Thomas-Yau conjecture Symplectic aspects

Yang Li

MIT

March 25, 2022

Question

What are the Floer theoretic obstructions to special Lagrangians?
(*i.e.* when can we rule out the existence of special Lagrangians in certain derived Fukaya category classes?)

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(time permitting) What is the functional governing the existence of special Lagrangians?

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Main message: you should look at $(n-1)$-dimensional moduli spaces of holomorphic curves, and the current swept out by the $(n+1)$-dimensional family.

Setting

- We work in the exact setting. The ambient manifold is a **Stein** complex manifold $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, with a nowhere vanishing holomorphic volume form Ω.
- The Lagrangians are **exact, compact, and almost calibrated**.

Setting

- We work in the exact setting. The ambient manifold is a **Stein** complex manifold $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, with a nowhere vanishing holomorphic volume form Ω.
- The Lagrangians are **exact, compact, and almost calibrated**.
- Recall exactness means

$$d\lambda = \omega, \quad df_L = \lambda|_L.$$

Caveat: we do not require f_L to take the same value at self intersections of immersed Lagrangians. There can be teardrop curves. Almost calibrated means

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Symplectic background: immersed Floer theory

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- For the Thomas-Yau conjecture, the embedded Lagrangians are not enough, eg. because the Lagrangian mean curvature flow can turn embedded Lagrangians into immersed Lagrangians.
- Floer cohomology of immersed Lagrangian is defined by Joyce-Akaho (*cf.* also Woodward-Palmer, ...).
- food for thought: Probably one needs to introduce more singular Lagrangians.

immersed Floer theory

- Typical immersed Lagrangian: union of several Lagrangians (with transverse intersections).
- The Floer degrees at intersection points: (different from Seidel convention!)

$$\mu_{L,L'}(p) = \frac{1}{\pi}\left(\sum_{1}^{n} \phi_i + \theta_L(p) - \theta_{L'}(p)\right),$$

where

$$T_pL = \mathbb{R}^n \subset \mathbb{C}^n, \quad T_pL' = (e^{i\phi_1}, \dots e^{i\phi_n})\mathbb{R}^n.$$

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- However, Lagrangian branes are boundary conditions of holomorphic curves, not just geometric Lagrangians. We need to remember some data at self intersection points, which encode how holo curves cross from one branch to another, and how these are weighted.

immersed Floer theory

- The self intersection data at degree one self intersection points are known as '**bounding cochains**'. To make Floer cohomology well defined, they need to satisfy the Mauer-Cartan equation (cancellation of obstructions)

$$m_0 + m_1(b) + m_2(b, b) + \ldots = 0 \in CF^2_{self-intersection}(L, L).$$

  Then m_1^b squares to zero, where

$$m_1^b(x) = \sum m(b, \ldots b, x, b, \ldots).$$

- Each self intersection has a Novikov exponent $f_{L_+}(p) - f_{L_-}(p)$. Bounding cochains involve only self intersections with **non-negative Novikov exponents**. This is needed in Floer theory to bound the energy of holomorphic discs.

Immersed Floer theory

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Immersed Floer theory

- Advantage of immersed Floer theory: no need to add twisted complexes in the construction of the Fukaya category, but simply take the union of the Lagrangians with some bounding cochain data.
- For instance, exact triangles $L_1 \to L \to L_2$ can be recast as saying L is isomorphic to an object supported on $L \cup L'$.
- Technical caveat: The $D^b Fuk(X)$ I am using does not add in idempotents. All branes are geometric, but we do not know the idempotent closedness (If Joyce's picture is right then it is, but we don't need it).

Symplectic backgrounds: open-closed map

A basic ingredient for the Thomas-Yau conjecture is that the central charge function

$$Z(L) = \int_L \Omega$$

needs to be well defined on the derived Fukaya category class. In fact there is a well defined map from the Grothendieck group of $D^b Fuk(X)$ to the middle homology:

$$L \mapsto [L] \in H_n(X).$$

This is known to experts as a special case of the **open-closed map**. In particular, isomorphism in $D^b Fuk(X)$ implies being homologous, and exact triangle $L_1 \to L \to L_2$ implies $[L] = [L_1] + [L_2]$.

Open-closed map

Question

Given L, L' isomorphic in $D^b Fuk$, why are they homologous?

- Oversimplified answer: take the generators $\alpha \in HF^0(L, L')$, and $\beta \in HF^0(L, L')$, whose compositions are the identities. The **moduli space** of (perturbed) holomorphic curves between intersections contributing to α, β are $(n-1)$-dimensional, so the **universal family** of these curves gives rise to an $(n+1)$-dimensional integration current $\mathcal{C}$. Its boundary is $L - L'$.

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- More accurately, one needs to take into account the bounding cochain data, and the difference between cohomological units and geometric units.

