Constructing compact, holonomy Spin(7) manifolds as generalised connected sums (of asymptotically cylindrical manifolds)

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The group $\text{Spin}(7)$

The standard action of $\text{GL}(8, \mathbb{R})$ on $\mathbb{R}^8$ induces an action on 4-forms. The stabilizer of

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is isomorphic to $\text{Spin}(7)$. (Here $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$.) The 4-form is self-dual $*\Phi_0 = \Phi_0$ ($*$ is the Hodge star of Euclidean $\mathbb{R}^8$).
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Also $\Phi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 + \text{Re } \theta_0$, where $\omega_0 = dx_1 \wedge dx_2 + \ldots + dx_7 \wedge dx_8$ and $\theta_0 = (dx_1 + idx_2) \wedge \ldots \wedge (dx_7 + idx_8)$. The stabilizer of pair $(\omega_0, \theta_0)$ in the $GL(8, \mathbb{R})$-action is $SU(4) \subset \text{Spin}(7)$. 
Torsion-free Spin(7)-structures

Let $M$ be an 8-manifold. A differential 4-form $\Phi$ on $M$ is *admissible* if it is pointwise equivalent to $\Phi_0$, via a linear isomorphism $\mathbb{R}^8 \to T_x M$, for each $x \in M$. Every admissible form induces a Spin(7)-structure on $M$, hence also a metric $g(\Phi)$ and an orientation on $M$ as Spin(7) $\subset SO(8)$. 

The holonomy of $g(\Phi)$ is contained in Spin(7) iff $d\Phi = 0$ (Fernandez); in this case the Spin(7)-structure $\Phi$ is said to be torsion-free and $(M, \Phi)$ is a Spin(7)-manifold. Then the metric $g(\Phi)$ is Ricci-flat (Bonan) and when its holonomy is all of Spin(7) the metric is not Kähler. There are similar type results for torsion-free $G_2$ structures on real 7-manifolds and torsion-free SU(4)-structures (Calabi–Yau structures) on complex 4-folds, in terms of closed forms, respectively, $\phi$, $*7\phi$ and $\omega$, $\theta$. 
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The *holonomy* of $g(\Phi)$ is contained in Spin(7) iff $d\Phi = 0$ (Fernandez); in this case the Spin(7)-structure $\Phi$ is said to be *torsion-free* and $(M, \Phi)$ is a Spin(7)-manifold. Then the metric $g(\Phi)$ is Ricci-flat (Bonan) and when its holonomy is all of Spin(7) the metric is *not* Kähler.
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There are similar type results for torsion-free $G_2$ structures on real 7-manifolds and torsion-free $SU(4)$-structures (Calabi–Yau structures) on complex 4-folds, in terms of closed forms, respectively, $\varphi, \ast_7 \varphi$ and $\omega, \theta$. 
Suppose that \((M, \Phi)\) is a compact Spin(7)-manifold. Then 
\[48\hat{A}(M) = 3\text{sign}(M) - \chi(M),\]
where \(\hat{A}(M)\) is the \(\hat{A}\)-genus of \(M\), 
\[\text{sign}(M) = b_4^+(M) - b_4^-(M)\]
is the signature of the intersection form on \(H^4(M, \mathbb{R})\) 
and \(\chi(M)\) is the Euler characteristic.

Conversely, if the holonomy group of a compact Spin(7)-manifold 
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If, in addition, $M$ is simply-connected, then the holonomy of the induced metric $g(\Phi)$ is determined by the $\hat{A}$-genus. In particular, the holonomy will be all of Spin(7) if and only if $\hat{A} = 1$. 
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Conversely, if the holonomy group of a compact Spin(7)-manifold $(M, \Phi)$ is $\text{Spin}(7)$, then $M$ is simply-connected. (Joyce)
The holonomy of asymptotically cylindrical Spin(7)-manifolds

**Theorem**

Assume that an asymptotically cylindrical Spin(7)-manifold \((M, \Phi)\) is simply-connected. Let \(Y\) denote its cross-section. Then the only possible holonomy of the metric \(g(\Phi)\) is:

- **\(\text{Spin}(7)\)** iff \(Y\) is connected and \(b_1(Y) = 0\);
- **\(G_2\)** iff \(M \cong \mathbb{R} \times Y\) and \(Y\) is a compact 7-manifold with holonomy \(G_2\);
- **\(SU(4)\)** iff \(b_1(Y) = 1\);
- **\(SU(2) \times SU(2)\)** iff \(M = S_1 \times S_2\) with \(S_1\) a K3 surface and \(S_2\) an asymptotically cylindrical Calabi–Yau surface.

