# On the Coupled Moduli Space of Exceptional Holonomy Manifolds with Instanton Bundles

Eirik Eik Svanes (Kings College London)

Based on work in collaboration with

B. Acharya, A. Ashmore, X. de la Ossa, E. Hardy,

M. Larfors, R. Minasian, C. Strickland-Constable.

arXiv:1402.1725, 1409.3347, 1509.08724,

1607.03473, 1704.08717, 17xx.xxxx, etc.

September 14th 2017, Simons Foundation, New York

# Introduction

Low energy limit of heterotic string theory is ten dimensional supergravity coupled to Yang-Mills gauge theory $\Rightarrow$ Easy to obtain nice particle physics [*Candelas etal '85, ..*].

This talk: Compactifications to four and three dimensions [*See talks by Acharya, Gukov*]

$$\mathcal{M}_{10} = \mathcal{M}_{10-d} \times X_d \ ,$$

where $\mathcal{M}_{10-d}$ is external spacetime, and $X_d$ is the internal (compact) geometry.

Supersymmetry: To lowest order $X$ is a torsion-free (special holonomy) manifold.
Heterotic: Spoiled by $\alpha'$-corrections! Harder to understand geometries and moduli space.

Low energy limit of heterotic string theory is ten dimensional supergravity coupled to Yang-Mills gauge theory $\Rightarrow$ Easy to obtain nice particle physics [*Candelas etal '85, ..*].

This talk: Compactifications to four and three dimensions [*See talks by Acharya, Gukov*]

$$\mathcal{M}_{10} = \mathcal{M}_{10-d} \times X_d \ ,$$

where $\mathcal{M}_{10-d}$ is external spacetime, and $X_d$ is the internal (compact) geometry.

Supersymmetry: To lowest order $X$ is a torsion-free (special holonomy) manifold. Heterotic: Spoiled by $\alpha'$-corrections! Harder to understand geometries and moduli space.

Why then heterotic?

- Good for Particle Physics: Easy to obtain Standard Model like physics.
- Similar structures often appear in geometries with more fibration structures [*See talks by Salamon, Foscolo, Haskins*].
- Natural generalizations of torsion free geometry with bundles, when bundle can back-react on the base. Mathematically interesting structures arise!
  - Local and global constructions, e.g. [*Fu-Yau '06, Andreas-GarciaFernandez '10, Halmagyi-Israel-EES '16, Acharya-EES '17, etc*].
  - Investigations into moduli [*Becker-Tseng '05, Anderson-Gray-Sharpe '14, delaOssa-EES '14, GarciaFernandez-Rubio-Tipler '15, delaOssa-Larfors-EES '17,* **This Talk!**].

3 Steps in understanding moduli of a stringy geometry:

3 Steps in understanding moduli of a stringy geometry:

- *Step 1:* Infinitesimal *massless* spectrum $T\mathcal{M}$. Identify differential $\mathcal{D}$ ($\mathcal{D}^2 = 0$). Tangent space of moduli space then given by cohomology

$$T\mathcal{M} = H^1_{\mathcal{D}}(\mathcal{Q}) \ .$$

Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf) naturally associated to the given moduli problem. [**Heterotic:** *Anderson-Gray-Sharpe '14, delaOssa-EES '14, delaOssa-Larfors-EES '16*].

3 Steps in understanding moduli of a stringy geometry:

- *Step 1:* Infinitesimal *massless* spectrum $T\mathcal{M}$. Identify differential $\mathcal{D}$ ($\mathcal{D}^2 = 0$). Tangent space of moduli space then given by cohomology

$$T\mathcal{M} = H^1_{\mathcal{D}}(\mathcal{Q}) \, .$$

  Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf) naturally associated to the given moduli problem. [**Heterotic:** *Anderson-Gray-Sharpe '14, delaOssa-EES '14, delaOssa-Larfors-EES '16*].

