E}$ lies in $\ker\left(\tau_\Lambda^{taut}\right)$

So cosection localisation gives class in

$$\underbrace{Z(\tau_\Lambda^{taut}) \cap \Lambda}_{O_\Lambda \cong Y} \cap \Gamma_s = Z(s) \subset Y$$

as required.

Pushforward to Y is $e(\Lambda^*) = (-1)^n \sqrt{e}(E)$.

§4(b) Localising [EG] class by isotropic cone

$$\begin{tikzcd} C \arrow[dr, "p"'] & \subset & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

isotropic means

$\mathcal{O}_C \xrightarrow{\tau_E^{taut}} p^*E$ is an

isotropic section

$\Rightarrow$ can localise as in (a) to its zeros

$\Rightarrow$ Gives $\sqrt{0_E^!}\, C$ class in

$$Z(t_E^{taut}) \cap C = 0_E \cap C = \operatorname{supp}_Y(C)$$

with correct properties.

DT^4 theory

X CY^4

$\mathcal{M}$ moduli of stable sheaves (fixed topological type)

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ X \times \mathcal{M} \xrightarrow{\pi} \mathcal{M} \end{array}$$

(twisted) universal sheaf

$$R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[3] \xrightarrow{At} \mathbb{L}_{\mathcal{M}}$$

obstruction theory

imperfect: in degs $-2, -1, 0$

$\wr\|$ Serre duality down π

$$R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0^\vee[-1]$$

Some homological algebra

$\Rightarrow$ can represent this by $\left\{E_0^* \xrightarrow{a^*} E_1 \xrightarrow{a} E_0\right\} \to \mathbb{1}_M$ where $E_1 \overset{Q}{\cong} E_1^*$

where E_1 is an $SO(m, \mathbb{C})$ bundle

Locally M cut out of smooth ambient space modelled on E_0^* by section s of $E_1^* \cong E_1$.

<u>Behrend-Fantechi:</u> Cone $C := \lim_{\lambda \to \infty} \Gamma_{\lambda s} \subset E_1$ glues globally.

Darboux local model $\Rightarrow$ s isotropic $\Rightarrow$ Cone C <u>isotropic</u>

$\Rightarrow \exists$ Localised $\sqrt{0_{E_1}^!}\, C =: [M]^{\mathrm{vir}} \in A_*(M, \mathbb{Z}[\tfrac{1}{2}])$.

$[M]^{vir}$ well defined, independent of choices.

Surely equal (in $H_*(M, \mathbb{Z}[\frac{1}{2}])$) to $[M]^{vir}_{BJ}$

Local holomorphic Kuranishi models glue in C^∞ sense to global chart;

<u>need</u> $\boxed{\lim_{\lambda\to\infty} [\Gamma_{\lambda s}] = C} \subset E_1$; we can

then homotop the cosection intersection with $L \subset E_1$ with BJ's intersection with $E^- \subset E_1$.

§5 Torus localisation

$T \curvearrowright (X, \Omega)$ preserving Ω

$\Rightarrow T \curvearrowright \mathcal{M}$ preserving self-duality of $R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})$

$\overset{\text{work}}{\Longrightarrow}$ On $\mathcal{M}^T \overset{\iota}{\hookrightarrow} \mathcal{M}$, splits into fixed + moving parts

$$(E_0^{fix})^* \to E_1^{fix} \to E_0^{fix} \quad \oplus \quad (E_0^{mov})^* \to E_1^{mov} \to E_0^{mov}$$

gives $[\mathcal{M}^T]^{vir}$ as before

This is N^{vir}. We would like to take $\frac{1}{2}$ of this.

The formula

Define $\sqrt{e_T(N^{\mathrm{vir}})} := e_T\big((E_0^{\mathrm{mov}})^*\big)\sqrt{e_T(E_1^{\mathrm{mov}})}$

Then $$\boxed{[M]^{\mathrm{vir}} = \iota_* \frac{[M^T]^{\mathrm{vir}}}{\sqrt{e_T(N^{\mathrm{vir}})}}}$$