Two assumptions

- **Automatic transversality assumption**: all the holomorphic curves (no perturbation!) involved in the construction of the 'bordism current' $\mathcal{C}$ are smooth points of the moduli space.

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- **Automatic transversality assumption**: all the holomorphic curves (no perturbation!) involved in the construction of the 'bordism current' $\mathcal{C}$ are smooth points of the moduli space.
- Generally speaking, there are many $(n-1)$-dim moduli spaces corresponding to the many Lagrangian intersection points in the HF^0 generators α, β. The moduli spaces come with orientations, and upon the evaluation of $\partial\Sigma \to L \cup L'$, we can compare this orientation with the orientation of L and L'.
- **Positivity condition**: all holomorphic curves contribute to $\partial\mathcal{C} = L - L'$ with the same orientation sign.

Morse theory analogy

In **Morse theory**, the fundamental class of a compact oriented manifold L can be viewed as follows:

- The generators of the zeroth and the n-th Morse cohomology are given by the sum of local maxima/minima.
- The fundamental cycle $[L] \in H_n(L)$ is the integration current swept out by the union of the $(n-1)$-dim moduli space of gradient flowlines between local maxima and local minima.
- Notice at each generic point on L, there is only one gradient flowline passing through. We *do not have cancellation* of $\pm$ oriented flowline contributions!

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Question

Is there a general criterion for the positivity condition in the Floer theory setting, eg. assuming almost calibrated Lagrangians etc?

A priori expectations for obstructions

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- The obstruction involves an **integration over a moduli space** of worldsheet instantons (ie holo curves), and the sign comes from the pointwise positivity of the integrand on the moduli space.
- The inputs from Floer theory are **exact triangles** in the derived Fukaya category.
- The role of holomorphic volume forms enters via **cohomological integrals**.

Looking for obstructions

Concretely: given an **exact triangle** $L_1 \to L \to L_2$ of exact, almost calibrated, compact, unobstructed Lagrangians. **Destabilizing condition**:

$$\hat{\theta}_1 = \arg \int_{L_1} \Omega > \hat{\theta}_2 = \arg \int_{L_2} \Omega.$$

Question

Does the existence of a destabilizing exact triangle rule out the possibility of L being a speical Lagrangian?

An unsatisfactory answer: if we know L_1, L_2 are represented by special Lagrangians of phase $\hat{\theta}_1, \hat{\theta}_2$, then

$$\sup_L \theta \geq \hat{\theta}_1, \quad \inf_L \theta \leq \hat{\theta}_2.$$

- Reason: if $\sup_L \theta < \hat{\theta}_1$, the formula for the Floer degrees of Lagrangian intersection points implies $CF^0(L_1, L) = 0$. Thus the holomorphic curves contributing to the bordism current $\mathcal{C}$ with $\partial\mathcal{C} = L - L_1 - L_2$ cannot pass from L_1 to L. Any curve passing through L_1 is stuck on L_1, which is impossible for almost calibrated Lagrangians due to the absence of $CF^{-1}(L_1, L_1)$ intersection points.

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- This answer is unsatisfactory because when we are looking for special Lagrangians, we are not supposed to assume the existence of any special Lagrangians.
- If one believes in Joyce's picture of Bridgeland stability, then one can consider the Harder-Narasimhan filtration of L_1, L_2, and a version of the above arguments suggests the existence of the destabilizing exact triangle indeed rules out L being special Lagrangian.

Looking for obstructions

Question

Can destabilizing exact triangles obstruct the existence of special Lagrangians without Lagrangian angle assumptions on L_1, L_2 beyond being almost calibrated?

- Answer: Yes, if we assume the **automatic transversality and the positivity condition** on the bordism current $\mathcal{C}$ between L and $L_1 + L_2$.
- Technique: **integration over moduli space**.

Theorem

Assume automatic transversality+ positivity condition+ destabilizing exact triangle. Then

$$\sup_L \theta \geq \hat{\theta}_1 > \hat{\theta}_2 \geq \inf_L \theta.$$

In particular L cannot be a special Lagrangian.

Moduli integral technique

- Strategy: express the period integrals $\int_{L_1} \Omega, \int_{L_2} \Omega$ in terms of integrals over the $n-1$ dim moduli spaces of holomorphic curves.
- Try to derive integral inequalities based on some pointwise inequality on the moduli space.

Recall some basic **deformation theory of holomorphic discs** $\Sigma \to X$ with boundary on Lagrangians (and corners at Lagrangian intersection points/bounding cochain elements):

- First order deformation vector fields are solutions to the extended linearized Cauchy-Riemann equation.
- Let $v_1, \ldots v_{n-1}$ be first order deformations of holomorphic curves. Then v_i define holomorphic vector fields in the normal bundle of the image of Σ.
- The $(1,0)$-form on Σ defined by $\Omega(\cdot, v_1, \ldots v_{n-1})$ is therefore holomorphic. It must be the differential of a **holomorphic function** F on the domain Σ.

