From gauge theory to calibrated geometry and back

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Two variational problems in geometry

Given a Riemannian manifold M, we are interested in minimizing

1. The Yang–Mills functional

$$\text{connection } A \mapsto \int_M |F_A|^2$$

2. The volume functional

$$\text{submanifold } Q \mapsto \mathrm{volume}(Q)$$

Classical examples

1. Electromagnetism; Hodge theory
2. Geodesics; minimal surfaces

Special holonomy manifolds (Calabi–Yau, G_2, Spin(7)) have natural calibrations, i.e. differential forms $\phi \in \Omega^k(M)$ such that

$$\mathrm{d}\phi = 0 \quad \text{and} \quad \phi(e_1, \dots, e_k) \leq \mathrm{vol}(e_1, \dots, e_k)$$

1. Instantons: connections A satisfying

$$F_A + *(F_A \wedge \phi) = 0 \quad \Longrightarrow \quad A \text{ is a Yang–Mills connection}$$

2. Calibrated submanifolds: $Q \subset M$ satisfying

$$\phi_{|Q} = \mathrm{vol}_Q \quad \Longrightarrow \quad Q \text{ is a minimal submanifold}$$

Calabi–Yau: holomorphic curves and surfaces, special Lagrangians

G_2: associative and coassociative submanifolds

Spin(7): Cayley submanifolds

The Euler–Lagrange equations for the Yang–Mills and volume functionals are non-linear generalizations of the Laplace equation

$$\Delta f = 0.$$

Instantons and calibrated submanifolds obey simpler, first order elliptic differential equations.

Analogy

Cauchy–Riemann equation $\overline{\partial} f = 0$ or Dirac equation $\not{D} f = 0$

$$\implies \Delta f = 0.$$

Motivations from low dimensions

1. Invariants of manifolds: Donaldson defined topological invariants of 4–manifolds by "counting" instantons

$$F_A + *F_A = 0.$$

More generally, there is Donaldson–Floer topological field theory

dimension	type of invariant
4	number
3	vector space
2	category

2. Algebraic geometry: Atiyah and Bott related flat connections

$$F_A = 0$$

to Mumford's theory of holomorphic vector bundles on curves.

3. Symplectic geometry: Monopoles on symplectic 4–manifolds correspond to pseudo-holomorphic curves, by work of Taubes.

4. Quantum field theory: Chern–Simons theory and quantum invariants of knots, Seiberg–Witten QFT, S–duality

We want to find similar beautiful structures in higher dimensions, for Calabi–Yau, G_2, and Spin(7) manifolds.

I will mention three fascinating but challenging problems.

1. Invariants

Problem (Donaldson–Thomas, 1998)
Define invariants by counting instantons / calibrated submanifolds.

dimension	holonomy	type of invariant
8	Spin(7)	number
7	G_2	vector space
6	SU(3)	category

Main difficulties

1. Bubbling phenomenon (Uhlenbeck): for a sequence (A_n) of instantons, energy $|F_{A_n}|^2$ can concentrate as $n \to \infty$ along codimension 4 subset $S \subset M$, which is calibrated (Tian)

   $\implies$ Gauge theory and calibrated geometry are closely related!

2. Singularities of instantons
3. Singularities of calibrated cycles

Interesting foundational questions in analysis: elliptic differential equations, geometric measure theory (cf. DeLellis, Naber, ...)

2. Algebraic geometry

Donaldson–Thomas invariants were rigorously constructed for Calabi–Yau three-folds using sheaf theory.

instanton $\Longleftrightarrow$ stable holomorphic bundle $\Longleftrightarrow$ sheaf of sections

calibrated surface $\Longleftrightarrow$ holomorphic curve $\Longleftrightarrow$ ideal sheaf

There is rich algebraic theory of moduli spaces of sheaves!

Connections with mirror symmetry and representation theory

Conjecture (Maulik–Nekrasov–Okounkov–Pandharipande, 2003)
Donaldson–Thomas invariants of Calabi–Yau three-folds are equivalent to Gromov–Witten invariants:

$$GW_A(u) = DT_A^{\mathrm{red}}(q) \qquad q = -e^{iu}$$

3. Symplectic geometry

Gromov–Witten invariants depend only on the symplectic structure.

Problem
Find a symplectic interpretation of Donaldson–Thomas invariants.

We should count instantons and embedded pseudo-holomorphic curves as in Taubes' work on symplectic 4–manifolds.

This approach can help us understand better the MNOP conjecture (cf. proof of Gopakumar–Vafa conjecture by Ionel–Parker, 2013) and shed light on higher rank invariants.

