Constructing compact, holonomy Spin(7) manifolds as generalised connected sums (of asymptotically cylindrical manifolds)

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The group Spin(7)

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is isomorphic to Spin(7). (Here $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$.) The 4-form is self-dual $*\Phi_0 = \Phi_0$ ($*$ is the Hodge star of Euclidean $\mathbb{R}^8$).

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$\Phi_0 = dx_1 \wedge \varphi_0 + *_7\,\varphi_0$ for some unique 3-form φ_0 on $\mathbb{R}^7$ (with coordinates $x_2, \ldots, x_8$), where $*_7$ is the Hodge star of $\mathbb{R}^7$. The subgroup of $GL(7,\mathbb{R})$ fixing φ_0 is the exceptional Lie group G_2.

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Also $\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \operatorname{Re}\theta_0$, where $\omega_0 = dx_1 \wedge dx_2 + \ldots + dx_7 \wedge dx_8$ and $\theta_0 = (dx_1 + idx_2) \wedge \ldots \wedge (dx_7 + idx_8)$. The stabilizer of pair (ω_0, θ_0) in the $GL(8,\mathbb{R})$-action is $SU(4) \subset \mathrm{Spin}(7)$.

Torsion-free Spin(7)-structures

Let M be an 8-manifold. A differential 4-form Φ on M is *admissible* if it is pointwise equivalent to Φ_0, via a linear isomorphism $\mathbb{R}^8 \to T_x M$, for each $x \in M$. Every admissible form induces a Spin(7)-*structure* on M, hence also a metric $g(\Phi)$ and an orientation on M as $\mathrm{Spin}(7) \subset SO(8)$.

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The *holonomy* of $g(\Phi)$ is contained in Spin(7) iff $d\Phi = 0$ (Fernandez); in this case the Spin(7)-structure Φ is said to be *torsion-free* and (M, Φ) is a Spin(7)-*manifold*. Then the metric $g(\Phi)$ is Ricci-flat (Bonan) and when its holonomy is all of Spin(7) the metric is *not* Kähler.

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There are similar type results for torsion-free G_2 structures on real 7-manifolds and torsion-free $SU(4)$-structures (Calabi–Yau structures) on complex 4-folds, in terms of closed forms, respectively, φ, $*_7\varphi$ and ω, θ.

The holonomy of a Spin(7)-metric $g(\Phi)$

Suppose that (M, Φ) is a *compact* Spin(7)-manifold. Then $48\hat{A}(M) = 3\,\mathrm{sign}(M) - \chi(M)$, where $\hat{A}(M)$ is the $\hat{A}$-genus of M, $\mathrm{sign}(M) = b^4_+(M) - b^4_-(M)$ is the signature of the intersection form on $H^4(M, \mathbb{R})$ and $\chi(M)$ is the Euler characteristic.

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Conversely, if the holonomy group of a compact Spin(7)-manifold (M, Φ) is *Spin*(7), then M is simply-connected. (Joyce)

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- Spin(7) *iff* Y *is connected and* $b^1(Y) = 0$*;*
- G_2 *iff* $M \cong \mathbb{R} \times Y$ *and* Y *is a compact 7-manifold with holonomy* G_2*;*
- $SU(4)$ *iff* $b^1(Y) = 1$*;*
- $SU(2) \times SU(2)$ *iff* $M = S_1 \times S_2$ *with* S_1 *a* K3 *surface and* S_2 *an asymptotically cylindrical Calabi–Yau surface.*

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The criterion for holonomy Spin(7) was first obtained by Nordström in 2008.

Examples for the last 3 cases are obtainable from known results of Joyce, A.K., Haskins–Hein–Nordström, Biquard–Minerbe.

