Thomas-Yau conjecture Variational method

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Setting

- We work in the exact setting. The ambient manifold is a **Stein Calabi-Yau** manifold $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, with a nowhere vanishing holomorphic volume form Ω.
- In particular $H_n(X)$ has no torsion, and higher homology vanishes.
- Assume the regularity scale of the Calabi-Yau manifold grows to $+\infty$ asymptotically near infinity.
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- In particular $H_n(X)$ has no torsion, and higher homology vanishes.
- Assume the regularity scale of the Calabi-Yau manifold grows to $+\infty$ asymptotically near infinity.
- The Lagrangians are **exact, compact, and almost calibrated**.
- Recall exactness means

$$d\lambda = \omega, \quad df_L = \lambda|_L.$$

Caveat: we do not require f_L to take the same value at self intersections of immersed Lagrangians. There can be teardrop curves. **Quantitatively almost calibrated** means

$$-\frac{\pi}{2} + \epsilon \leq \theta \leq \frac{\pi}{2} - \epsilon.$$

Background 1: Solomon functional

Jake Solomon introduced a functional (up to universal cover issue) among a fixed Hamiltonian isotopy class of Lagrangians, with the property that its first variation for the Hamiltonian deformation H is

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where $\hat{\theta} = \arg \int_L \Omega$.
Consequence: **critical points are formally special Lagrangians**.

Solomon functional

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Answer: suppose L is isomorphic to L_0 in D^bFuk, so in particular homologous. We take $\mathcal{C}$ so that $\partial\mathcal{C} = L - L_0$. Recall L is an exact Lagrangian with potential f_L.

$$\mathcal{S}(L) = \int_L f_L \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{L_0} f_{L_0} \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{\mathcal{C}} \lambda \wedge \mathrm{Im}(e^{-i\hat{\theta}}\Omega).$$

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Remark

(Homological property) Changing $\mathcal{C}$ by any exact integration current does not affect the functional.

Background: immersed Floer theory

- Typical immersed Lagrangian: union of several Lagrangians (with transverse intersections).
- The **Floer degrees** at intersection points: (different from Seidel convention!)

$$\mu_{L,L'}(p) = \frac{1}{\pi}(\sum_{1}^{n} \phi_i + \theta_L(p) - \theta_{L'}(p)),$$

where

$$T_pL = \mathbb{R}^n \subset \mathbb{C}^n, \quad T_pL' = (e^{i\phi_1}, \dots e^{i\phi_n})\mathbb{R}^n.$$

Immersed Floer

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- In Floer theory, Lagrangian branes are boundary conditions of holomorphic curves. The brane structure (grading, local system, spin structures) is responsible for extracting counts.
- In the immersed case, We need to remember some data at self intersection points, which encode how holo curves cross from one branch to another, and how these are weighted.

immersed Floer theory

- The self intersection data at degree one self intersection points are known as '**bounding cochains**'. To make Floer cohomology well defined, they need to satisfy the Mauer-Cartan equation (cancellation of obstructions)

$$m_0 + m_1(b) + m_2(b,b) + \ldots = 0 \in CF^2_{self-intersection}(L,L).$$

  Then m_1^b squares to zero, where

$$m_1^b(x) = \sum m(b, \ldots b, x, b, \ldots).$$

- Each self intersection has a Novikov exponent $f_{L_+}(p) - f_{L_-}(p)$. Bounding cochains involve only self intersections with **non-negative Novikov exponents**. This is needed in Floer theory to bound the energy of holomorphic discs.

Topological energy of holomorphic discs between L, L':

$$\int_{\Sigma} u^*\omega + \sum_{corners} f|_{-}^{+} = 0.$$

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There are two types of corners: self intersections on L, L', and $CF^*(L, L')$, $CF^*(L', L)$ corners.
The **positivity of Novikov exponent** says that the corners at bounding cochain elements located at the self intersections have $f|_-^+ \geq 0$. This entails that for fixed $CF^*(L, L'), CF^*(L', L)$ corners, there is a uniform energy bound independent of the number of bounding cochain insertions. This is a standard condition in Floer theory to ensure convergence of things like

$$m_0 + m_1(b) + m_2(b, b) + \dots.$$

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- In immersed Floer theory, cones/twisted complexes are tautologically represented by immersed Lagrangians with some bounding cochain elements supported at the intersection points.
- **exact triangles** $L_1 \to L \to L_2$ can be recast (roughly) as saying L is isomorphic to an object supported on $L_1 \cup L_2$.

