Metric SYZ conjecture

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- SYZ is physically motivated, and admits many interpretations. The strong version asserts that the SLag fibration exists globally; this would be much harder or perhaps false (*cf.* Joyce). Some people adopt much softer viewpoints (algebraic, symplectic, topological, mirror symmetry).

Weak SYZ conjecture

- 'Large complex structure limit' means a polarized algebraic family of n-dim CY manifolds $X \to S \setminus \{0\}$ over a punctured algebraic curve, whose 'essential skeleton' has dimension n. There are small variations on the definitions. For instance, it is common to require the family to have a semistable snc model $\mathcal{X} \to S$, as we will do.

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- The essential skeleton can be viewed as a simplicial subset of the dual complex of any snc model of the degenerating family. It is an important birational invariant independent of the snc models, and you can discover it yourself if you try to compute CY volume integrals. (More on this later).

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- The essential skeleton can be viewed as a simplicial subset of the dual complex of any snc model of the degenerating family. It is an important birational invariant independent of the snc models, and you can discover it yourself if you try to compute CY volume integrals. (More on this later).
- 'Generic' should at least mean a subset of large percentage of the measure. (Notice on a CY manifold there is a canonical measure up to scale). This talk is orthogonal to some of my previous talks about the metric model for the nongeneric regions in the 3-fold case.

Weak SYZ conjecture: precursors

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- HyperKähler case and Abelian variety fibrations (*cf.* Tosatti et al., Gross-Wilson). These are closely related to SYZ but not polarized.
- Nonarchimedean pluripotential theory developed by Boucksom et al. This has strong analogy with Kähler geometry, eg. there is a notion for psh functions, MA measure, and an analogue of the CY theorem. The technical foundations are however very different: this NA story is built on birational geometry, intersection theory, vanishing theorems etc, rather than analysis of differential operators. For instance, the NA MA equation is a priori not even a PDE, although conjecturally it should be equivalent to a real MA equation.

Weak SYZ conjecture

My previous work on the Fermat family:

$$X_s = \{Z_0 Z_1 \dots Z_{n+1} + e^{-s} \sum_0^{n+1} Z_i^{n+2} = 0\}, s \gg 1.$$

The result was that weak SYZ conjecture holds in this special case, for some subsequence of large enough s. The essential tool is complex pluripotential theory. No NA geometry was used in that work, and instead there was some combinatorial argument exploiting the discrete symmetry.

Weak SYZ conjecture: main theorem

More recently I showed

Theorem

Given a polarised algebraic degeneration of Calabi-Yau manifolds, near the large complex structure limit, then assuming some conjecture in NA geometry, the weak SYZ conjecture will follow.

Good features:

- General
- No need for subsequence
- Uniqueness of metric limit in some sense
- Clean approach, no messy combinatorics
- The NA conjecture is purely algebraic

Discussions

Unsatisfactory feature:

- No known way to verify the NA conjecture, even for K3 surfaces or the Fermat family (hopefully will change soon!)
- The region with SYZ fibration in my construction is not connected (roughly because I did not assume anything about codimension 1 faces of the essential skeleton). (In contrast, in my previous work on the Fermat case, we do know this connectedness).

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The plan for today is to first push as far as possible without really talking about NA geometry, so that you appreciate what we need for the purpose of SYZ, and then mention some basic features of NA so that it feels less arcane.

Weak SYZ conjecture: NA assumption

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Consider the Boucksom-Favre-Jonsson solution to the NA Calabi conjecture, represented by a potential function ϕ on the Berkovich space X^{an}. There exists a large enough semistable snc model $\mathcal{X}$, (whose dual complex $\Delta_{\mathcal{X}}$ necessarily contains the essential skeleton $Sk(X)$), such that over the n-dimensional open faces of $Sk(X)$, the solution ϕ *factorizes through the retraction map* $X^{an} \to \Delta_{\mathcal{X}}$.