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Let M be an 8-manifold with cylindrical end $M_\infty = \mathbb{R}_+ \times Y$ and suppose that an admissible 4-form $\tilde{\Phi} = \tilde{\Phi}(s)$ defines on M a Spin(7)*-structure whose restriction to the end $M_\infty \subset M$ is asymptotically cylindrical and torsion-free $d\tilde{\Phi}|_{M_\infty} = 0$.*

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Assumer further that for $0 < s < \kappa$ there exists $\psi = \psi(s) \in \Omega^4(M)$ with $d\tilde{\Phi} + d\psi = 0$ satisfying the estimates $\|\psi\|_{L^2(M)} \leq \lambda s^{13/3}$, $\|d\phi\| \leq \lambda s^{7/5}$ and such that the injectivity radius of $g(\tilde{\Phi})$ is $\geq \mu s$ and the curvature $\|R(g(\tilde{\Phi}))\|_{C^0(M)} \leq \mu^{-1} s^{-2}$.

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Then the equation $d\eta = d\psi + dF(\eta)$ has for each sufficiently small $s > 0$ a solution $\eta \in \Omega^4_-(M)$ exponentially decaying along the end M_∞ and satisfying $\|\eta\|_{C^0(M)} \leq K s^{1/3}$. Hence $d(\Theta(\tilde{\Phi} + \eta)) = 0$ and $\Phi = \Theta(\tilde{\Phi} + \eta)$ is an admissible 4-form inducing a torsion-free asymptotically cylindrical Spin(7)*-structure on all of M.*

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This extends a result proved by Dominic Joyce for *compact* 8-manifolds to non-compact 8-manifolds *with cylindrical ends*.

The gluing theorem

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For $i = 1, 2$, *let* (M_i, Φ_i) *be asymptotically cylindrical* Spin(7)-*manifolds with cross-sections* G_2-*manifolds* (Y_i, φ_i) *at infinity. Suppose there is an orientation-reversing diffeomorphism* $Y_2 \to Y_1$ *sending* $\varphi_1 \mapsto -\varphi_2$ *and* $*_7\varphi_1 \to *_7\varphi_2$. *Thus the generalized connected sum* $M = M_1 \#_Y M_2$ *has a natural family of* Spin(7)-*structures* $\tilde{\Phi}_T$, $T > 1$ *(with 'small' torsion).*

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Then there exists $T_0 > 1$ and, for each $T \geq T_0$, a form $\eta_T \in \Omega^4_-(M)$ with $\|\eta_T\|_{p,k} = O(e^{-\lambda T})$, such that the Spin(7)*-structure $\Phi_T = \Theta(\tilde{\Phi}_T + \eta_T)$ is torsion-free.*

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Furthermore, if $b^j(M_i) = b^j_c(M_i) = 0$ for $j = 1, 2, 3$ and $b^1(Y) = b^2(Y) = 0$ and $3\,\mathrm{sign}(M_i) - b^4(M_i) = 25$, then the holonomy of $g(\Phi_T)$ is Spin(7).

The 'Fano type' orbifold configuration (V, D, Σ, ρ)

We require these condisions

1. V is a compact Kähler complex 4-dimensional orbifold. The singular locus of V is non-empty and finite $\{p_1, \ldots, p_k\}$. Each singularity is isomorphic to $\mathbb{C}^4/\langle i \cdot \mathrm{Id}\rangle$ with isotropy group $\mathbb{Z}_4$.

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2. The smooth locus V^* of V is simply-connected and there exists a simply-connected smooth divisor $D \in |-K_V|$, with $D \subset V^*$ and $V^* \setminus D$ simply-connected. The self-intersection $D \cdot D$, in the sense of the Chow ring, is represented by an effective divisor $\Sigma = \sum_i n_i \Sigma_i$, where each Σ_i is a smooth connected complex surface in D.

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3. There is an antiholomorphic involution ρ of V fixing the singular locus and no other points in V and such that $\rho(D) = D$ and $\rho(\Sigma_i) = \Sigma_i$, for each i.

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- Blow up Σ in V, obtaining a Kähler orbifold $\tilde{V}$. Then the proper transform $\tilde{D}$ of D is an anticanonical divisor $\tilde{D} \in |-K_{\tilde{V}}|$ with holomorphically trivial normal bundle. Also, ρ lifts to an anticanonical involution of $\tilde{V}$ preserving $\tilde{D}$.

