Special holonomy and branes: observations and themes

Observations: physics $\rightsquigarrow$ maths (geometry)

(a) Physical theories + certain compactifications
  $\rightsquigarrow$ requirement for parallel spinor & Ricci-flat metric
  $\rightsquigarrow$ special holonomy

Key example for this talk: holonomy G_2 in 7 dim.

Other physical theories/compactifications $\rightsquigarrow$ geometries with torsion

e.g. $AdS_3 \times Y^7$
  $AdS_2 \times Y^9$ $\rightsquigarrow$ GK geometry

Heterotic string $\rightsquigarrow$ Hull–Strominger system

6D – Kinsella et al, Fino et al

7D – De La Ossa et al, L.–Sa Earp.

(b) Physical theory admitting brane solutions
  $\rightsquigarrow$ calibrated geometry (+ special connection)

(M^m, η) $\quad d\eta = 0 \quad N^n \subseteq M^m$
  $\uparrow$ n-form
$\eta|_N \leq \mathrm{vol}_N$

A calibrated is: $\eta|_N = \mathrm{vol}_N$.

G_2 (M^7, φ) φ 3-form

3 dim A^3 calibrated by φ (+ flat $U(1)$-connection)
  associative

4 dim C^4 calibrated by $*\varphi$ (+ ASD $U(1)$-connection)
  (coassociative

7 dim C^7 calibrated by $*\varphi$ (+ ASD $U(1)$-connection)
coassociative

sometimes "mirror"

(deformed) G_2-instantons / G_2-monopoles

F curvature of connection: $F \wedge *\varphi = 0$
$$\Updownarrow$$
$$F \wedge \varphi = -*F$$

Physics ⤳ Maths: "counting" calibrated cycles

Example: • Joyce (possible wall-crossing phenomena)

• Doan, Haydys, Walpuski
relation to this counting problems &
Seiberg-Witten equations with multiple spinors.

(c) Supersymmetry in effective physical theory
⤳ structure on moduli space of geometric objects
(e.g. Kähler or hyperkähler structure)

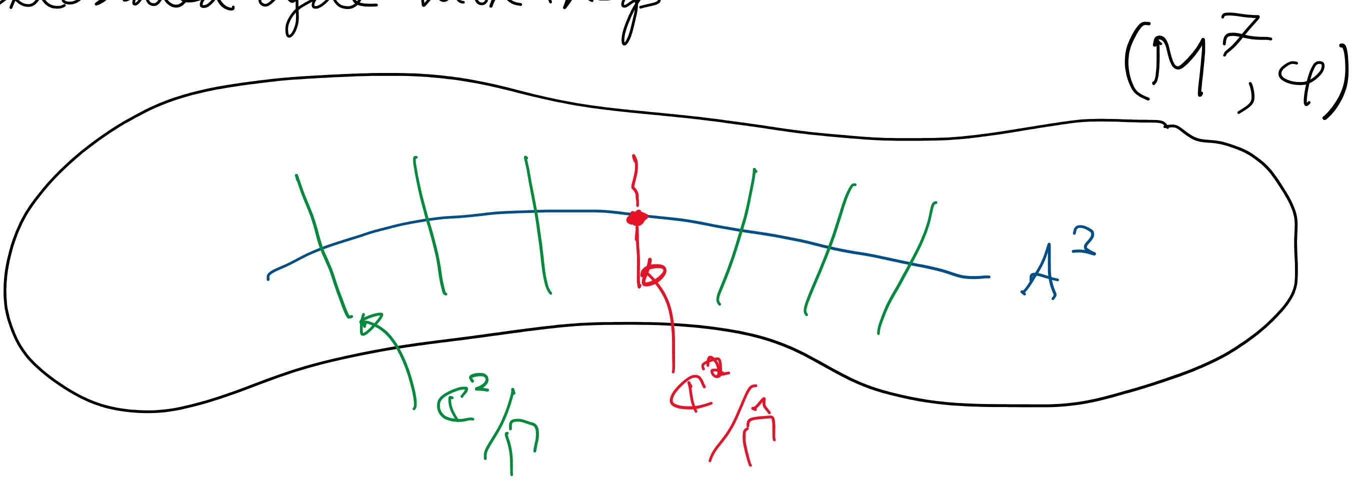
Examples: • Kähler structure on (complexified) moduli space
of G_2 spaces

• hyperkähler structure on moduli space $\mathcal{M}$ for Hitchin's
equations on Riem. surface Σ ⤳ Higgs bundles on Σ
metric behaviour near ends of $\mathcal{M}$ ⤳ generalized DT invariants.
[Gaiotto – Moore – Neitzke, Fredrickson – Mazzeo – ...]

Theme 1: Special holonomy spaces with singularities

[metric should blow up as we approach branes]

(a) Construct compact special holonomy spaces with conical (codim 7) sings. embedded in codim 4 calibrated cycle N with sings transverse to N



(b) Understand/classify/construct local models, i.e. noncompact, complete at ∞?