- *Step 2:* Understand geometry of $\mathcal{M}$ (e.g. Kähler metric [**Heterotic:** *Candelas etal '15*]). Higher order deformations, smoothness and obstructions (superpotential and Yukawa couplings). Maurer-Cartan elements,

$$\mathcal{D}\mathcal{X} + \tfrac{1}{2}[\mathcal{X}, \mathcal{X}] = 0 \, ,$$

  and associated differentially graded Lie algebra (or $L_\infty$-algebra). [**E.g.** *Tian-Toderov Lemma '87*, **Atiyah Algebroid:** Huang '94, **Heterotic:** *in progress '17*].

3 Steps in understanding moduli of a stringy geometry:

- *Step 1:* Infinitesimal *massless* spectrum $T\mathcal{M}$. Identify differential $\mathcal{D}$ ($\mathcal{D}^2 = 0$). Tangent space of moduli space then given by cohomology

$$T\mathcal{M} = H^1_{\mathcal{D}}(\mathcal{Q}) \, .$$

  Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf) naturally associated to the given moduli problem. [**Heterotic:** *Anderson-Gray-Sharpe '14, delaOssa-EES '14, delaOssa-Larfors-EES '16*].

- *Step 2:* Understand geometry of $\mathcal{M}$ (e.g. Kähler metric [**Heterotic:** *Candelas etal '15*]). Higher order deformations, smoothness and obstructions (superpotential and Yukawa couplings). Maurer-Cartan elements,

$$\mathcal{D}\mathcal{X} + \tfrac{1}{2}[\mathcal{X}, \mathcal{X}] = 0 \, ,$$

  and associated differentially graded Lie algebra (or $L_\infty$-algebra). [**E.g.** *Tian-Toderov Lemma '87*, **Atiyah Algebroid:** Huang '94, **Heterotic:** *in progress '17*].

- *Step 3:* Understand *quantum cohomology ring*. Include higher genus quantum effects. Construct topological theory of the corresponding structures, in analogy with e.g. topological A/B-models for CY's [*Witten '91*] and holomorphic Chern-Simons theory [*Donaldson-Thomas '98*]. Compute new invariants for structures, and relate to known invariants such as Gromov-Witten and Donaldson-Thomas invariants?

## The Heterotic $SU(3)$ System

$$\mathcal{M}_{10} = \mathcal{M}_4 \times X_6$$

Supersymmetric compactifications of the heterotic string to 4d Minkowski space require that the internal geometry $X$ be of Strominger-Hull type [*Strominger '86, Hull '86*]:

- The internal geometry admits an $SU(3)$-structure $(X, \Omega, \omega)$ satisfying

$$\mathrm{d}(e^{-2\phi}\Omega) = 0\,, \quad \mathrm{d}\left(e^{-2\phi}\omega\wedge\omega\right) = 0\,.$$

Supersymmetric compactifications of the heterotic string to 4d Minkowski space require that the internal geometry $X$ be of Strominger-Hull type [*Strominger '86, Hull '86*]:

- The internal geometry admits an $SU(3)$-structure $(X, \Omega, \omega)$ satisfying

$$\mathrm{d}(e^{-2\phi}\Omega) = 0\,, \quad \mathrm{d}\left(e^{-2\phi}\omega\wedge\omega\right) = 0\,.$$

- The bundle satisfies the holomorphic and Yang-Mills conditions

$$F\wedge\Omega = 0\,, \quad \omega\wedge\omega\wedge F = 0\,.$$

  Unique hermitian solutions $\Leftrightarrow$ bundle is poly-stable [*Donaldson '85, Uhlenbeck-Yau '86, Li-Yau '87*]. This talk: Assume *stable* bundle.

Supersymmetric compactifications of the heterotic string to 4d Minkowski space require that the internal geometry $X$ be of Strominger-Hull type [*Strominger '86, Hull '86*]:

- The internal geometry admits an $SU(3)$-structure $(X, \Omega, \omega)$ satisfying

$$\mathrm{d}(e^{-2\phi}\Omega) = 0\,, \quad \mathrm{d}\left(e^{-2\phi}\omega\wedge\omega\right) = 0\,.$$

- The bundle satisfies the holomorphic and Yang-Mills conditions

$$F\wedge\Omega = 0\,, \quad \omega\wedge\omega\wedge F = 0\,.$$

  Unique hermitian solutions $\Leftrightarrow$ bundle is poly-stable [*Donaldson '85, Uhlenbeck-Yau '86, Li-Yau '87*]. This talk: Assume *stable* bundle.