The criterion for holonomy \(\text{Spin}(7)\) was first obtained by Nordström in 2008. Examples for the last 3 cases are obtainable from known results of Joyce, A.K., Haskins–Hein–Nordström, Biquard–Minerbe.
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Let $M$ be an 8-manifold with cylindrical end $M_\infty = \mathbb{R}_+ \times Y$ and suppose that an admissible 4-form $\tilde{\Phi} = \tilde{\Phi}(s)$ defines on $M$ a Spin(7)-structure whose restriction to the end $M_\infty \subset M$ is asymptotically cylindrical and torsion-free $d\tilde{\Phi}|_{M_\infty} = 0$.

This extends a result proved by Dominic Joyce for compact 8-manifolds to non-compact 8-manifolds with cylindrical ends.
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Assume further that for $0 < s < \kappa$ there exists $\psi = \psi(s) \in \Omega^4(M)$ with $d\tilde{\Phi} + d\psi = 0$ satisfying the estimates $\|\psi\|_{L^2(M)} \leq \lambda s^{13/3}$, $\|d\phi\| \leq \lambda s^{7/5}$ and such that the injectivity radius of $g(\tilde{\Phi})$ is $\geq \mu s$ and the curvature $\|R(g(\tilde{\Phi}))\|_{C^0(M)} \leq \mu^{-1} s^{-2}$.
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Then the equation $d\eta = d\psi + dF(\eta)$ has for each sufficiently small $s > 0$ a solution $\eta \in \Omega^4_-(M)$ exponentially decaying along the end $M_\infty$ and satisfying $\|\eta\|_{C^0(M)} \leq Ks^{1/3}$. Hence $d(\Theta(\tilde{\Phi} + \eta)) = 0$ and $\Phi = \Theta(\tilde{\Phi} + \eta)$ is an admissible 4-form inducing a torsion-free asymptotically cylindrical $\text{Spin}(7)$-structure on all of $M$.
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Theorem\nFor $i = 1,2$, let $(M_i, \Phi_i)$ be asymptotically cylindrical $\text{Spin}(7)$-\ manifold with cross-sections $G_2$-manifolds $(Y_i, \varphi_i)$ at infinity.\nSuppose there is an orientation-reversing diffeomorphism $Y_2 \to Y_1$\nsending $\varphi_1 \mapsto -\varphi_2$ and $\star_7 \varphi_1 \to \star_7 \varphi_2$. Thus the generalized\nconnected sum $M = M_1 \# Y M_2$ has a natural family of\n$\text{Spin}(7)$-structures $\tilde{\Phi}_T$, $T > 1$ (with ‘small’ torsion).
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*For* $i = 1, 2$, *let* $(M_i, \Phi_i)$ *be asymptotically cylindrical Spin(7)-manifolds* with cross-sections $G_2$-manifolds $(Y_i, \varphi_i)$ *at infinity*. Suppose there is an orientation-reversing diffeomorphism $Y_2 \to Y_1$ sending $\varphi_1 \mapsto -\varphi_2$ and $*_7 \varphi_1 \to *_7 \varphi_2$. Thus the generalized connected sum $M = M_1 \# Y M_2$ has a natural family of Spin(7)-structures $\tilde{\Phi}_T$, $T > 1$ (with ‘small’ torsion).

Then there exists $T_0 > 1$ and, for each $T \geq T_0$, a form $\eta_T \in \Omega^4_-(M)$ with $\|\eta_T\|_{p,k} = O(e^{-\lambda T})$, such that the Spin(7)-structure $\Phi_T = \Theta(\tilde{\Phi}_T + \eta_T)$ is torsion-free.
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Theorem

For \( i = 1, 2 \), let \((M_i, \Phi_i)\) be asymptotically cylindrical Spin(7)-manifolds with cross-sections G2-manifolds \((Y_i, \varphi_i)\) at infinity. Suppose there is an orientation-reversing diffeomorphism \( Y_2 \to Y_1 \) sending \( \varphi_1 \mapsto -\varphi_2 \) and \( ^*7\varphi_1 \to ^*7\varphi_2 \). Thus the generalized connected sum \( M = M_1 \#_Y M_2 \) has a natural family of Spin(7)-structures \( \tilde{\Phi}_T, T > 1 \) (with ‘small’ torsion).

