Thomas-Yau conjecture backgrounds

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Special Lagrangian

- Let (X, ω, Ω) be a Kähler manifold with a nowhere vanishing holomorphic volume form. An n-dimensional submanifold (or some weaker notion, eg. integral current) is called **special Lagrangian**, if

$$\omega|_L = 0, \quad \mathrm{Im}(e^{-i\hat{\theta}}\Omega)|_L = 0.$$

Volume minimizer

- We assume the metric is Calabi-Yau. Then L is a minimal submanifold.
- In fact, any submanifold in the same homology class satisfies

$$\int_L Re(e^{-i\hat{\theta}}\Omega) \leq \int_L dvol = \mathrm{Vol}(L),$$

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  saturated precisely by special Lagrangians.
- Thus if a special Lagrangian exists then it is an **absolute volume minimizer**.

Almost calibrated Lagrangians

- Recall the **Lagrangian angle** is defined by

$$\Omega|_L = e^{i\theta} dvol_L.$$

  Here $\theta : L \to S^1$ is assumed to lift to $\mathbb{R}$ (**graded Lagrangians**).
- Special Lagrangians have constant phase angle $\theta = \hat{\theta}$.
- **Almost calibrated** means the Lagrangian angle is inside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus L is automatically graded.
- **Quantitative almost calibrated** means $\theta \in (-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon)$. It implies an apriori volume bound

$$\mathrm{Vol}(L) \leq \frac{1}{\sin \epsilon} \int_L \mathrm{Re}\Omega.$$

- However, the volume minimizer within a given homology class needs not be a special Lagrangian (Schoen, Wolfson). *'Direct minimization of volume is not good enough.'*
- The known construction techniques: high symmetry, gluing style constructions, integrable system (reduce to ODE or Riemann surface), Cartan-Kähler theory.
- Existence question is a major open problem in general.

What is Thomas-Yau conjecture?

- Thomas-Yau principle: **'The existence and uniqueness of unobstructed special Lagrangian branes should be governed by a stability condition on the (derived) Fukaya category.'**
- Thomas-Yau's main motivations: mirror analogy with stable vector bundles.
- Their main evidence: uniqueness theorem (further developed by Joyce-Imagi-Santos, Imagi, Abouzaid-Imagi).

Potential significance of the Thomas-Yau philosophy:

- Produce special Lagrangians.
- Mirror symmetry beyond homological mirror symmetry.
- (Far beyond the current technology) special Lagrangian enumerative invariants?

Caveats:

- The notion of stability is meant to be tentative in Thomas-Yau's proposal.
- The mirror version of stability is not really meant to be μ-stability for Hermitian-Yang-Mills connections. A slightly better mirror candidate is deformed Hermitian-Yang-Mills, though I expect it is also only approximate.

Joyce's update

The most significant progress since Thomas-Yau was the update by Dominic Joyce.

- Joyce says there should be a **Bridgeland stability condition** on the derived Fukaya category, such that the semistable objects of given phase $\hat{\theta}$ are represented by Lagrangian branes with arbitrarily small phase oscillation $|\theta - \hat{\theta}| \ll 1$. (Morally, represented by special Lagrangian branes, although these may be too singular).

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- Joyce says the way to construct this stability condition is to run **Lagrangian mean curvature flow** with surgery, and take the infinite time limit.
- Joyce says the role of unobstructed brane structure and the Fukaya category machinery is to rule out the worst singularities in the flow.

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- The subcategory generated by almost calibrated Lagrangians is supposedly the heart of a bounded t-structure, and in particular is an abelian category, and generates the entire $D^b Fuk$.
- Joyce hopes the Lagrangian mean curvature flow only encounters finitely many surgeries.

Question: Can we formulate the Thomas-Yau conjecture in a version circumventing these strong predictions?

Thomas-Yau conjecture

My attempted interpretation of Thomas-Yau:

- All Lagrangian branes involved are almost calibrated and unobstructed by assumption. They can be immersed (or perhaps more singular).
- We say L is **Thomas-Yau semistable** if for any exact triangle of almost calibrated branes

$$L_1 \to L \to L_2 \to L_1[1],$$

  we have the phase angle inequality

$$\hat{\theta}_1 = \int_{L_1} \Omega \leq \hat{\theta}_2 = \int_{L_2} \Omega.$$

Thomas-Yau conjecture

Thomas-Yau conjecture: consider the quantitatively almost calibrated Lagrangians inside a given $D^bFuk(X)$ class, which is nonempty by assumption. There is a special Lagrangian inside the geometric measure theoretic closure, if and only if the $D^bFuk(X)$ class is Thomas-Yau semistable.