- The Neveu-Schwarz field strength $H$ satisfies anomaly cancellation condition

$$i(\partial - \bar{\partial})\omega = H = \mathrm{d}B + \tfrac{\alpha'}{4}\omega_{CS}(A)\,.$$

  Gauge invariance $\Rightarrow$ Kalb-Ramond field $B$ transforms [*Green-Schwarz '84*]

$$\omega_{CS}(A) \to \omega_{CS}(A) + \mathrm{d}\omega_2(A,g) \quad \Rightarrow \quad B \to B - \tfrac{\alpha'}{4}\omega_2(A,g) + \mathrm{d}\lambda\,,$$

  where $g \in \Omega^0(\mathrm{End}(V))$, $\lambda \in \Omega^1(X)$. This is the Green-Schwarz Mechanism.

Infinitesimal moduli preserving SUSY conditions $\Leftrightarrow$ *Massless* fields in 4d theory

This talk: Assume $X$ satisfies $\partial\overline{\partial}$-lemma (e.g. smooth $\alpha' \to 0$ Calabi-Yau limit exists).

Infinitesimal moduli preserving SUSY conditions $\Leftrightarrow$ *Massless* fields in 4d theory

This talk: Assume $X$ satisfies $\partial\overline{\partial}$-lemma (e.g. smooth $\alpha' \to 0$ Calabi-Yau limit exists).

The infinitesimal moduli space of the Strominger-Hull system is given by [*DelaOssa-EES '14, Anderson etal '14, Garcia-Fernandez etal '15*]

$$T\mathcal{M} = H^{(0,1)}_{\overline{D}}(Q)\;, \quad Q = T^{*(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X$$

where $\overline{D}$ is nilpotent by the supersymmetry conditions, and defines $Q$ as a double extension.

| | | |
|---|---|---|
| Infinitesimal moduli preserving SUSY conditions | $\Leftrightarrow$ | *Massless* fields in 4d theory |

This talk: Assume $X$ satisfies $\partial\overline{\partial}$-lemma (e.g. smooth $\alpha' \to 0$ Calabi-Yau limit exists).

The infinitesimal moduli space of the Strominger-Hull system is given by [*DelaOssa-EES '14, Anderson etal '14, Garcia-Fernandez etal '15*]

$$T\mathcal{M} = H^{(0,1)}_{\overline{D}}(Q)\,, \quad Q = T^{*(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X$$

where $\overline{D}$ is nilpotent by the supersymmetry conditions, and defines $Q$ as a double extension. To disambiguate

$$H^{(0,1)}_{\overline{D}}(Q) \cong H^{(0,1)}_{\overline{\partial}}(T^{*(1,0)}X) \oplus \text{ker}(\mathcal{H})\,,$$

where $\mathcal{H} \in \text{Ext}^1(Q_1, T^{*(1,0)}X)$ is given by the anomaly cancellation condition. $Q_1$ is the Atiyah extension with extension class given by $\mathcal{F}$ (the curvature) [*Atiyah '57*]

$$0 \to \text{End}(V) \to Q_1 \to T^{(1,0)}X \to 0\,.$$

We thus have $(\overline{\partial}_1 = \overline{\partial} + \mathcal{F})$

$$\text{ker}(\mathcal{H}) \subseteq H^{(0,1)}_{\overline{\partial}_1}(Q_1) \cong H^{(0,1)}(\text{End}(V)) \oplus \text{ker}(\mathcal{F})\,, \quad \text{ker}(\mathcal{F}) \subseteq H^{(0,1)}_{\overline{\partial}}(T^{(1,0)}X)\,.$$

*Generic* higher deformations of $SU(3)$ system $\Rightarrow$ complicated $L_\infty$-algebra.

*Clues from physics:* 4d theory is an $N = 1$ supergravity $\Rightarrow$ complex field space equipped with Kähler metric [*Candelas etal '15*], and *holomorphic* superpotential $W$ [*Becker etal '03, Cardoso etal '03, Lukas etal '05*]. A *finite holomorphic* deformation can be represented as

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q) \ ,$$

where $\alpha \in \Omega^{(0,1)}(\mathrm{End}(V))$, $x \in \Omega^{(0,1)}(T^{*(1,0)}X/\partial\text{-exact})$, $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)|_{\mathrm{ker}(\Delta)}$.