Weak SYZ conjecture: NA assumption

- Morally, this is saying the information of the NA CY solution is no more than the value on the essential skeleton. (The assumption above is technically weaker).
- Christian Vilsmeier has a recent result which says the above assumption implies a real MA equation on ϕ on the open n-dimensional faces of $Sk(X)$.

Easy complex geometry

Basic feature of the complex geometry: near the large complex structure limit, the generic region (in the measure theoretic sense) is locally modelled on $(\mathbb{C}^*)^n$.

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Let's compute the CY volume form. Given a semistable snc model $\mathcal{X} \to S$, with a holomorphic volume form Ω, which can only vanish somewhere on the central fibre. The holomorphic volume form on the CY manifolds X_t is

$$\Omega_t = \Omega/dt.$$

The semistable snc condition means $\mathcal{X} \to S$ locally looks like

$$t = z_0 \dots z_k, \quad k \leq n.$$

The z_i are the local defining functions of the components of the central fibre, which are divisors on the total space $\mathcal{X}$.

Easy complex geometry

Now using

$$\Omega \sim z_0^{a_0} \dots z_k^{a_k} dz_0 \wedge dz_1 \wedge \dots dz_n, \quad a_i \geq 0.$$

we see

$$\Omega_t \sim z_0^{a_0} \dots z_k^{a_k} d \log z_1 \wedge \dots d \log z_k \wedge dz_{k+1} \dots dz_n.$$

You can compute the volume integral $\int_{X_t} \Omega_t \wedge \overline{\Omega}_t$ in polar coordinates in these charts. You easily see that the dominant contribution comes from the case $a_0 = \dots = 0$, in which case the local volume is $O(|\log|t||^k)$. The large complex structure limit says $\max k = n$. This means the generic region is modelled on

$$t = z_0 \dots z_n,$$

with holomorphic volume form $\sim \prod_1^n d \log z_i$.

Easy complex geometry

- Formally, the dual complex $\Delta_{\mathcal{X}}$ associated to a snc model $\mathcal{X}$ is the simplicial complex whose vertices v_i correspond to the components E_i of the central fibre, and we attach a simplex with vertices v_i for $i \in J$ if $\cap_{i \in J} E_i \neq \emptyset$.

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- The condition $a_0 \ldots = 0 = \min a_i$ precisely singles out the *essential skeleton* from the dual complex of the snc model. The advantage is that the snc model is highly nonunique (you can always blow up further), but $Sk(X)$ is a well defined birational invariant. This is not surprising because the CY volume measure does not care about the birational transform of the central fibre.

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- The condition $a_0 \ldots = 0 = \min a_i$ precisely singles out the *essential skeleton* from the dual complex of the snc model. The advantage is that the snc model is highly nonunique (you can always blow up further), but $Sk(X)$ is a well defined birational invariant. This is not surprising because the CY volume measure does not care about the birational transform of the central fibre.
- There is a 'hybrid topology' in which you can remove the central fibre $\mathcal{X}_0$ from $\mathcal{X}$ and replace it with $\Delta_{\mathcal{X}}$. Intuitively, the measure theoretic limit of the family is highly non-algebraic. (If you follow this idea further, you will discover NA geometry...)

Semiflat metric and SLag

- CY metrics have an important dimensional reduction. Take a function ϕ on (a torus invariant subset of) $(\mathbb{C}^*)^n$, so $\phi = u \circ \text{Log}$. Then ϕ is psh iff u is convex, and ϕ satisfies complex MA iff u satisfies real MA.

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  Metrics from this dim reduction are called semiflat, because the restriction to torus fibres are flat. The metrics on fibres can vary.

Semiflat metric and SLag

Really we want to prove that on local charts in the generic region the metric is C^∞ approximately

$$\omega_t \approx \frac{\sqrt{-1}}{|\log|t||^2} \sum \frac{\partial^2 u}{\partial x_i \partial x_j} d\log z^i \wedge d\overline{\log} z^j.$$

Here $x_i = \frac{\log|z_i|}{\log|t|}$. This scaling convention of the Kähler class is compatible with a finite diameter Gromov Hausdorff limit as $t \to 0$.

