G_2-instantons, the heterotic G_2 system and generalized geometry

Jason D. Lotay

Oxford

13 May 2024

(Joint with A.A. da Silva Jr, M. García-Fernández and H.N. Sá Earp)

Motivation

Heterotic String Theory on $\mathbf{M}^{9,1} = B^{2,1} \times M^7 \rightsquigarrow$

- G_2-structure 3-form φ on M^7 with torsion H_φ
- connection θ on principal bundle $P \to M^7$, curvature F_θ

satisfying heterotic G_2 system: coupled PDE system for (φ, θ)

$$\Rightarrow \qquad F_\theta \wedge *\varphi = 0 \qquad (G_2\text{-instanton})$$

Question

What does the heterotic G_2 system mean geometrically?

Overview

Main results

Solution (φ, θ) to heterotic G_2 system $\rightsquigarrow$

- G_2-instanton on $TM \oplus \operatorname{ad} P$ (cf. De La Ossa–Larfors–Svanes)
- generalized Ricci-flat metric on $E = TM \oplus \operatorname{ad} P \oplus T^*M$
- solution to Killing spinor equations

Coupled G_2-instantons

Generalize heterotic G_2 system $\rightsquigarrow$

- (φ, H, θ): G_2-structure φ, 3-form H, connection θ
- examples (cf. Fino–Martín-Merchan–Raffero, Ivanov–Ivanov)

G_2-structures: torsion and connections

Recall:

- torsion forms of G_2-structure φ on M^7

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \quad d * \varphi = 4\tau_1 \wedge *\varphi + *\tau_2$$

- $\varphi \rightsquigarrow$ metric $g_\varphi \rightsquigarrow$ Levi-Civita connection ∇_φ
- φ torsion-free $\Rightarrow \mathrm{Hol}(g_\varphi) \subseteq G_2 \Rightarrow \nabla_\varphi$ G_2-instanton
- ∇ connection on $TM \rightsquigarrow$ torsion is section of $T^*M \otimes \Lambda^2 T^*M$

Lemma (Friedrich–Ivanov)

$\tau_2 = 0 \Leftrightarrow \exists\ \nabla^+_\varphi$ on TM with $\nabla^+_\varphi \varphi = 0$ and totally skew torsion

Moreover, ∇^+_φ exists $\Rightarrow$ unique and torsion H_φ

$$\nabla^+_\varphi = \nabla_\varphi + \frac{1}{2} g_\varphi^{-1} H_\varphi, \qquad H_\varphi = \frac{1}{6}\tau_0\varphi + *(\tau_1 \wedge \varphi) - \tau_3$$

Heterotic G_2 system

- φ G_2-structure on M^7 with $\tau_2 = 0$: $\mathrm{d} * \varphi = 4\tau_1 \wedge *\varphi$

  $\rightsquigarrow H_\varphi = \frac{1}{6}\tau_0\varphi + *(\tau_1 \wedge \varphi) - \tau_3$

- θ connection on principal K-bundle $P \to M$,

  $\langle\cdot,\cdot\rangle_P$ non-degenerate, symmetric, bilinear form on $\mathfrak{k}$

Definition

(φ, θ) *solution to heterotic* G_2 *system* $\Leftrightarrow$

- $7\tau_0 = 12\lambda \in \mathbb{R}, \quad 2\tau_1 = \mathrm{d}\mu, \quad \tau_2 = 0$ *(Torsion)*
- $F_\theta \wedge *\varphi = 0$ *(G_2-instanton)*
- $\mathrm{d}H_\varphi = \langle F_\theta \wedge F_\theta\rangle_P$ *(Anomaly)*

Observations

- Anomaly $\Rightarrow$

$$p_1(P) = \kappa[\langle F_\theta \wedge F_\theta \rangle_P] = \kappa[\mathrm{d}H_\varphi] = 0 \in H^4(M)$$

- $7\tau_0 = 12\lambda$, $\mathbf{M}^{9,1} = B^{2,1} \times M^7$

  $\Rightarrow B^{2,1}$ is AdS_3 or $\mathbb{R}^{2,1}$ with cosmological constant $-\lambda^2$

- If $TM \oplus E$ associated bundle to P can choose

$$\langle \cdot, \cdot \rangle_P = \alpha'(\mathrm{tr}_E - \mathrm{tr}_{TM})$$

  for $\alpha' > 0$ "small" $\rightsquigarrow$ usual formulation in physics literature

- "simple" solution: φ torsion-free, $\theta = (\nabla_\varphi, \nabla_\varphi)$ on $TM \oplus TM$

Connections and curvature

Goal: define G_2-instanton $\mathbf{D}_{(\varphi,\theta)}$ on $TM \oplus \operatorname{ad} P$ from (φ, θ)

φ G_2-structure with $\tau_2 = 0 \rightsquigarrow$

$$\nabla^{\pm}_{\varphi} = \nabla_{\varphi} \pm \frac{1}{2} g_{\varphi}^{-1} H_{\varphi}$$