Then there exists \( T_0 > 1 \) and, for each \( T \geq T_0 \), a form \( \eta_T \in \Omega^4_-(M) \) with \( \|\eta_T\|_{p,k} = O(e^{-\lambda T}) \), such that the Spin(7)-structure \( \Phi_T = \Theta(\tilde{\Phi}_T + \eta_T) \) is torsion-free.

Furthermore, if \( b^j(M_i) = b^j_c(M_i) = 0 \) for \( j = 1, 2, 3 \) and \( b^1(Y) = b^2(Y) = 0 \) and \( 3 \text{ sign}(M_i) - b^4(M_i) = 25 \), then the holonomy of \( g(\Phi_T) \) is Spin(7).
The ‘Fano type’ orbifold configuration \((V, D, \Sigma, \rho)\)

We require these conditions

1. \(V\) is a compact Kähler complex 4-dimensional orbifold. The singular locus of \(V\) is non-empty and finite \(\{p_1, \ldots, p_k\}\). Each singularity is isomorphic to \(\mathbb{C}^4/\langle i \cdot \text{Id}\rangle\) with isotropy group \(\mathbb{Z}_4\).
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2. The smooth locus \(V^*\) of \(V\) is simply-connected and there exists a simply-connected smooth divisor \(D \in |-K_V|\), with \(D \subset V^*\) and \(V^* \setminus D\) simply-connected. The self-intersection \(D \cdot D\), in the sense of the Chow ring, is represented by an effective divisor \(\Sigma = \sum_i n_i \Sigma_i\), where each \(\Sigma_i\) is a smooth connected complex surface in \(D\).
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3. There is an antiholomorphic involution \(\rho\) of \(V\) fixing the singular locus and no other points in \(V\) and such that \(\rho(D) = D\) and \(\rho(\Sigma_i) = \Sigma_i\), for each \(i\).
From Fano type orbifolds to compact $Spin(7)$-manifolds

- Blow up $\Sigma$ in $V$, obtaining a Kähler orbifold $\tilde{V}$. Then the proper transform $\tilde{D}$ of $D$ is an anticanonical divisor $\tilde{D} \in | - K_{\tilde{V}}|$ with holomorphically trivial normal bundle. Also, $\rho$ lifts to an anticanonical involution of $\tilde{V}$ preserving $\tilde{D}$. 

Then $\tilde{V} \setminus \tilde{D}$ admits an asymptotically cylindrical Calabi–Yau structure $\omega, \theta$. (Tian–Yau, A.K., Haskins–Hein–Nordström) Moreover, $\tilde{\rho}^* (\omega) = -\omega$ and $\tilde{\rho}^* (\theta) = \bar{\theta}$, so the $Spin(7)$-structure $\Phi_{CY} = \frac{1}{2} \omega \wedge \omega + \text{Re} \theta$ is invariant under $\tilde{\rho}$.

The quotient $M_0 = (\tilde{V} \setminus \tilde{D}) / \tilde{\rho}$ can be made into a smooth, simply-connected 8-manifold $M$ by replacing neighbourhoods of the singular points by a certain ALE $Spin(7)$-spaces (given by Dominic Joyce). We obtain on $M$, by patching, an asymptotically cylindrical $Spin(7)$-structure $\tilde{\Phi}$ satisfying the hypotheses of existence theorem, so $\tilde{\Phi}$ can be perturbed into a torsion-free asymptotically cylindrical $Spin(7)$-structure $\Phi$.

For a pair $M, M'$ as above with isomorphic $(\tilde{D}, \rho)$ $(\tilde{D}', \rho')$ the gluing theorem applies to give a compact $Spin(7)$-manifold.
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- For a pair $M, M'$ as above with isomorphic $(D, \rho)$ $(D', \rho')$ the gluing theorem applies to give a compact $Spin(7)$-manifold.
Examples: asymptotically cylindrical Spin(7)-manifolds from complex orbifolds

A weighted projective space $V = \mathbb{C}P^4_{1,1,1,1,4}$ has just one singular point $p_0 = [0, 0, 0, 0, 1]$.
Let $D = \{z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 = 0\}$ and $\Sigma = D \cap D'$, where $D' = z_0^8 - z_1^8 + 2z_2^8 - 2z_3^8 + i z_4^2$, and
\[ \rho: [z_0, \ldots, z_4] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4]. \]
This configuration produces a Spin(7)-manifold $M_1$ with $b^4(M) = 839$, signature $(488, 200)$ and cross-section satisfying $b^3(Y) = 151$. 
Examples: asymptotically cylindrical Spin(7)-manifolds from complex orbifolds

A weighted projective space $V = \mathbb{C}P^4_{1,1,1,1,4}$ has just one singular point $p_0 = [0, 0, 0, 0, 1]$. Let $D = \{z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 = 0\}$ and $\Sigma = D \cap D'$, where $D' = z_0^8 - z_1^8 + 2z_2^8 - 2z_3^8 + i z_4^2$, and $\rho : [z_0, \ldots, z_4] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4]$. This configuration produces a Spin(7)-manifold $M_1$ with $b^4(M) = 839$, signature $(488, 200)$ and cross-section satisfying $b^3(Y) = 151$.

