Heterotic Distance conjectures and Symplectic Cohomology

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Based on work with: Javier Murgas-Ibarra,
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Jim Simons has had a large impact on my career:

- Postdoc in Simons Collab.: Sep '17 – Dec '19.

  $\rightsquigarrow$ Job at UiS

Analogs of Chern-Simons theory appears everywhere in my work:

- In particular, in the study of heterotic moduli:

$$S(y) = \int_X (\langle y, \bar{D} y \rangle + \tfrac{1}{3} \langle y, [y, y] \rangle \wedge \Omega$$

Thank You Jim!

(Heterotic) String Compactification

Let (X, ω, Ω) be an SU(3)-structure manifold:

$$\omega \wedge \Omega = 0 , \qquad i \frac{\Omega \wedge \bar{\Omega}}{\|\Omega\|_\omega^2} = \frac{1}{6} \omega \wedge \omega \wedge \omega = (*1)_\omega$$

10d spacetime: $\mathcal{M}_{10} = \mathcal{M}_4 \times X$.

The effective 4d physics on $\mathcal{M}_4$ is decided from the internal compact geometry X.

4d superpotential [Cardoso et al '03, Gurrieri et al '04]:

$$W = \int_X (H + i d\omega) \wedge \Omega$$

$$H := dB + \frac{\alpha'}{4}\left(\omega_{CS}(A) - \omega_{CS}(\tilde{\nabla})\right) \quad \text{(NS-Flux)}$$

A : Connection on $E_8 \times E_8 / SO(32)$ gauge bundle

$\tilde{\nabla}$: Connection on $\mathrm{End}(TX)$.

B : Heterotic Kalb-Ramond 2-form. Note: H is global, so B transforms under gauge transformations [Green-Schwarz '84].

Note :
$$dH = \frac{\alpha'}{4}\left(\mathrm{tr}\, F(A)\wedge F(A) - \mathrm{tr}\, R(\hat{\nabla})\wedge R(\hat{\nabla})\right)$$

(Anomaly / B.I.)

Vacuum Solutions:

Classical 4d (Susy) Minkowski vacua require:

$$\delta W = W = 0.$$

$\Rightarrow$ - X is complex : $\quad d\Omega = 0$

- Anomaly constraint : $\quad i(\partial - \bar{\partial})\omega = H$

- The bundle is holomorphic : $\quad F(A) \wedge \Omega = 0$

These three equations are referred to as F-term constraints.

Additionally, there are D-term constraints:

- Yang-Mills: $F(A) \wedge \omega \wedge \omega = 0$
- Conformally balanced: $d(\|\Omega\|_\omega \, \omega \wedge \omega) = 0$

These equations are collectively referred to as The Hull-Strominger system [Hull '86, Strominger '86].

Note:

- EOM: Require $\tilde{\nabla}$ to be an instanton:

(*) $R(\tilde{\nabla}) \wedge \Omega = 0 \quad , \quad \omega \wedge \omega \wedge R(\tilde{\nabla}) = 0.$

- $R(\tilde{\nabla})$ must be type $(1,1)$. $\Rightarrow$ Strominger
  Set $\tilde{\nabla}$ = Chern connection.

BUT; generically Chern is not an instanton.

- Hall: $\tilde{\nabla} = \nabla^H$ (Hall connection).

This is an instanton ($\Rightarrow$ satisfies EOM) to appropriate order in α'.

However: Equations only close modulo higher orders!

Common Solution: Promote $\tilde{\nabla}$ to a new "gauge field", satisfying its own instanton condition $(*)$.

Physically problematic: $\rightsquigarrow$ Introduce spurious new degrees of freedom?

The effective theory

From hereon: $\alpha' = 0$, and (X, ω, Ω) is a Calabi-Yau.

Deform superpotential around a CY background:

$$\Delta W = \int_X \left(\mu^a \bar{\partial} x_a + (H_{ab} + \partial x_{ab}) \mu^a \mu^b \right) \wedge \Omega$$

- $x \in \Omega^{(0,1)}(T^{*(1,0)}X)$: complexified hermitian deformation ($x = \delta\omega + i\,\delta B$).

- $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$: Beltrami differential ($\mu = \delta J$).

- $H \in \mathcal{H}^{(2,1)}(X)$ can either be thought of as a background flux, or a backgound complex structure deformation.

$$H = \frac{1}{2} h^a \wedge \Omega_{abc} dz^{bc} \quad , \quad h^a \in \mathcal{H}^{(0,1)}_{\bar\partial}(T^{(1,0)}X).$$

Treat ΔW as an effective theory:

Note: ΔW is quadratic in μ.

- "Large H" $\leftrightarrow$ "large mass" for μ.