Note: $R^{+}_{\varphi}(X, Y, Z, W) = R^{-}_{\varphi}(Z, W, X, Y) + \frac{1}{2}\mathrm{d}H_{\varphi}(X, Y, Z, W)$

θ connection $\rightsquigarrow$ $\mathbf{F}$ 1-form with values in $\mathrm{Hom}(TM, \operatorname{ad} P)$

$$(i_X \mathbf{F})(Y) = F_{\theta}(X, Y)$$

$\mathbf{F}^{\dagger}$ 1-form with values in $\mathrm{Hom}(\operatorname{ad} P, TM)$

$$(i_X \mathbf{F}^{\dagger})(u) = g_{\varphi}^{-1} \langle i_X F_{\theta}, u \rangle_P$$

Coupled G_2-instantons

Recap: $(\varphi, \theta) \rightsquigarrow \nabla^{\pm}_{\varphi}$ on TM and $\mathbf{F}, \mathbf{F}^{\dagger}$ Hom-valued 1-forms

Define connection $\mathbf{D}_{(\varphi,\theta)}$ on $TM \oplus \operatorname{ad} P$ by:

$$\mathbf{D}_{(\varphi,\theta)} = \begin{pmatrix} \nabla^{-}_{\varphi} & \mathbf{F}^{\dagger} \\ -\mathbf{F} & d_{\theta} \end{pmatrix}$$

Theorem (cf. De La Ossa–Larfors–Svanes)

(φ, θ) *solves heterotic* G_2 *system* $\Rightarrow$

$$F_{\mathbf{D}_{(\varphi,\theta)}} \wedge *\varphi = 0 \qquad (G_2\text{-instanton})$$

Question

Where does this connection on $TM \oplus \operatorname{ad} P$ *come from?*

Generalized geometry

Key object: $E = TM \oplus T^*M$ with non-degenerate pairing

$$\langle X+\xi, X+\xi\rangle_E = \xi(X)$$

and bracket

$$[X+\xi, Y+\eta]_E = [X,Y] + \mathcal{L}_X\eta - i_Y \mathrm{d}\xi$$

Note: closed 3-form $H \rightsquigarrow$ can add $H(X,Y,.)$ to $[X+\xi, Y+\eta]_E$

Observation: $E = TM \oplus \mathrm{ad}\, P \oplus T^*M$

$\rightsquigarrow$ modify pairing using $\langle\cdot,\cdot\rangle_P$ and bracket using F_θ and 3-form H satisfying

$$\mathrm{d}H = \langle F_\theta \wedge F_\theta\rangle_P$$

$\rightsquigarrow$ string algebroid (E, H, θ)

Connection revisited

$(E = TM \oplus \operatorname{ad} P \oplus T^*M, H, \theta)$ string algebroid

$\varphi \rightsquigarrow g_\varphi \rightsquigarrow$ splitting $E = V_+ \oplus V_-$

$$V_+ = \{X + g_\varphi(X,\cdot) \,:\, X \in TM\} \cong TM$$
$$V_- = \{X + u - g_\varphi(X,\cdot) \,:\, X \in TM, u \in \operatorname{ad} P\} \cong TM \oplus \operatorname{ad} P$$

Note: $\langle\cdot,\cdot\rangle_E|_{V_+}$ positive definite $\rightsquigarrow D_+^- : \Gamma(V_-) \to \Gamma(V_+^* \otimes V_-)$

$$\langle v_+, D_+^- v_- \rangle_E = \pi_{V_-}[v_+, v_-]_E$$

Lemma

Identify $V_+ \cong TM$, $V_- \cong TM \oplus \operatorname{ad} P$ *and choose* $H = H_\varphi \Rightarrow$

$$D_+^- = \mathbf{D}_{(\varphi,\theta)}$$

Generalized metrics

Recap: string algebroid (E, H_φ, θ) and $\varphi \rightsquigarrow$ splitting $E = V_+ \oplus V_-$ with $V_+ \cong TM$, $\langle\cdot,\cdot\rangle_E|_{V_+}$ positive definite

$\rightsquigarrow$ generalized metric $\mathbf{G}$:

- $\mathbf{G} : E \to E$ orthogonal for $\langle\cdot,\cdot\rangle_E$
- $\mathbf{G}^2 = \mathrm{id}$
- $V_\pm$ ± 1-eigenspace

$\rightsquigarrow$ generalized Ricci curvature $\mathrm{Ric}^+_{\mathbf{G}} \in \Gamma(V_- \otimes V_+)$

Theorem

(φ, θ) satisfies heterotic G_2 system $\Rightarrow$

$$\mathrm{Ric}^+_{\mathbf{G}} = 0$$

Killing spinors

Recall: φ G_2-structure $\leftrightarrow$ nowhere vanishing spinor η

$$\varphi(X,Y,Z) = (X\cdot Y\cdot Z\cdot\eta,\eta)$$

3-form $H \rightsquigarrow$

$$\nabla^t_H = \nabla_\varphi + \frac{t}{2} g_\varphi^{-1} H$$

$\rightsquigarrow$ $\nabla^\pm_H$ for $t=\pm 1$ and $\not{D}^{1/3}_H$ Dirac operator associated to $t = 1/3$