- In clockwise order (in my conventions), the Lagrangian boundary of Σ encounters L, L_2, L_1. (More generally, there is a possibility to skip L_1 or L_2.)
- The function F is a holomorphic map from the disc Σ to $\mathbb{C}$. Along $\partial\Sigma$, the **incline angle** of dF is equal to the **Lagrangian angle** of the Lagrangian boundary condition.
- Feature of **almost calibrated Lagrangians+ positivity condition**: clockwise along $\partial\Sigma$, the function ReF *increases* on the L boundary portion, and *decreases* on the $L' = L_1 \cup L_2$ boundary portion.

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- Consequence by elementary complex analysis: the image $F(\Sigma) \subset \mathbb{C}$ lies **above** its $L_1 \cup L_2$ portion, and **below** the L portion.

Remark

(Partial justification for automatic transversality assumption) In fact, by some index computation, one can show that

- Either F is constant on Σ,
- Or $\Sigma \to X$ is a smooth point of the moduli space, and morever dF has no zero inside the interior or the boundary of Σ and only vanishes to minimal order at the corner.

- Since the current $\partial\mathcal{C}$ sweeps out almost every point on $L - L_1 - L_2$ precisely once in the sense of counting, the period integral $\int_{L_i} \Omega$ can be expressed as an integral over the $n-1$ dim moduli space. Notice F is proportional to $v_1 \wedge \dots v_{n-1}$, which means its proper interpretation is a family of complex valued volume forms on the moduli space.
- For each corner of Σ mapping to the $CF^1(L_2, L_1)$ point, we can find some point on the L portion of $\partial\Sigma$, with the same value of ReF, and **bigger** value of ImF.
- When this fact is **integrated over the moduli space**, it says that there is a subset A of L, with

$$\text{Re} \int_A \Omega = \text{Re} \int_{L_1} \Omega > 0, \quad \text{Im} \int_A \Omega \geq \text{Im} \int_{L_1} \Omega.$$

This implies $\sup_L \theta > \hat{\theta}_1 = \arg \int_{L_1} \Omega$.

Solomon functional

Jake Solomon introduced a functional among a fixed Hamiltonian isotopy class of Lagrangians, with the property that its first variation for the Hamiltonian deformation H is

$$\delta\mathcal{S} = \int_L H \mathrm{Im}(e^{-i\hat{\theta}}\Omega),$$

where $\hat{\theta} = \arg \int_L \Omega$.

Question

Can we make sense of this functional for Lagrangians inside a fixed derived category class?

Solomon functional

Answer 1: suppose L is isomorphic to L_0 in $D^b Fuk$, so in particular homologous. We take $\mathcal{C}$ so that $\partial\mathcal{C} = L - L_0$. Recall L is an exact Lagrangian with potential f_L.

$$\mathcal{S}(L) = \int_L f_L \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{L_0} f_{L_0} \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{\mathcal{C}} \lambda \wedge \mathrm{Im}(e^{-i\hat{\theta}}\Omega).$$

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Remark

Changing $\mathcal{C}$ by any exact integration current does not affect the functional.

Remark

This formula is more useful for the variational method, and is the starting point of the more geometric measure theoretic aspects.

Solomon functional

Answer 2 (equivalent answer, under the automatic transversality assumption) The Solomon functional can be expressed as an integral over the $n-1$ dim moduli spaces $\mathcal{M}$ of holomorphic discs

$$\mathcal{S}(L) = \int_{\mathcal{M}} \mathcal{I},$$

$$\mathcal{I} = \operatorname{Im} \int_{\Sigma} e^{-i\hat{\theta}} F\omega + \operatorname{Im} \sum_{corners} e^{-i\hat{\theta}} Ff|_{-}^{+},$$

where $f|_{-}^{+}$ signifies the jump in the Lagrangian potentials at the corner, and F is the holomorphic function on Σ constructed from

$$dF = \Omega(\cdot, v_1, \ldots, v_{n-1}).$$

Solomon functional

Consequence of moduli space integral formula for the Solomon functional: if L_0 is a special Lagrangian, and L is almost calibrated, and assuming automatic transversality+positivity condition on the bordism current, then

$$\mathcal{S}(L) \geq \mathcal{S}(L_0).$$

- Reason: $\mathrm{Im}(e^{-i\hat{\theta}}F)$ is zero on the L_0 boundary portion of Σ, and non-negative on Σ. Morever, the bounding cochain elements on L satisfy the Novikov exponent positivity $f|_{-}^{+} \geq 0$.
- Moral: special Lagrangians should be absolute minimizers of the Solomon functional.