The SLag fibration in such regions is more or less for free; it is a small deformation of the log map. The construction uses no more than McLean deformation theory; then you check the independence of the chart.

Reduction to potential estimate

- We have argued that SLag fibration is a consequence of C^∞ metric asymptote. Now we claim it is enough to prove a C^0-estimate on the potential in the generic region.
- This means we want the local potential ϕ_t to satisfy

$$|\phi_t - u| \to 0, \quad t \to 0,$$

  where u solves the real MA equation $\det(D^2 u) = \text{const}$.
- Solutions of real MA have automatically very good regularity, so u is C^∞ if we discard some subset with Hausdorff $(n-1)$-measure zero.

Reduction to potential estimate

This reduction uses a highly nontrivial result of Savin:
Savin's small perturbation theorem roughly says that for a large class of fully nonlinear 2nd order elliptic equation, including complex MA, if u is a smooth solution in B_2, and v is another (viscosity) solution with $\|u - v\|_{C^0} \ll 1$, then v has C^∞ bounds and $u - v$ is C^∞-small.

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We apply Savin to the local universal cover of the annuli in $(\mathbb{C}^*)^n$.
The effect is that the high regularity of u is transferred to the CY local potential ϕ_t for small t.

Potential estimate: two strategies (To use NA, or not to use NA, that is the question...)

The strategies of my two papers bifurcate on the potential estimate.

- Either you can produce u by arguing a priori that ϕ_t converge to a subsequential limit, and argue that the limit satisfies real MA;
- Or you can produce the limit u from some other methods (eg. NA geometry), and argue try to compare ϕ_t with u.

Pluripotential theory toolbox

How do you estimate potentials?

- Given a psh function, you can use mean value theorem to achieve upper bounds.
- The core of Kolodziej's method is to achieve lower bounds on Kähler potentials, given some integral bound on the volume density. Technically, the main thing you need is a Skoda type inequality. This holds even in highly degenerate settings, and I verified a uniform version for arbitrary polarized algebraic degenerations of Calabi-Yau manifolds in an auxiliary paper. Unlike the upper bound, this lower bound requires a global argument.

Pluripotential theory toolbox

How do you compare potentials? On a fixed Kähler manifolds, given two potentials ϕ, ψ with density bounds, with suitable normalisation,

- If you know $\phi - \psi$ is small in L^1, you can conclude the smallness in L^∞.
- If you know the two volume densities are close in L^1, you can also conclude $\phi - \psi$ is small in L^∞.

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There are uniform versions of these in degenerate settings.
I emphasize that you need the two potentials to be globally defined, even though you just want to compare them in the generic region. Global positivity is hard to achieve!

Strategy I: a priori limit

In my strategy I, no NA geometry is needed, and I produce u a priori as follows:

- On $(\mathbb{C}^*)^n$, if you have a psh function, you can average in the T^n direction to produce a convex function on $\mathbb{R}^n$. A priori the deviation between the two functions is small in a very strong integral sense. Intuitively, 'near the large complex structure limit any Kähler potential looks almost convex'.
- Patch together the convex functions into a global Kähler potential. This step is nontrivial and is the place to really use the Fermat hypersurfaces via tropical combinatorics. Global positivity is the key difficulty here.

Strategy I: a priori limit

- Then one appeals to pluripotential theory to argue the local CY potentials are C^0-close to convex functions.
- Use uniform Lipschitz bounds on convex functions to extract subsequential limit u.
- Argue u satisfies real MA.

Strategy II: limit via NA geometry

Regardless of where u comes from, we want the following features:

- The function u should solve the real MA equation.
- There should be a global Kähler potential ψ on X_t for $|t| \ll 1$, which is C^0-close to u on the generic region of X_t. I emphasize that 'preserving positivity' is the key difficulty.
- Morever, we want the volume density of ψ to be close to being CY in some L^1-sense. This property is not independent, and in some sense can be arranged by a further regularisation of solutions to the first two requirements.