*Generic* higher deformations of $SU(3)$ system $\Rightarrow$ complicated $L_\infty$-algebra.

*Clues from physics:* 4d theory is an $N = 1$ supergravity $\Rightarrow$ complex field space equipped with Kähler metric [*Candelas etal '15*], and *holomorphic* superpotential $W$ [*Becker etal '03, Cardoso etal '03, Lukas etal '05*]. A *finite holomorphic* deformation can be represented as

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q) \ ,$$

where $\alpha \in \Omega^{(0,1)}(\mathrm{End}(V))$, $x \in \Omega^{(0,1)}(T^{*(1,0)}X/\partial\text{-exact})$, $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)|_{\mathrm{ker}(\Delta)}$.

Deform superpotential $\Rightarrow$ interesting generalisation of holomorphic Chern-Simons theory [*Ashmore-delaOssa-Minasian-StricklandConstable-EES '17*]

$$\Delta W = \int_X \left(\langle y, \overline{D}y\rangle + \tfrac{1}{3}\langle y, [y, y]\rangle\right) \wedge \Omega \ ,$$

Equation of motion reads

$$\overline{D}y + \tfrac{1}{2}[y, y] = \partial_a\text{-exact} \ ,$$

the Maurer-Cartan equation in the heterotic DGLA.

*Generic* higher deformations of $SU(3)$ system $\Rightarrow$ complicated $L_\infty$-algebra.

*Clues from physics:* 4d theory is an $N = 1$ supergravity $\Rightarrow$ complex field space equipped with Kähler metric [*Candelas etal '15*], and *holomorphic* superpotential $W$ [*Becker etal '03, Cardoso etal '03, Lukas etal '05*]. A *finite holomorphic* deformation can be represented as

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q) ,$$

where $\alpha \in \Omega^{(0,1)}(\text{End}(V))$, $x \in \Omega^{(0,1)}(T^{*(1,0)}X/\partial\text{-exact})$, $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)|_{\ker(\Delta)}$.

Deform superpotential $\Rightarrow$ interesting generalisation of holomorphic Chern-Simons theory [*Ashmore-delaOssa-Minasian-StricklandConstable-EES '17*]

$$\Delta W = \int_X \left(\langle y, \overline{D}y\rangle + \tfrac{1}{3}\langle y, [y, y]\rangle\right) \wedge \Omega ,$$

Equation of motion reads

$$\overline{D}y + \tfrac{1}{2}[y, y] = \partial_a\text{-exact} ,$$

the Maurer-Cartan equation in the heterotic DGLA.

Future: Compute correlation functions weighted by $\Delta W$, and define new invariants for heterotic geometry in spirit of [*Donaldson-Thomas '98, Donaldson-Segal '09*]? Is there a heterotic version of holomorphic linking [*Frenkel-Khesin-Todorov '97, Thomas '97*]?

# The Heterotic $G_2$ System

$$\mathcal{M}_{10} = \mathcal{M}_3 \times Y_7$$

Supersymmetry requires the internal geometry $Y$ to have a $G_2$-structure $\varphi$ ( $\psi = *\varphi$).
Torsion classes (decomposed into irreducible representations of $G_2$)

$$\mathrm{d}\varphi = \tau_1\psi + 3\,\tau_7 \wedge \varphi + *\tau_{27}$$
$$\mathrm{d}\psi = 4\,\tau_7 \wedge \psi + *\tau_{14}\ .$$

Note that $\mathrm{d}\varphi = \mathrm{d}\psi = 0 \;\Leftrightarrow\; \nabla^{LC}\varphi = 0$ ($G_2$ *holonomy*).

