Deformed $G_2$-instantons

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(Joint work with Goncalo Oliveira)
Overview

Deformed $G_2$-instantons

- special connections in 7 dimensions
- "mirror" to calibrated cycles $\sim$ enumerative invariants?
- critical points of Chern–Simons-type functional $\mathcal{F}$

Results

- first non-trivial examples
- examples detect different $G_2$-structures (including nearly parallel and isometric)
- deformation theory: obstructions and topology of moduli space
- relation to $\mathcal{F}$
Deformed $G_2$-instantons

$X^7$, $\varphi$ $G_2$-structure, $d \star \varphi = 0$, connection $A$ on bundle over $X$

**Definition** (J.-H. Lee–N.C. Leung)

A deformed $G_2$-instanton ($dG_2$) $\iff$ curvature $F_A$ satisfies

$$F_A \wedge \star \varphi + \frac{1}{6} F_A^3 = 0$$

$\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, $\varphi(u, v, w) = g_{\mathbb{R}^7}(u \times v, w) \rightsquigarrow$ “mirror calibrated cycles”

- $A = i(a_j(x)dx_j + u_k(x)dy_k)$ $dG_2$-instanton $\iff$
  Graph$(u)$ associative and $i(a_jdx_j)$ flat

- $A = i(v_j(y)dx_j + b_k(y)dy_k)$ $dG_2$-instanton $\iff$
  Graph$(v)$ coassociative and $i(b_kdy_k)$ anti-self-dual (ASD)

$\rightsquigarrow$ primary interest in $U(1)$-connections
Lower dimensions

4 dimensions: $\pi: X^7 \to Z^4$, $Z$ ASD Einstein, connection $B$ on $Z$

Lemma

$\pi^* B$ dG$_2$-instanton $\iff B$ ASD $\iff \pi^* B$ G$_2$-instanton

6 dimensions: $\pi: X^7 \to Y^6$, $Y$ Calabi–Yau 3-fold
$\omega$ Kähler form on $Y$, connection $B$ on $Y$

Lemma

$\pi^* B$ dG$_2$-instanton $\iff B$ deformed Hermitian–Yang–Mills

$F_B^{(0,2)} = 0$ and $\text{Im} \left((\omega + F_B)^3\right) = 0$.

● Conjecture: existence of dHYM $\iff$ stability condition
3-Sasakian 7-manifolds

**Definition**

\[(X^7, g^{ts}) \text{ 3-Sasakian } \iff (\mathbb{R}^+ \times X^7, g = dr^2 + r^2 g^{ts}) \text{ hyperkähler}\]

**Fact:** \(\exists\) infinitely many 3-Sasakian 7-manifolds

**Examples:** \(S^7,\) Aloff–Wallach \((SU(3) \times SU(2))/ (U(1) \times SU(2))\)

- \(V^3 \to X^7 \to Z^4,\) \(V = SU(2)/\Gamma,\) \(Z\) ASD Einstein
- left-invariant coframe \(\eta_1, \eta_2, \eta_3\) on \(V\)
- \(\omega_1, \omega_2, \omega_3\) orthogonal self-dual 2-forms on \(Z\) with length 2
- \(\sim\) two 1-parameter families of 3-forms for \(t > 0:\)
  \[
  \varphi_t^\pm = \pm t^3 \eta_1 \wedge \eta_2 \wedge \eta_3 - t \eta_1 \wedge \omega_1 - t \eta_2 \wedge \omega_2 \mp t \eta_3 \wedge \omega_3
  \]
- \(\varphi_t^\pm\) induces \(g_t = t^2 g_V + g_Z \Rightarrow \varphi_t^+\) and \(\varphi_t^-\) isometric
- \(\varphi^{ts} = \varphi_1^-\) nearly parallel inducing \(g^{ts}\)
- \(\varphi^{np} = \varphi_1^{+1/\sqrt{5}}\) nearly parallel inducing \(g^{np}\) “squashed” Einstein metric, cone has \(\text{Hol} = \text{Spin}(7)\)
Non-trivial examples

- $(X^7, g^{ts})$ 3-Sasakian, $\varphi_t^{\pm}$ inducing $g_t$

- $a = (a_1, a_2, a_3) \in \mathbb{R}^3 \leadsto A = i(a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3)$ connection on trivial line bundle, $r = |a|$ “distance to trivial connection”

Theorem (L.–Oliveira)

A $dG_2$-instanton on $(X^7, \varphi_t^{\pm}) \iff$

![Diagram showing $r$ vs $t$ with critical points and distances](image)
Theorem (L.–Oliveira)

- Non-trivial dG\(_2\)-instantons constructed for \(\varphi^{ts}\) and \(\varphi^{np}\) are obstructed
- Trivial dG\(_2\)-instanton unobstructed for \(\varphi^{ts}\) and \(\varphi^{np}\) but obstructed for \(\varphi^{\pm}_{1/\sqrt{2}}\)

**Proof:** (Kawai–Yamamoto) \(\leadsto\) unobstructed \(\iff\) rigid and isolated

At \(t = \frac{1}{\sqrt{2}}\): \(\exists\) infinitesimal deformation of trivial dG\(_2\)-instanton \(\Rightarrow\) obstructed (proof uses functional \(\mathcal{F}\))

**Corollary**

Moduli space of dG\(_2\)-instantons on trivial line bundle for \(\varphi^{ts}\) and \(\varphi^{np}\) contains at least two components of different dimensions
Chern–Simons-type functional

- $A_0$ reference connection on line bundle $L$ on $(X^7, \varphi)$, $d \ast \varphi = 0$
- $A$ connection on $L$
- $\leadsto A = A_0 + s(A - A_0)$ connection on $L$ over $X \times [0, 1]$
- $\leadsto$ curvature $F$

Proposition (Karigiannis–N.C. Leung)

A dG$_2$-instanton $\iff$ A critical point of functional

$$\mathcal{F}(A) = \int_{X \times [0,1]} e^{F + \ast \varphi}$$

Recall: our examples $A = i(a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3)$

$\leadsto$ restriction of $\mathcal{F}$ is function of two variables $x$ and $y$:

$$x = a_3 \quad \text{and} \quad y^2 = a_1^2 + a_2^2$$
$F$ for $\varphi^+_1$ and $\varphi^{ts} = \varphi^-_1$ with level sets

- trivial connection only critical point $\Rightarrow$ local maximum
- otherwise trivial connection saddle point
- non-trivial $dG_2$-instantons are local maxima
$\mathcal{F}$ for $\varphi^{np} = \varphi^+_{1/\sqrt{5}}$ and $\varphi^-_{1/\sqrt{5}}$ with level sets

- trivial connection is local minimum
- continuous families of $dG_2$-instantons are local maxima
- two isolated examples are saddle points
Questions

- non-trivial $dG_2$-instantons for holonomy $G_2$-manifolds?
- (adiabatic) limits of $dG_2$-instantons?
- dependence of moduli space on $G_2$-structure?
- “mirror count”?
- applications of $\mathcal{F}$ to compactness or deformation theory?
- Spin(7) version?