Some recent developments

1. Constructions

   Instantons on known G_2 and Spin(7) manifolds (Walpuski, Sá Earp, Menet, Nordström, Tanaka); similarly for calibrated submanifolds (discussed in Mark Haskins' talk)

2. Analytic foundations

   Gluing theorems for associatives (Joyce, Nordström) and instantons (Walpuski); orientations (Cao, Joyce, Upmeier, Tanaka); new ideas on counting problems (Joyce, Haydys, Walpuski)

3. Singularities

   Local models (Bryant, Li, Joyce); deformation theory (Wang, Waldron); relations to algebraic geometry (Jacob, Walpuski, Chen, Sun), singularities in gauge theory (Haydys, Walpuski, Doan)

4. Dualities

   G_2 fibrations (Li, Donaldson, Scaduto), twisted connected sums (Acharya, Braun, Svanes, Valandaro)

Instantons → calibrated submanifolds → monopoles

Donaldson–Segal, Haydys, Walpuski

$(M^7, \phi) = G_2$–manifold

When ϕ varies, G_2 instantons can bubble along associative $S \subset M$
$\implies$ counting G_2 instantons does not yield invariants of M

Idea
Count also associatives S with weights. Schematically,

$$\mathrm{DT}(M, \phi) = \sum_{A \text{ instanton on } (M,\phi)} \mathrm{sign}(A) + \sum_{S \subset M \text{ calibrated by } \phi} w(S, \phi)$$

We want $w(S, \phi)$ to change by ± 1 when bubbling along S happens.

Haydys and Walpuski proposed to construct $w(S, \phi)$ by counting monopoles, solutions to generalized Seiberg–Witten equations on S.

This proposal connects special holonomy geometry with problems in low-dimensional topology, especially with

1. work of Taubes on flat $\mathrm{SL}(2, \mathbb{C})$ connections, related 3–manifolds invariants of Abouzaid–Manolescu;
2. new approaches to Khovanov homology via gauge theory by Witten and via symplectic geometry by Seidel–Smith.

Counting associatives

joint work with Thomas Walpuski

The proposal is interesting even when we ignore instantons.
The naive count of associatives is not an invariant as ϕ varies.
We expect that these transitions can occur:

1. S_1, $S_2 \rightsquigarrow S_1 \# S_2$ (Joyce–Nordström crossing)
2. S_1, $S_2 \rightsquigarrow S_3$ (Harvey–Lawson smoothing of cone singularities);
3. $S_1 \rightsquigarrow kS_2$ for $k > 1$ (degeneration to multiple cover).

The idea is to equip each S with a weight given by counting solutions to generalized Seiberg–Witten equations on S.
For the usual Seiberg–Witten invariant (provided $b_1 > 1$):

$$w(S_1 \# S_2) = 0$$
$$w(S_1) + w(S_2) = w(S_3)$$

Multiple covers are harder. To understand this better, consider a dimensional reduction of this proposal to $S^1 \times$CY3.

Towards symplectic invariants

M = Calabi–Yau three-fold, or symplectic 6–manifold with $c_1 = 0$

Theorem (Doan–Walpuski)
For a generic compatible J there are finitely many closed, embedded J–holomorphic curves in a given class $A \in H_2(M, \mathbb{Z})$. If A is primitive, a signed count of genus g curves $n_{g,A}$ does not depend on J and fits into the Gopakumar–Vafa formula. For g large, $n_{g,A} = 0$ (finiteness part of the GV conjecture).

Proof uses work / ideas of DeLellis et al., Taubes, Wendl, Zinger.

If $A \in H_2(M, \mathbb{Z})$ is divisible, the naive count depends on J.

Suppose $A = 2B$ with B primitive.
As J_t varies, we can have $[\tilde{C}_t] = A$ and $\tilde{C}_t \to 2C$ with $[C] = B$.

Consider

$$DT_A(M, J) = \sum_{[\tilde{C}]=A} SW_1(\tilde{C}) + \sum_{[C]=B} SW_2(C, J)$$

$SW_1(C) =$ count of solutions to the abelian vortex equations on C

$SW_2(C, J) =$ count of solutions to non-abelian vortex equations similar to Hitchin's equations and depending on J

$$\begin{cases} \bar{\partial}_{J,A}\xi = 0, \quad \bar{\partial}_A \alpha = 0, \quad \bar{\partial}_A \beta = 0 \\ [\xi \wedge \xi] + \alpha \cdot \beta = 0 \\ i * F_A + [\xi \wedge \xi^*] + \alpha\alpha^* - \beta^*\beta = 0 \end{cases}$$

$SW_2(C, J)$ changes whenever there exists a J–holomorphic section

$$C \to \mathrm{Sym}^2 N_{C/M}.$$

This is Hitchin's spectral curve construction:

$$\xi \in \Gamma(N \otimes \mathrm{End} E)$$
$$[\xi \wedge \xi] = 0,$$
$$[\xi \wedge \xi^*] = 0.$$

Such a section can be deformed to a J–holomorphic curve $\tilde{C}$ with $[\tilde{C}] = 2B = A$. Thus, $DT_A(M, J)$ should not depend on J.

To prove invariance, we need to study carefully the compactness problem for these equations (following Taubes, Walpuski–Zhang).

After a long journey we come back to where we started: gauge theory on Riemann surfaces.

Michael Atiyah and Raoul Bott