Supersymmetry requires the internal geometry $Y$ to have a $G_2$-structure $\varphi$ ( $\psi = *\varphi$). Torsion classes (decomposed into irreducible representations of $G_2$)

$$\mathrm{d}\varphi = \tau_1 \psi + 3\, \tau_7 \wedge \varphi + *\tau_{27}$$
$$\mathrm{d}\psi = 4\, \tau_7 \wedge \psi + *\tau_{14} \,.$$

Note that $\mathrm{d}\varphi = \mathrm{d}\psi = 0 \;\Leftrightarrow\; \nabla^{LC}\varphi = 0$ ($G_2$ *holonomy*).

Supersymmetry requires [*Papadopoulos etal '05, Lukas etal '10, Lüst et at '12, ..*],

$$H = H(\tau_i) = -\tau_{27} + \tfrac{1}{6}\tau_1\varphi - \tau_7 \lrcorner \psi \,, \quad \tau_{14} = 0$$
$$F \wedge \psi = 0 \,.$$

Note, this is an *integrable* $G_2$-structure $\Rightarrow$ can define differential complex [*Reyes-Carrion '93, Fernandez etal '98*]

$$0 \to \Omega^0_1 \xrightarrow{\check{\mathrm{d}}} \Omega^1_7 \xrightarrow{\check{\mathrm{d}}} \Omega^2_7 \xrightarrow{\check{\mathrm{d}}} \Omega^3_1 \to 0 \,.$$

$\check{\mathrm{d}} = \pi \circ \mathrm{d}$, where $\pi$ is the appropriate $G_2$ projection. Note the close analogy with the Dolbeault complex.

The complexes generalize to bundle-valued ones whenever $F$ is an instanton, and they are elliptic [*Reyes-Carrion '93*].

# Infinitesimal Moduli of $G_2$-Holonomy Manifolds with Instantons

$\mathcal{O}(\alpha'^0)$ Heterotic system reduces to a $G_2$-holonomy manifold with an instanton bundle. Infinitesimal geometric moduli counted by:

$$[\Delta_{t\,n}{}^m\, \mathrm{d}x^n] \in H^1_{\check{\mathrm{d}}_\nabla}(TY) \cong \mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

with $\check{\mathrm{d}}_\nabla$ the Levi-Civita connection, and $\mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)$ denotes harmonic forms.

$\mathcal{O}(\alpha'^0)$ Heterotic system reduces to a $G_2$-holonomy manifold with an instanton bundle. Infinitesimal geometric moduli counted by:

$$[\Delta_{t\,n}{}^m\, \mathrm{d}x^n] \in H^1_{\check{\mathrm{d}}_\nabla}(TY) \cong \mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

with $\check{\mathrm{d}}_\nabla$ the Levi-Civita connection, and $\mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)$ denotes harmonic forms. Harmonic forms decompose as

$$\mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY) = \mathcal{H}^{1\,(s)}_{\check{\mathrm{d}}_\nabla}(TY) \oplus \mathcal{H}^{1\,(a)}_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

where the symmetric/anti-symmetric representations correspond to metric moduli [*Joyce '94*] and $B$-field moduli [*deBoer etal '08*]

$$\mathcal{H}^{1\,(s)}_{\check{\mathrm{d}}_\nabla}(TY) \cong H^3(Y)\ , \quad \mathcal{H}^{1\,(a)}_{\check{\mathrm{d}}_\nabla}(TY) \cong H^2(Y)\ .$$

$\mathcal{O}(\alpha'^0)$ Heterotic system reduces to a $G_2$-holonomy manifold with an instanton bundle. Infinitesimal geometric moduli counted by:

$$[\Delta_{t\,n}{}^m \, \mathrm{d}x^n] \in H^1_{\check{\mathrm{d}}_\nabla}(TY) \cong \mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

with $\check{\mathrm{d}}_\nabla$ the Levi-Civita connection, and $\mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY)$ denotes harmonic forms. Harmonic forms decompose as

$$\mathcal{H}^1_{\check{\mathrm{d}}_\nabla}(TY) = \mathcal{H}^{1\,(s)}_{\check{\mathrm{d}}_\nabla}(TY) \oplus \mathcal{H}^{1\,(a)}_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

where the symmetric/anti-symmetric representations correspond to metric moduli [*Joyce '94*] and $B$-field moduli [*deBoer etal '08*]

$$\mathcal{H}^{1\,(s)}_{\check{\mathrm{d}}_\nabla}(TY) \cong H^3(Y)\ , \quad \mathcal{H}^{1\,(a)}_{\check{\mathrm{d}}_\nabla}(TY) \cong H^2(Y)\ .$$