Another example: a hypersurface $V = \{f_8(z) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 + z_5^2 = 0\} \subset \mathbb{C}P^5_{1,1,1,1,4,4}$ has two singular points $p_{\pm} = [0, 0, 0, 0, i, \pm 1]$. Take $D = V \cap \{z^4 + z^5 = 0\}$ and $\Sigma = D \cap \{z_4 - z_5 = 0\}$ and $\rho : [z_0, \ldots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4]$. A Spin(7)-manifold $M_2$ we then obtain has $b^4(M_2) = 455$ and signature $(232, 72)$. The cross-section $Y$ of $M_2$ is isomorphic (as a $G_2$-manifold) to the cross section of $M_2$. 
Properties of the Spin(7)-manifolds $M_i$

The cross-section of $M_i$ is a 7-manifold $Y \cong (D \times S^1)/\rho$ with $\rho$ acting on $S^1$ by reflection. Then $b^1(Y) = 0$ and any torsion-free Spin(7)-structure $\Phi$ on $M_i$ induces a metric with holonomy Spin(7). The limit $G_2$-structure on $Y$ ‘at infinity’ induces a metric on $Y$ with holonomy $\mathbb{Z} \ltimes SU(3)$ (not $G_2$ as $\pi_1(Y)$ is infinite).
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These holonomy Spin(7) manifolds are simply-connected and have Betti numbers

$b^1(M_i) = b^1_c(M_i) = b^2(M_i) = b^2_c(M_i) = b^3(M_i) = b^3_c(M_i) = 0$

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By the inverse Hurewicz theorem, $M_i$ are 3-connected.
Compact Spin(7)-manifolds

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The two asymptotically cylindrical examples $M_1, M_2$ satisfy the topological hypotheses of the gluing theorem. The map $(y, e^{is}) \mapsto (y, e^{-is})$ of $D \times S^1$ descends to a $G_2$-compatible orientation-reversing isometry of $Y = (D \times S^1)/\rho$. We obtain, by application of the gluing theorem, three compact examples $M_{ij}$ of 8-manifolds with holonomy Spin(7). These are 3-connected and have $b_4(M_{ij}) = b_4(M_i) + b_4(M_j)$ and $b_4^+\pm(M_{ij}) = b_4^+\pm(M_i) + b_4^+\pm(M_j) + b_4^+(Y)$. The values of $(b_4^+, b_4^-)$ are $(615, 295), (871, 423), (1127, 551)$. The first two are deformation equivalent to Spin(7)-manifolds found by Dominic Joyce. The third example, with $b_4 = 1678$, is topologically new. Previously, Joyce (1999) and Clancy (2011) found 15 compact Spin(7)-manifolds with $b_1 = b_2 = b_3 = 0$, by a different method. Of these, 5 have $b_4 < 1678$ and 10 have $b_4 > 1678$. 
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An example of Spin(7)-manifolds constructed by Joyce from Calabi–Yau 4-orbifolds

A hypersurface $W = \{z_0^8 + \ldots + z_5^4 = 0\}$ of weighted degree 8 in weighted projective space $\mathbb{C}P^5_{1,1,1,1,2,2}$ is invariant under a holomorphic involution $\beta : [z_0, \ldots, z_5] \mapsto [iz_0, iz_1, iz_2, iz_3, z_4, z_5]$. 

Dominic Joyce proved that the desingularization $M$ of the orbifold $\tilde{W}/\sigma$ is a holonomy Spin(7)-manifold which has $b_1 = b_2 = b_3 = 0$ and the signature $(b_4^+ , b_4^-) = (615, 295)$. 
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A hypersurface $W = \{z_0^8 + \ldots + z_5^4 = 0\}$ of weighted degree 8 in weighted projective space $\mathbb{CP}_1,1,1,1,2,2$ is invariant under a holomorphic involution $\beta : [z_0, \ldots, z_5] \mapsto [iz_0, iz_1, iz_2, iz_3, z_4, z_5]$. The singular locus of $W/\beta$ consists of 4 isolated $\mathbb{Z}_4$-orbifold singularities and $W/\beta \cap \{z_4 = z_5 = 0\}$ a copy of the octic Fermat surface $\Sigma_8 \subset \mathbb{CP}^3$. Let $\tilde{W}$ be the blow-up of $\Sigma_8$ in $W/\beta$.