Integrate out μ gives a schematic effective theory:

$$S_{eff} = \mathrm{Tr}\,\mathrm{Log}\,(H + \partial x) + \int_X \bar\partial x\,(H + \partial x)^{-1}\,\bar\partial x$$

The Quadratic effective action is:

$$S^{(2)}_{eff} = \int_X \mathrm{Tr}\left(\left(H^{-1}\partial x\right)^2\right) + \int_X \bar{\partial} x \left(H^{-1}\right) \bar{\partial} x$$

- What is H^{-1}?

In ΔW, H is viewed as a matrix acting on elements of $\Omega^{(0,1)}(T^{(1,0)}X)$.

It turns out that H is invertible whenever

$$0 \neq h^a \wedge h^b \wedge h^c \, \Omega_{abc} = K \, \bar{\Omega}$$ [Strominger '90, Candelas-delaOssa '91]

Constant $K \propto \mathrm{Yuk}(h,h,h)$ (Special Geometry)

$\Rightarrow$ H invertible $\Leftrightarrow$ $\mathrm{Yuk}(h,h,h) \neq 0$.

The Large distance Conjecture

Distance Conjecture [Ooguri-Vafa '06]:

> There should be a tower of light modes coming down at an exponential rate as we approach a large distance in any direction in moduli space.

Often:

Light States $\longleftrightarrow$ Eigenmodes of elliptic operator on internal manifold.

For example:

The infinite volume limit gives a tower of light Kaluza-Klein modes.

Write $H = c H_0$, for $H_0 \in \mathcal{H}^{(2,1)}$ fixed.

$$\Rightarrow S_{eff}^{(2)} = \frac{1}{c^2}\int_X \mathrm{Tr}\left((H_0^{-1}\partial x)^2\right) + \frac{1}{c}\int_X \bar{\partial} x \,(H_0^{-1})\, \bar{\partial} x$$

As $c \to \infty$, we expect two towers of states becoming light at $\mathcal{O}(c^{-2})$ and $\mathcal{O}(c^{-1})$.

Where is the exponential decay?

To see exponential decay, we need to go to geodesic coordinates:

At large c, the prepotential becomes

$$\mathcal{G} \propto \mathrm{Yuk}(h,h,h) \propto c^3 + \text{lower orders}$$

Leading to a Kähler potential:

$$\mathcal{K} = -\operatorname{Log}\left(\bar{c}c^2 + c^2\bar{c}\right) + \text{lower orders}$$

Setting $c = e^{i\phi}\lambda$

$$\Rightarrow \quad K_{\lambda\lambda} = \frac{3}{\lambda^2} .$$

Switching to flat or

"geodesic" coordinates T:

$$\Rightarrow \quad \lambda = |c| = \mathrm{Exp}(-T)$$

where T parametrises the geodesic distance.

Elliptic Complexes:

What are the elliptic operators whose eigenmodes become light?

⇝ Tied to gauge structure of effective action.

Consider E.g.

$$\int_X \bar{\partial}x \, (H_0^{-1}) \, \bar{\partial}x = \int_X \bar{\partial}x \wedge *_0 \bar{\partial}x = -\int_X x \wedge \partial *_0 \bar{\partial}x$$

where $*_0$ is some "Hodge-dual" like operator defined by H_0.

This action has a naive gauge symmetry:

$$\delta x = \bar\partial \alpha \quad , \quad \alpha \in \Omega^{(1,0)}.$$

Fits into differential complex:

$$0 \to \Omega^{(1,0)} \xrightarrow{\bar\partial} \Omega^{(1,1)} \xrightarrow{\bar\partial *_0 \bar\partial} \Omega^{(2,2)} \xrightarrow{\bar\partial} \Omega^{(2,3)} \to 0$$

However, this complex is NOT elliptic!

We like elliptic complexes:
- Hodge decomposition
- Elliptic Laplacians
- BV-quantisation

Interlude: Symplectic Cohomology:

Given a Calabi-Yau 3-fold X, with a Kähler form ω.

Any form has a Lefschetz decomposition. For example:

$$\alpha \in \Omega^3 \Rightarrow \alpha = \alpha_P + \omega \wedge \beta.$$

where $\alpha_P \in P^3$ is primitive: $\omega \lrcorner \alpha_P = 0$.

Restricting to primitive forms for simplicity, We can decompose de Rham as [Tseng-Yau '09, '10]:

$$d = d_+ + \omega \wedge d_-$$

where :

$$d_+ : P^n \to P^{n+1}$$

$$d_- : P^n \to P^{n-1}$$

Note: $d^2 = 0$

$\Rightarrow \quad d_+^2 = 0 , \quad d_-^2 = 0 , \quad \omega \wedge (d_+ d_- + d_- d_+) = 0.$

d_+ and d_- form part of a primitive elliptic complex:

$$0 \to P^0 \xrightarrow{d_+} P^1 \xrightarrow{d_+} P^2 \xrightarrow{d_+} P^3 \xrightarrow{d_+ d_-} P^3 \xrightarrow{d_-} P^2 \xrightarrow{d_-} P^1 \xrightarrow{d_-} P^0 \to 0$$

Back to Heterotic:

In our case, to find the corresponding elliptic gauge complex, define

$$\hat{x}^{ab}{}_{\bar{i}} = \bar{\Omega}^{abc} x_{a\bar{c}} \in \hat{\Omega}^{(2,1)}$$

Note that $h_{\bar{a}}{}^{b} \in \hat{\Omega}^{(1,1)}$. We think of h as a (complexified) "Kähler form".