An infinitesimal deformation of the instanton condition gives

$$\check{\mathrm{d}}_A \partial_t A = \pi_7\left(F_{mn} \mathrm{d}x^m \wedge \Delta^n\right) = \check{\mathcal{F}}(\Delta_t)\ .$$

$\check{\mathcal{F}}$ defines $\mathcal{Q}_1 = \mathrm{End}(V) \oplus TY$ as an extension (analogy with Atiyah map), and

$$T\mathcal{M}_1 = H^1_{\check{\mathrm{d}}_A}(\mathrm{End}(V)) \oplus \ker(\check{\mathcal{F}}) = H^1_{\check{\mathrm{d}}_1}(\mathcal{Q}_1)\ , \quad \ker(\check{\mathcal{F}}) \subseteq H^1_{\check{\mathrm{d}}_\nabla}(TY)\ ,$$

as expected. Here $\check{\mathrm{d}}_1 = \check{\mathrm{d}}_\nabla + \check{\mathcal{F}}$.

Including $\alpha'$ effects, the heterotic $G_2$ system naturally gives rise to a differential

$$\check{\mathcal{D}} = \begin{pmatrix} \check{\mathrm{d}}_A & \check{\mathcal{F}} \\ \check{\mathcal{F}} & \check{\mathrm{d}}_\theta \end{pmatrix} : \check{\Omega}^p \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} \to \check{\Omega}^{p+1} \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} ,$$

where the map $\check{\mathcal{F}} : \check{\Omega}^p(\mathrm{End}(V)) \to \check{\Omega}^{p+1}(TY)$ is given by

$$\check{\mathcal{F}}(\alpha)^m = \frac{\alpha'}{4} \pi \left[ \mathrm{tr} \left( g^{mn} F_{nq} \mathrm{d}x^q \wedge \alpha \right) \right] ,$$

and $\pi$ denotes the appropriate projection. The connection $\mathrm{d}_\theta$ has conncetion symbols

$$\theta_{mn}{}^p = \Gamma_{nm}{}^p = \Gamma^{LC\,p}_{nm} + \tfrac{1}{2} H_{nm}{}^p(\tau_i) .$$

For integrable $G_2$-structures $\Gamma$ is the unique metric connection with totally anti-symmetric torsion preserving the $G_2$ structure.

Including $\alpha'$ effects, the heterotic $G_2$ system naturally gives rise to a differential

$$\check{\mathcal{D}} = \begin{pmatrix} \check{\mathrm{d}}_A & \check{\mathcal{F}} \\ \check{\mathcal{F}} & \check{\mathrm{d}}_\theta \end{pmatrix} : \check{\Omega}^p \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} \to \check{\Omega}^{p+1} \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} ,$$

where the map $\check{\mathcal{F}} : \check{\Omega}^p(\mathrm{End}(V)) \to \check{\Omega}^{p+1}(TY)$ is given by

$$\check{\mathcal{F}}(\alpha)^m = \frac{\alpha'}{4} \pi \left[ \mathrm{tr} \left( g^{mn} F_{nq} \mathrm{d}x^q \wedge \alpha \right) \right] ,$$

and $\pi$ denotes the appropriate projection. The connection $\mathrm{d}_\theta$ has conncetion symbols

$$\theta_{mn}{}^p = \Gamma_{nm}{}^p = \Gamma_{nm}^{LC\,p} + \tfrac{1}{2} H_{nm}{}^p(\tau_i) .$$

For integrable $G_2$-structures $\Gamma$ is the unique metric connection with totally anti-symmetric torsion preserving the $G_2$ structure. We have the following theorem:

Theorem [*delaOssa-Larfors-EES*'17] : *Given a $G_2$ structure with a bundle $(Y, \varphi, A)$, we can construct the operator $\check{\mathcal{D}}$. This operator is nilpotent if and only if $(Y, \varphi, A)$ satisfies the heterotic $G_2$ system. Furthermore, the infinitesimal moduli of the heterotic $G_2$ system is*

$$T\mathcal{M} = H^1_{\check{\mathcal{D}}}(\mathcal{Q}_1) .$$

Note: To show [$\check{\mathcal{D}}^2 = 0 \;\Rightarrow\;$ *heterotic $G_2$ system*] requires the $\alpha'$ expansion.