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- Then $\tilde{V} \setminus \tilde{D}$ admits an asymptotically cylindrical Calabi–Yau structure (ω, θ). (Tian–Yau, A.K., Haskins–Hein–Nordström)

  Moreover, $\tilde{\rho}^*(\omega) = -\omega$ and $\tilde{\rho}^*(\theta) = \bar{\theta}$, so the Spin(7)-structure $\Phi_{\mathrm{CY}} = \frac{1}{2}\omega \wedge \omega + \operatorname{Re}\theta$ is invariant under $\tilde{\rho}$.

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- The quotient $M_0 = (\tilde{V} \setminus \tilde{D})/\tilde{\rho}$ can be made into a smooth, simply-connected 8-manifold M by replacing neighbourhoods of the singular points by a certain ALE Spin(7)-spaces (given by Dominic Joyce). We obtain on M, by patching, an asymptotically cylindrical Spin(7)-structure $\tilde{\Phi}$ satisfying the hypotheses of existence theorem, so $\tilde{\Phi}$ can be perturbed into a torsion-free asymptotically cylindrical Spin(7)-structure Φ.

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- For a *pair* M, M' as above with isomorphic (D, ρ) (D', ρ') the gluing theorem applies to give a compact *Spin*(7)-manifold.

Examples: asymptotically cylindrical Spin(7)-manifolds from complex orbifolds

A weighted projective space $V = \mathbb{C}P^4_{1,1,1,1,4}$ has just one singular point $p_0 = [0,0,0,0,1]$.
Let $D = \{z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 = 0\}$ and $\Sigma = D \cap D'$, where $D' = z_0^8 - z_1^8 + 2z_2^8 - 2z_3^8 + i\, z_4^2$, and
$\rho : [z_0, \ldots, z_4] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4]$.
This configuration produces a Spin(7)-manifold M_1 with $b^4(M) = 839$, signature $(488, 200)$ and cross-section satisfying $b^3(Y) = 151$.

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Another example: a hypersurface
$V = \{f_8(z) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 + z_5^2 = 0\} \subset \mathbb{C}P^5_{1,1,1,1,4,4}$ has two singular points $p_\pm = [0,0,0,0,i,\pm 1]$. Take
$D = V \cap \{z^4 + z^5 = 0\}$ and $\Sigma = D \cap \{z_4 - z_5 = 0\}$ and
$\rho : [z_0, \ldots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4]$.
A Spin(7)-manifold M_2 we then obtain has $b^4(M_2) = 455$ and signature $(232, 72)$. The cross-section Y of M_2 is isomorphic (as a G_2-manifold) to the cross section of M_2.

Properties of the Spin(7)-manifolds M_i

The cross-section of M_i is a 7-manifold $Y \cong (D \times S^1)/\rho$ with ρ acting on S^1 by reflection. Then $b^1(Y) = 0$ and any torsion-free Spin(7)-structure Φ on M_i induces a metric with holonomy Spin(7). The limit G_2-structure on Y 'at infinity' induces a metric on Y with holonomy $\mathbb{Z} \ltimes SU(3)$ (not G_2 as $\pi_1(Y)$ is infinite).

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By the inverse Hurewicz theorem, M_i are 3-connected.

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We obtain, by application of the gluing theorem, three compact examples M_{ij} of 8-manifolds with holonomy Spin(7). These are 3-connected and have

$$b^4(M_{ij}) = b^4(M_i) + b^4(M_j) \text{ and}$$
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The values of (b^4_+, b^4_-) are $(615, 295)$, $(871, 423)$, $(1127, 551)$. The first two are deformation equivalent to Spin(7)-manifolds found by Dominic Joyce. The third example, with $b^4 = 1678$, is topologically *new*.

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Previously, Joyce (1999) and Clancy (2011) found 15 compact Spin(7)-manifolds with $b^1 = b^2 = b^3 = 0$, by a different method. Of these, 5 have $b^4 < 1678$ and 10 have $b^4 > 1678$.

An example of Spin(7)-manifolds constructed by Joyce from Calabi–Yau 4-orbifolds

A hypersurface $W = \{z_0^8 + \ldots + z_5^4 = 0\}$ of weighted degree 8 in weighted projective space $\mathbb{C}P^5_{1,1,1,1,2,2}$ is invariant under a holomorphic involution $\beta : [z_0, \ldots, z_5] \mapsto [iz_0, iz_1, iz_2, iz_3, z_4, z_5]$.