G_2 setting: Joyce–Karigiannis, Foscolo–Haskins–Nordström

A^3 associative $\subseteq (M^7, \varphi)$ G_2 orbifold

transverse to A we have $\mathbb{C}^2/\mathbb{Z}_2$

compact G_2 spaces with codim 7 sings, cone $C(S^3 \times S^3)$

We can resolve to get smooth compact G_2 space if $\exists$ harmonic 1-form λ on A with no zeros.

Both of these constructions are related to coassociative Kovalev–Lefschetz fibrations.

Kovalev–Lefschetz fibrations.

Karigiannis–L.: explicit description for $C(\mathbb{CP}^3)$

$$C(\mathbb{CP}^3) \supset \begin{cases} T^*S^2 & x \neq 0 \\ \mathbb{C}^2/\pm 1 & x = 0 \end{cases}$$
$$\downarrow$$
$$\mathbb{R}^3 \ni x$$

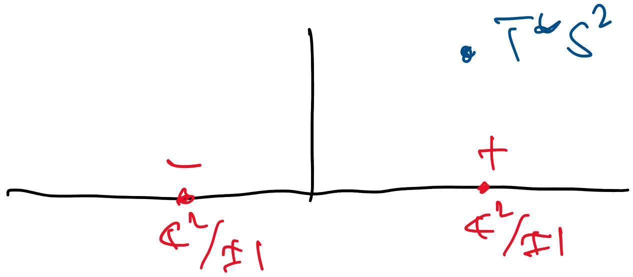
There is an extra $U(1)$-symmetry: $\mathbb{R}^3 \to H^2$

$\lambda = df \to$

• T^*S^2 on we can relate induced hypersymplectic str. to Eguchi–Hanson (squashed)

Q: Can we replace T^*S^2, $\mathbb{C}^2/\pm 1$ by other (singular) ALE gravitational instantons?
[cf. Acharya–Foscolo–Salamon, Acharya–Witten]

"Suggests": there could exist $SO(3)\times U(1)$-invariant G_2-space



Q: Local models from Higgs bundles?

Theme 2: 11D supergravity coupled to 7D SUSY Yang-Mills (Zhang.)

$$M^{11} = Y^7 \times \mathbb{C}^2/\Gamma$$
local

(a) A_n case: $\Gamma = \mathbb{Z}_{n+1}$ done by L. Anderson et al.

(a) A_n case: $\Gamma = \mathbb{Z}_{n+1}$ done by L. Anderson et al.
no Kähler potential on cx G_2 moduli space for
T^7/Γ_{T^7} & related to $\widetilde{T^7}/\Gamma_{T^7}$ smooth G_2 space

(b) Link to obstructions to existence of smooth G_2 space, i.e. zeros of harmonic 1-form λ.

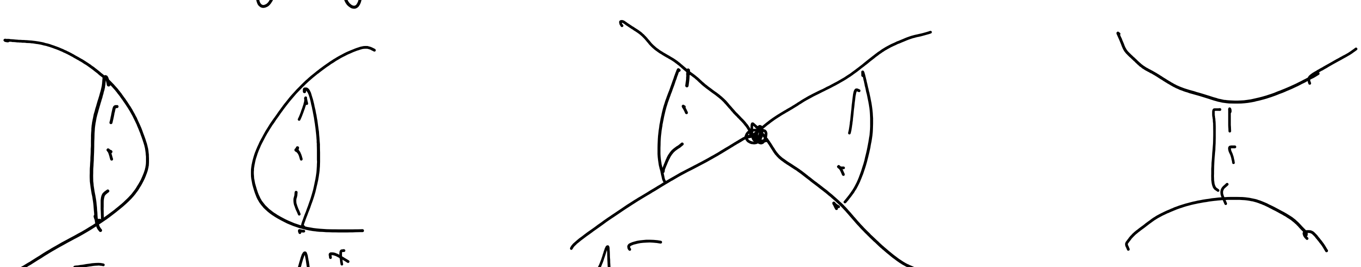
Theme 3 : Gauge theory by brane construction.

(a) Coulomb branch ⟷ "resolution" of branes, e.g. separating configs of n branes

(b) Higgs branch ⟷ "smoothing/deformation" of branes

(c) SW-theory models ⟷ intersecting branes.

Q: Is there a relation to Joyce–Nordström crossing of associative 3-folds?



G_2-Hull-Strominger. M^7

$d^*\varphi = 0$

$d\varphi = \tau_0 * \varphi + * \tau_3$, $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$

$d\varphi = \tau_0 * \varphi + * \tau_3$ $\quad$ $e_8 \wedge e_8$ a $\mathrm{Spin}(32)/\mathbb{Z}_2$

Connection Θ on TM $\quad$ $F_\Theta \wedge * \varphi = 0$

Connection A on E $\quad$ $F_A \wedge * \varphi = 0$

$$H = c_0 \tau_0 \varphi + c_1 \tau_3$$

$$dH = \frac{\alpha'}{4} \mathrm{tr}\left(F_\Theta^2 - F_A^2\right)$$

$C(\mathbb{CP}^3)$ has isometry group $Sp(2)$

$\parallel 2$

$\mathbb{H}^2/U(1)$