The G_2 Laplacian flow

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Outline

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- **Introduction**

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- **Foundational results**

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- **Recent progress**

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- **Recent progress**
- **Outlook**

Deforming G_2 structures

(M^7, φ_0) compact

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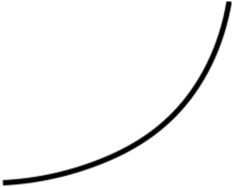

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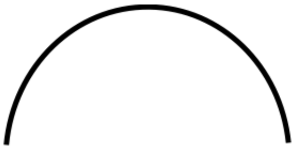

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- Uses Dirichlet problem for maximal spacelike submanifolds (Y. Li 2018)

Future directions

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Singularities

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