Closed G_2-structures with conformally flat metric

Special Holonomy: Progress and Open Problems 2020

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G_2-structures

A G_2-structure $\varphi \in \Omega^3(M)$ on a 7-manifold M defines a Riemannian metric g_φ and associated volume form vol_φ on M via the formula

$$g_\varphi(u,v)\,\mathrm{vol}_\varphi = \tfrac{1}{6}\,(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$

Special conditions satisfied by φ are reflected in the geometry of g_φ. For example, if φ is torsion-free, then g_φ has holonomy group contained in G_2 and it follows that g_φ is Ricci-flat.

In the study of manifolds with special holonomy, it is often necessary to consider G_2-structures satisfying first-order conditions weaker than the torsion-free condition. In this context, there are two natural cases of interest:

- *Closed* G_2*-structures*, those satisfying $d\varphi = 0$.
- *Coclosed* G_2*-structures*, those satisfying $d *_\varphi \varphi = 0$.

Suppose φ satisfies one of these weaker conditions. What can we say about g_φ?

Given a Riemannian metric g on M, is there a G_2-structure φ on M satisfying one of these weaker conditions with $g = g_\varphi$?

Locally, every Riemannian metric g is induced by some G_2-structure φ. Let $e_1, \ldots, e_7$ be a (locally defined) orthonormal coframe for g. The G_2-structure

$$\varphi = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356}$$

satisfies $g_\varphi = g$.

For the generic metric g there will not exist (even locally) a closed or coclosed G_2-structure φ with $g = g_\varphi$.

A Riemannian metric on a 7-manifold depends, modulo diffeomorphism, on 21 functions of 7 variables.

Locally, a closed G_2-structure is exact, $\varphi = d\beta$, and this can be used to show that closed G_2-structures depend, modulo diffeomorphism, on 8 functions of 7 variables.

Similarly, coclosed G_2-structures depend, modulo diffeomorphism, on 13 functions of 7 variables.

Thus, the prescribed metric problems are overdetermined systems of PDE and generically don't have solutions. An exterior differential systems analysis of these systems yields relatively complicated *integrability conditions* arising from the Bianchi identities.

Closed G_2-structures

For the rest of the talk we will consider only closed G_2-structures.

Bryant has shown that the scalar curvature of g_φ is nonpositive. If τ denotes the torsion 2-form of φ, i.e., the 2-form satisfying $d *_\varphi \varphi = \tau \wedge \varphi$, then

$$\mathrm{Scal}(g_\varphi) = -\tfrac{1}{2}|\tau|_\varphi^2.$$

More generally, certain components of the Riemannian curvature tensor of g_φ are determined by τ.

Let $\mathcal{K}$ denote the space of algebraic curvature tensors on $\mathbb{R}^7$.There is an SO(7)-irreducible decomposition

$$\mathcal{K} \cong \underbrace{\mathbb{R}}_{\text{scalar}} \oplus \underbrace{\mathrm{Sym}_0^2(\mathbb{R}^7)}_{\text{traceless Ricci}} \oplus \underbrace{\mathcal{W}}_{\text{Weyl}}.$$

The first two spaces remain irreducible under the action of G_2, but the space of algebraic Weyl tensors has an irreducible decomposition

$$\mathcal{W} \cong \mathrm{Sym}_0^2(\mathbb{R}^7) \oplus \mathbb{R}^{64} \oplus \mathbb{R}^{77}.$$

Thus, there are five basic tensors that can be formed from the Riemannian curvature on a manifold with G_2-structure:

$$\mathrm{Scal}(g_\varphi),\ \mathrm{Ric}^0(g_\varphi),\ W_{27},\ W_{64},\ W_{77}.$$

For a closed G_2-structure, the first Bianchi identity implies:

- Bryant's formula for $\mathrm{Scal}(g_\varphi)$.
- A formula also due to Bryant for $\mathrm{Ric}^0(g_\varphi)$ in terms of τ and $d\tau$.
- A formula for W_{27} in terms of τ and $d\tau$.
- A formula for W_{64} in terms of $\nabla\tau$.

In fact, using Cartan's method of equivalence, it is possible to show that $d\tau$, W_{64}, and W_{77} form a complete set of second-order diffeomorphism invariants for a closed G_2-structure φ.

Global issues

It is currently unknown whether the existence of a closed G_2-structure on a simply-connected compact spin 7-manifold implies any other topological restrictions. In particular, it is not known if the 7-sphere S^7 admits a closed G_2-structure.