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If we have these three properties the C^0-convergence of local CY potentials to u can be deduced by the pluripotential theory machinery.

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The difficulty here is that the manifolds are highly degenerate, and technically I need to adapt Kolodziej's argument to a version where the two potentials being compared appear asymmetrically (to address the difficulty that ψ does not have good density bound in the nongeneric region).

Why NA? And what is it?

- The main unsatisfactory feature of snc models is that they are highly nonunique, because you can keep blowing up to get higher and higher models. NA geometry from one perspective means to consider all snc models of a given degeneration family all at once. More precisely, if one model dominates another $\mathcal{X}' \to \mathcal{X}$, then there is a natural retraction map $\Delta_{\mathcal{X}'} \to \Delta_{\mathcal{X}}$, and the Berkovich space X^{an} is the inverse limit of these dual complexes.
- There are two comparison maps between X^{an} and $\Delta_{\mathcal{X}}$: an embedding map $\Delta_{\mathcal{X}} \to X^{an}$ and a retraction map $X^{an} \to \Delta_{\mathcal{X}}$. One thinks of dual complexes as finite approximations of X^{an}.

Why NA? And what is it?

- There is a GAGA principle which means the concept of line bundles and sections on X^{an} are equivalent to the usual algebro-geometric notions for X base changed to the formal disc.
- There is an analogue for metrics on line bundles, and a subtle analogue for psh functions/semipositive metrics. There are several equivalent definitions; the most intuitive one is to say that continuous semipositive metrics are precisely those that arise as C^0-limits of (the NA analogue of) Fubini-Study metrics.
- There is a hybrid topology which allows one to say X_t converge to X^{an} as $t \to 0$.
- For the purpose of SYZ it is important that semipositive metrics on X^{an} can be grafted to X_t up to C^0-small error, and *preserving positivity*. This uses Fubini-Study approximation.

Why NA? And what is it?

- There is a theory of NA MA measures. The standard definition is a little counterintuitive at first sight, as it is based on intersection theory, not differential operators. However, its formal properties are very similar to the usual complex MA operator.
- If the potential factors through the retraction map to the dual complex of some semistable snc model, then it is known that the NA MA measure agrees with the real MA measure. This result has a localized version.
- Boucksom-Favre-Jonsson proved the NA version of the Calabi conjecture.

Why NA?

- To conclude, if we assume the B-F-J solution factors through some retraction map at least over the n-dim open faces of $Sk(X)$, then we would get a solution of real MA equation, and crucially this can also be grafted to X_t up to C^0-error while *preserving positivity*.

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- To conclude, if we assume the B-F-J solution factors through some retraction map at least over the n-dim open faces of $Sk(X)$, then we would get a solution of real MA equation, and crucially this can also be grafted to X_t up to C^0-error while *preserving positivity*.
- These properties are sufficient to imply the weak SYZ conjecture.

For the future

- In my opinion the main merit of the NA formalism is to provide an answer to the question: what is the tropical limit notion for Kähler potentials?
- However, it is difficult to check in practice, and not elementary enough. A more elementary version of the question is: what is a convex function on a polyhedral set?
- My Fermat hypersurface paper depends on an ad hoc answer to this question via tropical combinatorics. One may expect a general answer is related to the NA conjecture I assumed.

For the future

One also would like to know beyond the generic region, and beyond the large complex structure limit case. Deeper study of the B-F-J solution may be useful for these questions:

- What can we say about singularities of the B-F-J solution? eg. Does it have Hausdorff codimension 2? Does it agree with the standard topology of $\Delta_{\mathcal{X}}$? These are highly relevant for the Kontsevich-Soibelman conjecture, and for the strong SYZ conjecture (that asks for the SYZ fibration to exist globally, not just generically).
- What about the case where $Sk(X)$ has dimension $0 < m < n$? What geometric information does the NA MA equation give? (cf. the last chapter in my paper for the conjectural picture).