Including $\alpha'$ effects, the heterotic $G_2$ system naturally gives rise to a differential

$$\check{\mathcal{D}} = \begin{pmatrix} \check{\mathrm{d}}_A & \check{\mathcal{F}} \\ \check{\mathcal{F}} & \check{\mathrm{d}}_\theta \end{pmatrix} : \check{\Omega}^p \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} \to \check{\Omega}^{p+1} \begin{pmatrix} \mathrm{End}(V) \\ TY \end{pmatrix} ,$$

where the map $\check{\mathcal{F}} \ : \ \check{\Omega}^p(\mathrm{End}(V)) \to \check{\Omega}^{p+1}(TY)$ is given by

$$\check{\mathcal{F}}(\alpha)^m = \frac{\alpha'}{4}\,\pi \left[ \mathrm{tr} \left( g^{mn} F_{nq} \mathrm{d}x^q \wedge \alpha \right) \right] ,$$

and $\pi$ denotes the appropriate projection. The connection $\mathrm{d}_\theta$ has conncetion symbols

$$\theta_{mn}{}^p = \Gamma_{nm}{}^p = \Gamma_{nm}^{LC\,p} + \tfrac{1}{2} H_{nm}{}^p(\tau_i) \ .$$

For integrable $G_2$-structures $\Gamma$ is the unique metric connection with totally anti-symmetric torsion preserving the $G_2$ structure. We have the following theorem:

Theorem [*delaOssa-Larfors-EES*'17] : *Given a $G_2$ structure with a bundle $(Y, \varphi, A)$, we can construct the operator $\check{\mathcal{D}}$. This operator is nilpotent if and only if $(Y, \varphi, A)$ satisfies the heterotic $G_2$ system. Furthermore, the infinitesimal moduli of the heterotic $G_2$ system is*

$$T\mathcal{M} = H^1_{\check{\mathcal{D}}}(\mathcal{Q}_1) \ .$$

Note: To show [$\check{\mathcal{D}}^2 = 0 \ \Rightarrow$ *heterotic $G_2$ system*] requires the $\alpha'$ expansion.

Cor.: *Strominger-Hull system $\Leftrightarrow$ A particular Holomorphic Yang-Mills connection on $Q$.*

Heterotic geometries give nice generalizations of torsion-free geometries when bundles are included, but the moduli problem is hard.

Heterotic geometries give nice generalizations of torsion-free geometries when bundles are included, but the moduli problem is hard.

For the future:

- Compute $H^1_{\check{\mathcal{D}}}(\mathcal{Q}_1)$ in explicit examples? Hard even for $G_2$-holonomy (no clear algebraic methods). Perhaps make progress for instantons on twisted connected sums [*SaEarp-Walpuski '11, '13*], or homogeneous geometries for torsional examples?
- Further investigation into higher order deformations and obstructions of the heterotic $SU(3)$ and $G_2$ systems. What is the $L_\infty$-algebra of the $G_2$ system, and what are the *integrable* deformations?
- Understand the geometric properties of the $G_2$ structure moduli space. No known natural complex structure or Kähler structure, in contrast to the $SU(3)$ system [*Candelas etal '15*], and M-theory compactifications on $G_2$ manifolds [Gukov '99, Hitchin '00, Beasley-Witten '02, Acharya-Gukov '04, Karigiannis-Leung '07].
- Quantum corrections: Is there a quasi-topological action governing the heterotic $G_2$ system? Compute invariants for heterotic geometries such as generalisations of Gromov-Witten and Donaldson-Thomas invariants?
- What is the significance of $\Delta W$ in terms of heterotic M-theory duality? Can $\Delta W$ say something about the M-theory superpotential or vice versa?

# Thank you!



Thank you for your attention!