In light of this lack of knowledge, it might be worth investigating if the special properties of g_φ imply any restriction on the topology of M.

In this direction, the following two results, valid for closed G_2-structures on a compact manifold, are interesting:

$$\int_M |\operatorname{Ric}^0(g_\varphi)|^2 \operatorname{vol}_\varphi \geq \frac{4}{21}\int_M \operatorname{Scal}(g_\varphi)^2 \operatorname{vol}_\varphi \quad \text{(Bryant)}.$$

$$\langle p_1(M) \cup [\varphi], [M] \rangle$$
$$= -\frac{1}{8\pi^2}\int_M \left\{ |W_{77}|^2 - \tfrac{1}{2}|W_{64}|^2 - \tfrac{9}{7}|\operatorname{Ric}^0(g_\varphi)|^2 + \tfrac{45}{784}\operatorname{Scal}(g_\varphi)^2 \right\} \operatorname{vol}_\varphi$$

(Cleyton & Ivanov).

Closed G_2-structures with conformally flat metric

What can we say about a closed G_2-structure φ if g_φ is conformally flat?

Conformal flatness is equivalent to the vanishing of the Weyl curvature of g_φ, so this condition amounts to a system of 168 second order PDE for φ.

The natural approach to a problem of this type is to make use of moving frames and the theory of exterior differential systems.

Structure equations

Let φ be a closed G_2-structure. Let $\mathcal{B}$ be the G_2-coframe bundle of φ, with tautological 1-form ω.

On $\mathcal{B}$, the Levi-Civita connection of g_φ splits as

$$\theta + T(\omega),$$

where θ is a $\mathfrak{g}_2$-valued connection 1-form on $\mathcal{B}$ representing the *natural* connection on $\mathcal{B}$, and T is a section of an associated bundle modeled on $\mathfrak{g}_2 \cong \Lambda^2_{14}$ and representing the torsion 2-form τ.

Cartan's first structure equation for $\mathcal{B}$ reads

$$d\omega = -\theta \wedge \omega + T\left(\omega \wedge \omega\right),$$

while Cartan's second structure equation and $d^2\omega = 0$ imply

$$dT + \theta \cdot T = \left(W_{27} + W_{64} + T^2 \text{ terms}\right)(\omega),$$
$$d\theta + \theta \wedge \theta = \left(W_{27} + W_{64} + W_{77} + T^2 \text{ terms}\right)(\omega \wedge \omega).$$

If g_φ is conformally flat, then $W_{27} = W_{64} = W_{77} = 0$, and the structure equations become:

$$\begin{aligned} d\omega + \theta \wedge \omega &= T\,(\omega \wedge \omega)\,, \\ dT + \theta \cdot T &= \left(T^2 \text{ terms}\right)(\omega), \\ d\theta + \theta \wedge \theta &= \left(T^2 \text{ terms}\right)(\omega \wedge \omega). \end{aligned} \tag{1}$$

Conversely, if N is a 21-manifold endowed with a coframe (ω, θ) and a function $T : N \to \mathbb{R}^{14}$ satisfying the structure equations (1), then N can be identified with the G_2-coframe bundle of a closed G_2-structure φ with g_φ conformally flat.

Thus, the problem has been reduced to a prescribed coframing problem. Such problems are the subject of powerful existence and uniqueness theorems dating back to the work of Élie Cartan on generalisations of Lie's third theorem. In this case, existence and uniqueness is guaranteed as long as the equation $d^2 = 0$ is formally satisfied (which it is).

Existence and uniqueness theorem

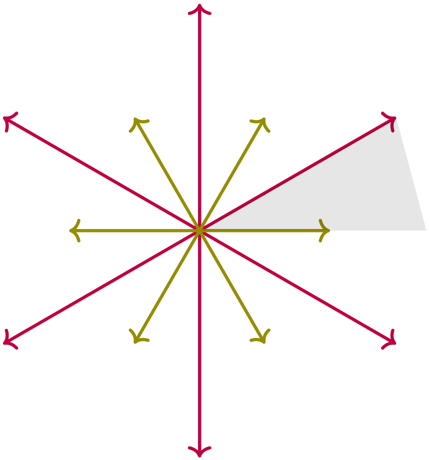
Theorem (B. 2020)

For any $T_0 \in \mathfrak{g}_2$ there exists a closed G_2-structure φ with g_φ conformally flat on a neighbourhood U of $0 \in \mathbb{R}^7$ whose G_2-coframe bundle contains a $u_0 \in \pi^{-1}(0)$ for which $T(u_0) = T_0$.
Any two C^2 closed G_2-structures satisfying these properties are isomorphic on a neighbourhood of $0 \in \mathbb{R}^7$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\;T\;} & \mathfrak{g}_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\;[T]\;} & \mathfrak{g}_2/\mathrm{G}_2 \end{array}$$

The set of isomorphism classes of germs of closed G_2-structures with g_φ conformally flat is in bijection with the orbit space $\mathfrak{g}_2/\mathrm{G}_2 = \mathbb{R}^2/\mathrm{D}_6$.