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The singular locus of W/β consists of 4 isolated $\mathbb{Z}_4$-orbifold singularities and $W/\beta \cap \{z_4 = z_5 = 0\}$ a copy of the octic Fermat surface $\Sigma_8 \subset \mathbb{C}P^3$. Let $\tilde{W}$ be the blow-up of Σ_8 in W/β.

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The anti-holomorphic involution $\sigma : [z_0, \ldots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4]$ of W fixes only its isolated singular points, commutes with β and acts freely on Σ_8. Thus σ lifts to an anti-holomorphic involution of the blow-up $\tilde{W}$.

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Dominic Joyce proved that the desingularization M of the orbifold $\tilde{W}/\sigma$ is a holonomy Spin(7) manifold which has $b^1 = b^2 = b^3 = 0$ and the signature $(b^4_+, b^4_-) = (615, 295)$.

Connecting $M_2 \# M_2$ with Joyce's Spin(7)-manifold

Now, in our generalized connected sum $M_2 \# M_2$ each M_2 arises from a degree 8 hypersurface V in $\mathbb{C}P^5_{1,1,1,1,4,4}$. The natural map $\pi : [z_0, z_1, z_2, z_3, z_4, z_5] \in \mathbb{C}P^5_{1,1,1,1,2,2} \to [z_0, z_1, z_2, z_3, z_4^2, z_5^2] \in \mathbb{C}P^5_{1,1,1,1,4,4}$ induces a 2-to-1 branched covering

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The path of Spin(7)-metrics on $M_2 \# M_2$ defined by the gluing theorem correspond to 'pulling apart' M at the pre-image of E. An asymptotic path of Spin(7)-metrics is obtained by *first* making a compact Calabi–Yau 4-orbifold by gluing the asymptotically cylindrical pieces $\tilde{V} \setminus \tilde{D}$ and *then* applying Joyce's construction. For 'large' values of neck length, the two Spin(7) 4-forms are in the same coordinate patch of the Spin(7)-moduli space.

Example of a different type of asymptotically cylindrical, holonomy *Spin*(7) manifold

We modify an example of Dominic Joyce's first construction of *Spin*(7)-manifolds, resolving singularities of T^8/Γ. The group Γ generated by

$$\begin{aligned}
\alpha :&(x_1,\ldots,x_7) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),\\
\beta :&(x_1,\ldots,x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),\\
\gamma :&(x_1,\ldots,x_7) \mapsto (\frac{1}{2} - x_1, \frac{1}{2} - x_2, x_3, x_4, \frac{1}{2} - x_5, \frac{1}{2} - x_6, x_7, x_8),\\
\delta :&(x_1,\ldots,x_7) \mapsto (-x_1, x_2, \frac{1}{2} - x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7, x_8).
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Consider Γ acting on $(T^7 \times \mathbb{R}_{x_8})$, then the quotient is an Spin(7)-orbifold with a cylindrical end. The cross-section is an orbifold T^7/Γ' for the subgroup $\Gamma' \subset \Gamma$ generated by β, γ, δ isomorphic to $\mathbb{Z}_2^3$ and fixing x_8. This 7-orbifold is isomorphic to one used by Dominic Joyce to construct a compact, holonomy G_2 manifold $X \to T^7/\Gamma'$ by resolving singularities.

It can be shown that a resolution $M \to (X \times \mathbb{R})/\langle\alpha\rangle$ has a structure of asymptotically cylindrical Spin(7)-manifold. The argument is similar in spirit to that in a joint paper with Johannes Nordström (2010) constructing asymptotically cylindrical, holonomy G_2 7-manifolds (from quotients of $T^6 \times \mathbb{R}$).

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Geometrically this may be interpreted via 'pulling apart' a compact $Spin(7)$-manifold obtained from T^8/Γ into two asymptotically cylindrical pieces each isomorphic to M. The cross-section at inftity in the present example is a simply-connected 7-manifold with holonomy *equal* to G_2.

Compact $Spin(7)$-manifolds may be ontained from pairs of asymptotically cylindrical pieces like M, with isomorphic cross-sections at infinity.