Theorem

(θ,φ) solution to heterotic G_2 system $\Rightarrow$ η satisfies for $H = H_\varphi$:

$$\nabla^+_\varphi\eta = 0, \quad F_\theta\cdot\eta = 0, \quad (\not{D}^{1/3}_\varphi - \mathrm{d}\mu)\cdot\eta = \lambda\eta$$

(Killing spinor equations with parameter λ)

Note: there is a converse result, where we do not assume $H = H_\varphi$

Coupled G_2-instantons revisited

φ G_2-structure on M, H 3-form on M, θ connection on $P \to M$ such that

$$\mathrm{d}H = \langle F_\theta \wedge F_\theta \rangle_P$$

$\rightsquigarrow$ connection $\mathbf{D}_{(\varphi,H,\theta)}$ on $TM \oplus \mathrm{ad}\, P$:

$$\mathbf{D}_{(\varphi,H,\theta)} = \begin{pmatrix} \nabla_H^- & \mathbf{F}^\dagger \\ -\mathbf{F} & d_\theta \end{pmatrix}$$

Definition

(φ, H, θ) *coupled G_2-instanton if*

$$F_{\mathbf{D}_{(\varphi,H,\theta)}} \wedge *\varphi = 0$$

Example: Hopf surface

- $\kappa \in \mathbb{R}^+ \setminus \{1\} \rightsquigarrow \mathbb{Z}$ acts on $(\mathbb{C}^2)^*$: $n \cdot (z_1, z_2) = \kappa^n(z_1, z_2)$
- $N^4 = (\mathbb{C}^2)^*/\mathbb{Z} \cong S^1 \times S^3$ diagonal Hopf surface
- SU(2)-structure (ω, Ψ):

$$\mathrm{d}\omega = \tau_1 \wedge \omega, \quad \mathrm{d}\Psi = \tau_1 \wedge \Psi, \quad \mathrm{d}\mathrm{d}^c\omega = 0,$$

  $\tau_1 \neq 0$ but $\mathrm{d}\tau_1 = 0 \rightsquigarrow$ twisted Calabi–Yau

- G_2-structure φ on $M^7 = T^3 \times N^4$ with

$$\tau_0 = 0, \quad \mathrm{d}\tau_1 = 0, \quad [\tau_1] \neq 0, \quad \tau_2 = 0, \quad H_\varphi = \mathrm{d}^c\omega \neq 0$$

- $\mathrm{d}H_\varphi = 0 \rightsquigarrow$ coupled G_2-instanton $(\varphi, H_\varphi, 0)$

Example: Calabi–Eckmann $S^3 \times S^3$

- $N^6 = ((\mathbb{C}^2)^* \times (\mathbb{C}^2)^*)/\mathbb{C}^* \cong S^3 \times S^3$
- Calabi–Eckmann SU(3)-structure (ω, Ψ)

$$\mathrm{d}\omega = \tau_1 \wedge \omega, \quad \mathrm{d}\Psi = \tau_1 \wedge \Psi, \quad \mathrm{d}\mathrm{d}^c\omega = 0$$

but $\mathrm{d}\tau_1 \neq 0$

- G_2-structure φ on $M^7 = S^1 \times N^6$ with

$$\tau_0 = 0, \quad \mathrm{d}\tau_1 \neq 0, \quad \tau_2 = 0, \quad H_\varphi = \mathrm{d}^c\omega \neq 0$$

- $\mathrm{d}H_\varphi = 0 \rightsquigarrow$ coupled G_2-instanton $(\varphi, H_\varphi, 0)$

Example S^7

- S^7 round $\rightsquigarrow$ nearly parallel G_2-structure φ:

$$\mathrm{d}\varphi = 4 * \varphi$$

- $\tau_0 = 4, \quad \tau_1 = 0, \quad \tau_2 = 0, \quad H_\varphi = \frac{2}{3}\varphi \quad \rightsquigarrow \quad \mathrm{d}H_\varphi \neq 0$
- $\theta = \nabla^+_\varphi$ G_2-instanton and

$$\mathrm{d}H_\varphi = -\alpha' \operatorname{tr} F_\theta \wedge F_\theta$$

  for $\alpha' > 0$

- $(\varphi, \nabla^+_\varphi)$ solves heterotic G_2 system

  $\rightsquigarrow$ coupled G_2-instanton $(\varphi, H_\varphi, \nabla^+_\varphi)$

Summary

Solutions to heterotic G_2 system:

G_2-structure φ on M^7 & connection θ on $P \to M \rightsquigarrow$

- G_2-instanton on $TM \oplus \text{ad } P$
- generalized Ricci-flat metric on string algebroid $(TM \oplus \text{ad } P \oplus T^*M, H_\varphi, \theta)$
- solution to Killing spinor equations with parameter λ

Introduced coupled G_2 instantons $(\varphi, H, \theta) \rightsquigarrow$ examples

- diagonal Hopf surface $S^1 \times S^3$ times T^3
- Calabi–Eckmann $S^3 \times S^3$ times S^1
- round S^7