The anti-holomorphic involution $\sigma : [z_0, \ldots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4]$ of $W$ fixes only its isolated singular points, commutes with $\beta$ and acts freely on $\Sigma_8$. Thus $\sigma$ lifts to an anti-holomorphic involution of the blow-up $\tilde{W}$. 

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Connecting $M_2\# M_2$ with Joyce’s Spin(7)-manifold

Now, in our generalized connected sum $M_2\# M_2$ each $M_2$ arises from a degree 8 hypersurface $V$ in $\mathbb{C}P^{5}_{1,1,1,1,4,4}$. The natural map $\pi : [z_0, z_1, z_2, z_3, z_4, z_5] \in \mathbb{C}P^{5}_{1,1,1,1,2,2} \rightarrow [z_0, z_1, z_2, z_3, z_4^2, z_5^2] \in \mathbb{C}P^{5}_{1,1,1,1,4,4}$ induces a 2-to-1 branched covering

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with branch locus a surface $\Sigma_8 \subset V$. It lifts to a branched covering $\tilde{\pi} : \tilde{W} \rightarrow \tilde{V}$ between the respective blow-ups, with branch locus the exceptional divisor $E$ on $\tilde{V}$. Furthermore, the anti-holomorphic involution $\rho$ in the construction of $M_2$ corresponds to the $\sigma$ above.
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The path of Spin(7)-metrics on $M_2 \# M_2$ defined by the gluing theorem correspond to ‘pulling apart’ $M$ at the pre-image of $E$. An asymptotic path of Spin(7)-metrics is obtained by first making a compact Calabi–Yau 4-orbifold by gluing the asymptotically cylindrical pieces $\tilde{V} \setminus \tilde{D}$ and then applying Joyce’s construction. For ‘large’ values of neck length, the two Spin(7) 4-forms are in the same coordinate patch of the Spin(7)-moduli space.
Example of a different type of asymptotically cylindrical, holonomy Spin(7) manifold

We modify an example of Dominic Joyce’s first construction of Spin(7)-manifolds, resolving singularities of $T^8/\Gamma$. The group $\Gamma$ generated by

- $\alpha : (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8)$,
- $\beta : (x_1, \ldots, x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8)$,
- $\gamma : (x_1, \ldots, x_7) \mapsto (\frac{1}{2} - x_1, \frac{1}{2} - x_2, x_3, x_4, \frac{1}{2} - x_5, \frac{1}{2} - x_6, x_7, x_8)$,
- $\delta : (x_1, \ldots, x_7) \mapsto (-x_1, x_2, \frac{1}{2} - x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7, x_8)$.

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Consider $\Gamma$ acting on $(T^7 \times \mathbb{R}_{x_8})$, then the quotient is an $Spin(7)$-orbifold with a cylindrical end. The cross-section is an orbifold $T^7/\Gamma'$ for the subgroup $\Gamma' \subset \Gamma$ generated by $\beta, \gamma, \delta$ isomorphic to $\mathbb{Z}_2^3$ and fixing $x_8$. This 7-orbifold is isomorphic to one used by Dominic Joyce to construct a compact, holonomy $G_2$ manifold $X \to T^7/\Gamma'$ by resolving singularities.
It can be shown that a resolution $M \to (X \times \mathbb{R})/\langle \alpha \rangle$ has a structure of asymptotically cylindrical Spin(7)-manifold. The argument is similar in spirit to that in a joint paper with Johannes Nordström (2010) constructing asymptotically cylindrical, holonomy $G_2$ 7-manifolds (from quotients of $T^6 \times \mathbb{R}$).
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Geometrically this may be interpreted via ‘pulling apart’ a compact $Spin(7)$-manifold obtained from $T^8/\Gamma$ into two asymptotically cylindrical pieces each isomorphic to $M$. The cross-section at infinity in the present example is a simply-connected 7-manifold with holonomy equal to $G_2$.

Compact $Spin(7)$-manifolds may be obtained from pairs of asymptotically cylindrical pieces like $M$, with isomorphic cross-sections at infinity.