Primitive decomposition:

$$\hat{x} = \hat{x}_{p} + h \wedge \hat{v} \quad , \quad \hat{x}_{p} \in \hat{P}^{(2,1)} \quad , \quad \hat{v} \in \hat{\Omega}^{(1,0)} = \hat{P}^{(1,0)}$$

"wedge product" on $\hat{\Omega}^{(p,q)}$

Note: $\hat{x}_r$ is primitive : $h_a{}^{\bar{c}}\, \hat{x}_r{}^{ab}{}_{\bar{c}} = 0$

where $h_a{}^{\bar{c}}$ is the inverse of $h_{\bar{a}}{}^{b}$:

$$h_a{}^{\bar{c}} h_{\bar{c}}{}^{b} = \delta_a{}^{b} .$$

As in symplectic cohomology, decompose

$$\bar{\partial} = \bar{\partial}_+ + h \wedge \bar{\partial}_-$$

$$\bar{\partial}_+ : \hat{P}^{(r,q)} \longrightarrow \hat{P}^{(r,q+1)}$$

$$\bar{\partial}_- : \hat{P}^{(r,q)} \longrightarrow \hat{P}^{(r-1,q)}$$

Again, nilpotency of $\bar\partial$ gives:

$\bar\partial^2 = 0 \Rightarrow$

$$\bar\partial_+^2 = 0$$

$$\bar\partial_-^2 = 0$$

$$h\wedge(\bar\partial_+\bar\partial_- + \bar\partial_-\bar\partial_+) = 0$$

Note: The last equality implies $\bar\partial_+, \bar\partial_-$ anti-commute on $\hat{P}^{(p,q)}$ for $p+q<3$.

The action decomposes as:

$$\int_X \bar{\partial} x (H_0^{-1}) \bar{\partial} x$$

$$\propto \int_X \hat{x}_\mu \tilde{\partial}_+ \bar{\partial}_- \hat{x}_\mu + \int_X h \wedge h \wedge \hat{v} \tilde{\partial}_+ \bar{\partial}_- \hat{v}$$

Gauge symmetry:

$$\delta \hat{x}_\mu = \tilde{\partial}_+ \hat{\alpha} \; ; \quad \hat{\alpha} \in \hat{P}^{(2,0)}$$

$$\delta \hat{v} = \bar{\partial}_- \hat{\beta} \; ; \quad \hat{\beta} \in \hat{P}^{(2,0)}$$

$$\delta \hat{\beta} = \bar{\partial}_- \gamma \; ; \quad \hat{\gamma} \in \hat{P}^{(3,0)}$$

We get elliptic gauge complexes:

$$0 \to \hat{P}^{(2,0)} \xrightarrow{\bar{\partial}_+} \hat{P}^{(2,1)} \xrightarrow{\bar{\partial}_+ \bar{\partial}_-} \hat{P}^{(1,2)} \xrightarrow{\bar{\partial}_-} \hat{P}^{(0,2)} \to 0$$

$$\hat{P}^{(2,0)} \ni \hat{\alpha}, \qquad \hat{P}^{(2,1)} \ni \hat{\chi}_\mu$$

$$0 \to \hat{P}^{(3,0)} \xrightarrow{\bar{\partial}_-} \hat{P}^{(2,0)} \xrightarrow{\bar{\partial}_-} \hat{P}^{(1,0)} \xrightarrow{\bar{\partial}_+ \bar{\partial}_-} \hat{P}^{(0,1)} \xrightarrow{\bar{\partial}_+} \hat{P}^{(0,2)} \xrightarrow{\bar{\partial}_+} \hat{P}^{(0,3)} \to 0$$

$$\hat{P}^{(3,0)} \ni \hat{\gamma}, \qquad \hat{P}^{(2,0)} \ni \hat{\beta}, \qquad \hat{P}^{(1,0)} \ni \hat{\nu}$$

We should use these complexes to:

- Do Hodge theory
- Define Elliptic Laplacians
- BV-quantise actions.

Note: The second complex indicates the theory is 1-reducible:

=> There is a "gauge of gauge" symmetry.

Outlook:

- Hodge theory: Hodge decomposition, Laplacians, ...

- BV-quantisation: Compute "Ray-Singer" torsion of complexes. Relations to mirror symmetry / other dualities?

- Our setup only makes sense if

$$\mathrm{Yuk}(u,u,u) \neq 0 .$$

  What about vanishing Yukawa coupling?

- $\alpha' \neq 0$? Couplings to gauge sector, world-sheet instantons, Heterotic G_2, F-theory, ...

Thank You :)