Thus, to classify closed G_2-structures with g_φ conformally flat it suffices to produce enough examples to fill out $\mathfrak{g}_2/G_2$:



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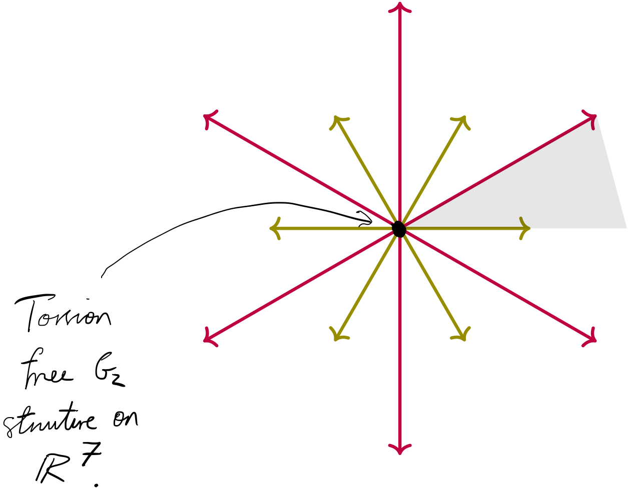

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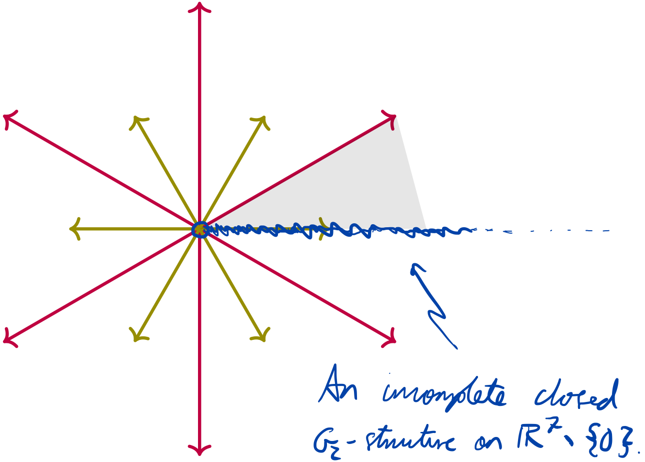

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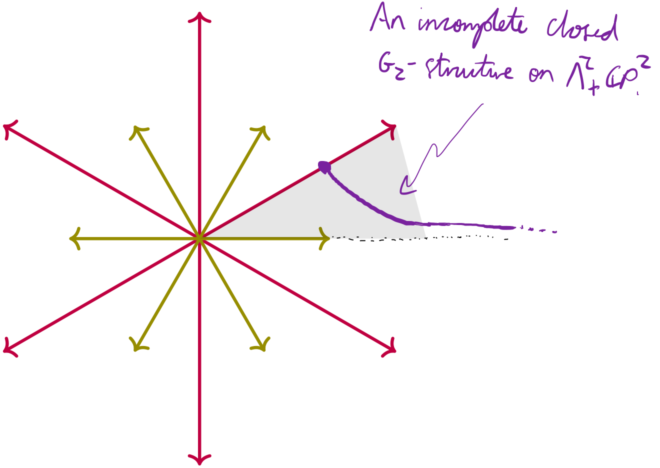

Classification theorem

Theorem (B. 2020)

A closed G_2*-structure* φ *with (locally) conformally flat induced metric* g_φ *is, up to constant rescaling, locally equivalent to one of three examples:* $\mathbb{R}^7, \mathbb{R}^7 \setminus \{0\}$, *and* $\Lambda^2_+ \mathbb{CP}^2$.

Corollary

Let (M, φ) *be a 7-manifold endowed with a closed* G_2*-structure such that* g_φ *is conformally flat and complete. Then* φ *is locally equivalent to the flat* G_2*-structure* ϕ *on* $\mathbb{R}^7$ *and* M *is a quotient of* $\mathbb{R}^7$ *by a discrete group of* G_2*-automorphisms.*

Thank you for your attention!