Singularities in the G_2-Laplacian flow

Jason D. Lotay

Oxford

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Finite-time singularities

G_2-Laplacian flow on M^7 = gradient flow of Hitchin volume functional on $[\varphi]_+$

$$\frac{\partial\varphi}{\partial t} = \Delta_\varphi\varphi = \mathrm{d}\mathrm{d}^*_\varphi\varphi = \mathrm{d}\tau \quad \text{and} \quad \mathrm{d}\varphi = 0$$

Question: Why do finite-time singularities occur on compact M?

Ricci/mean curvature flow: shrinking spheres, shrinking cylinders

Fact: No known finite-time singularities on compact M!

- (L.–Wei) φ "close to torsion-free" $\Rightarrow$ no singularities
- (L.–Wei) $|\Delta_\varphi\varphi|$ must blow up and lower bound for blow-up rate of

$$\Lambda = \sup_M \sqrt{|\mathrm{Rm}|^2 + |\nabla\tau|^2} \quad \text{as } t \to T \rightsquigarrow$$

Type I: $\lim_{t\nearrow T} \sup_M (T-t)\Lambda < \infty$ **Type II:** $\lim_{t\nearrow T} \sup_M (T-t)\Lambda = \infty$

Solitons

Type I: $\lim_{t\nearrow T}\sup_M (T-t)\Lambda < \infty$ **Type II:** $\lim_{t\nearrow T}\sup_M (T-t)\Lambda = \infty$

Expect finite-time Type I/II singularities modelled by shrinking/steady solitons (maybe gradient $v = \nabla f$?)

$$\Delta_\varphi \varphi = \lambda\varphi + \mathrm{d}(v \lrcorner \varphi) \quad \lambda \leq 0$$

- **Note:** G_2-cones $\tau = 0$ are shrinking (and stationary) solitons
- (Lin) Compact $\lambda \leq 0 \Rightarrow$ torsion-free $\tau = 0$
- (L.–Wei) $v = 0$, $\lambda \leq 0 \Rightarrow$ torsion-free $\tau = 0$
- (G. Chen) $|\tau|^2$ (equivalently scalar curvature) "does not blow-up too quickly" as $t \to T \Rightarrow$ singularity modelled by complete G_2-manifold with maximal volume growth

Dimensional reduction: symmetries

"0 dimensions": homogeneous case

- (Lauret–Nicolini) complete shrinking solitons, not gradient
- many complete steady examples, not gradient

1 dimension: cohomogeneity one

- (Haskins–Nordström) $\Lambda^2_-(\mathcal{S}^4)$, $\Lambda^2_-(\mathbb{CP}^2)$
  - complete gradient shrinking solitons
  - complete gradient steady solitons on $\Lambda^2_-(\mathbb{CP}^2)$
- (Fowdar) $\mathbb{S}(\mathcal{S}^3)$, $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant $\rightsquigarrow$
  - 1-parameter family of cones with invariant closed G_2-structure
  - invariant smooth soliton $\Rightarrow$ torsion-free

6 dimensions: $\mathcal{S}^1$-invariance (c.f. Apostolov–Salamon)

- (Fowdar) complete gradient shrinking solitons from hyperkähler 4-manifolds
- Example: $\mathbb{R} \times N$ where N is T^2-bundle over T^4

Dimensional reduction: calibrated fibrations

3 dimensions: coassociative fibrations, spacelike mean curvature flow $B^3 \hookrightarrow H^2(F^4)$ (c.f. Donaldson)

- (Lambert–L. in progress) general coassociative fibrations $\rightsquigarrow$ study infinite-time singularity of Bryant/Fernandez example
- $F = K3 \rightsquigarrow$ suggests finite-time conical singularities, which are then preserved by flow
- (Lambert–L.–Moore in progress) G_2-Laplacian flow with conical singularities

4 dimensions: associative fibrations, hypersymplectic flow X^4, quadratic closed G_2-structures

- (Fine–Yao) scalar curvature bound (pointwise/integral) $\Rightarrow$ no finite-time singularity for hypersymplectic flow
- (Ball) many complete gradient steady solitons

Open problems

- Construct compact examples of G_2-Laplacian flows with finite-time singularities
- Are tangent flows (gradient?) shrinking solitons?
- Does $|\tau|^2$ (equivalently scalar curvature) stay bounded at finite-time singularity?
- Role of (bound on) Hitchin volume functional?
- Role of calibrated submanifolds?