Immersed Floer theory

- Technical caveat: The $D^bFuk(X)$ I am using does not add in idempotents. All branes are geometric, but we do not know the idempotent closedness (If Joyce's picture is right then it is, but we don't need it).

Thomas-Yau semistability:

- All Lagrangians involved are exact, almost calibrated, compact, and unobstructed.
- An exact triangle $L_1 \to L \to L_2 \to L_1[1]$ is called destabilizing if
$$\hat{\theta}_1 = \arg \int_{L_1} \Omega > \hat{\theta}_2 = \arg \int_{L_2} \Omega.$$
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Question

Assume a $D^b Fuk(X)$ class has some quantitatively almost calibrated representative L_0, and assume Thomas-Yau semistability, can we produce a special Lagrangian?

Caveats:

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- The derived Fukaya category needs to include immersed Lagrangians, but we do not know how much more we need to include also.
- The special Lagrangian may have worse singularities than can be treated by current Floer theory. It would belong to some geometric measure theoretic closure, and we are hesitant whether it can be regarded as a representative of the same $D^b Fuk$ class.

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Remark

Remember: the Solomon functional depends not just on the Lagrangian, but also on the potential f_L. In the sketch proof below, we will modify the potential to make the Solomon functional go to $-\infty$ in the unstable case.

- All Lagrangians are assumed to be quantitatively almost calibrated.
- Suppose we have an exact triangle $L_1 \to L \to L_2$, then $L_1 \cup L_2$ equipped with some bounding cochain, is an immersed Lagrangian brane in the same $D^b Fuk$ class as L.

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- Subtle point: the bounding cochain element in $CF^1(L_2, L_1)$ is subject to the positive Novikov exponent requirement.
- Fix the underlying L_1, L_2. We can fix the potential on L_1, and add arbitrarily large **positive** constant the potential on L_2. (Adding large negative constant would violate Novikov positivity).

- The effect of adding constant to the potential of L_2 is to change the Solomon functional by

$$c\mathrm{Im}(e^{-i\hat{\theta}}\int_{L_2}\Omega), c \gg 1.$$

- In the destabilising case,

$$\hat{\theta}_2 = \arg\int_{L_2}\Omega < \hat{\theta} = \arg\int_{L}\Omega,$$

the Solomon functional becomes very **negative**.

- Conclusion: Thomas-Yau semistability is (essentially) necessary for the Solomon functional to be bounded from below.

Main themes of today:

- Thomas-Yau semistability is also **sufficient** for the Solomon functional to be **bounded below** (assuming the Lagrangians are sufficiently smooth).
- There is a flat norm topology of Lagrangian integral currents, so that we can prove **precompactness** in the variational program, to produce a minimizer of the Solomon functional in the geometric measure theoretic closure.
- Not known: whether the minimizer is actually special Lagrangian. (Problem: we lack techniques to produce Lagrangian competitors.)

Intuition about Lagrangian potential:

- The immersed Lagrangian may have several connected components.
- On each component the potential f_L has a priori bounded oscillation.
- Lagrangian potentials can be far separated between different components.

A priori estimates

Crucial assumption: quantitative almost calibratedness. Consider L in the same homology class as L_0.

- **Total volume bound**:

$$Vol(L) \leq \frac{1}{\sin \epsilon} \int_L \mathrm{Re}\Omega \leq C.$$

- **Volume lower bound** for both extrinsic and intrinsic balls: if $P \in L$, then for r up to the regularity scale of the ambient manifold,

$$Vol(B(r) \cap L) \geq C^{-1} r^n.$$

  (main ingredient: Neves's application of isoperimetric theorem+ quantitative almost calibratedness+coarea formula).

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- Any connected component has a lower bound on its volume. Consequently, the **number of connected components** is bounded.
- The total volume bound shows only finitely many possibilities can appear for the **homology classes of connected components** of L.

A priori estimates

- The **intrinsic distance diameter** of any connected component is bounded. (Otherwise you can inscribe many disjoint intrinsic balls, which takes up too much volume).
- The **potential f_L oscillates by a bounded amount** on each connected component. (Because $df_L = \lambda|_L$ is bounded.)

- Now decompose L according to the range of the potential f_L. We have $L = L_1 \cup L_2 \cup \dots L_N$, such that the range of f_L is connected on each L_i, and

$$\sup_{L_i} f_{L_i} < \inf_{L_{i+1}} f_{L_{i+1}}.$$

- **Novikov positivity** requirement implies L is of twisted complex form. In other words, as a Lagrangian brane L fits into

$$0 = \mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \dots \mathcal{E}_N = L,$$

with exact triangles

$$\mathcal{E}_{i-1} \to \mathcal{E}_i \to L_i \to \mathcal{E}_{i-1}[1].$$

Solomon functional bounded below

An **elementary functional** built from period integrals:

$$\bar{\mathcal{S}} = \sum_{1}^{N-1} (\sup_{L_i} f_L - \sup_{L_{i+1}} f_L) \mathrm{Im}(e^{-i\hat{\theta}} \int_{\mathcal{E}_i} \Omega).$$

- Can show an a priori bound $|\mathcal{S} - \bar{\mathcal{S}}| \leq C$. (**The difference between Solomon functional and the elementary functional is bounded**).
- Assuming **Thomas-Yau semistability**, then

$$\mathrm{Im}(e^{-i\hat{\theta}} \int_{\mathcal{E}_i} \Omega) \leq 0.$$

  Thus $\bar{S} \geq 0$, so **the Solomon functional is bounded from below**.

Geometric measure theory

Main tools in geometric measure theory:

- Federer-Fleming compactness under mass bound.
- Allard compactness under mass bound+ total variation bound.
- Almgren regularity.

Geometric measure theory

- We can make weak sense of the Lagrangian potential $f_L \in L^\infty$: for any test $(n-1)$-form χ:

$$\int_L \lambda \wedge \chi = -\int_L f_L d\chi.$$

- Assuming quantitative almost calibratedness, we have uniform mass bound, so the underlying Lagrangian integral currents converge up to subsequence in the flat norm.
- Can show: if in a sequence f_L is **uniformly bounded in L^∞**, then the exact Lagrangian condition passes to the limit, and **the Solomon functional is continuous** under convergence.

Minimizer

- Assuming Thomas-Yau semistability, then we can first modify the Lagrangian potential in the minimizing sequence, subject to the Novikov positivity requirement, so that f_L becomes uniformly bounded.
- Consequence: **the Solomon functional has a minimizer** in the flat norm closure of the quantitatively almost calibrated Lagrangian branes in the fixed D^bFuk class.

Question

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- Answer: If you know any small Hamiltonian deformations of L remains within the class of quantitatively almost calibrated Lagrangians ('enough competitors'), then from the first variation formula
$$\int_L H \mathrm{Im} e^{-i\hat{\theta}}\Omega = 0,$$
you can conclude it is special Lagrangian.
- Problem: small perturbations may no longer satisfy $|\theta| \leq \frac{\pi}{2} - \epsilon$. Do you have any idea?

Alternative heuristic justification:

- Observe the Solomon functional is monotone decreasing in time for almost calibrated Lagrangians. If it is constant, then the Lagrangian should be special.
- Suppose the LMCF can be extended to suitable Lagrangian integral currents (or by using some approximation arguments), then one expects the minimizer of the Solomon functional to be special Lagrangian.

Question

Can we assign any reasonable brane structure on the (likely singular) special Lagrangian current?

More precisely,

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Is there a suitable notion of distance on the brane structures, such that every sequence of suitably smooth Lagrangians converging to the special Lagrangian current, has a Cauchy subsequence?

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Do bordism currents have good convergence behaviour under the weak convergence of Lagrangians?