◊ Mirror symmetry for G_2 manifolds ◊

based on

- [1602.03521]
- [1701.05202]+[1706.xxxxx]
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Strings, T-duality & Mirror Symmetry

Type II String Theories and T-duality

Superstring theories on different backgrounds can give rise to equivalent physics: '*string dualities*'

'T-duality':

$$\text{IIA on } \mathbb{R}^{1,8} \times S^1 \quad \cong \quad \text{IIB on } \mathbb{R}^{1,8} \times S^1$$

where $r_{IIA} = r_{IIB}^{-1}$. It is crucial that strings can wind around S^1 !

For type II strings on T^2: T-duality along one S^1 swaps volume with complex structure.

This can be discussed at various levels:

- effective field theory
- worldsheet CFT
- full string theory $\supset$ CFT
- topological String Theory

String Theory on K3 Surfaces

CFT's have an (intrinsically defined) 'moduli space' = moduli space of background (metric plus B-field) on which string propagates

The CFT of type IIA string theory on a K3 surface S has a moduli space which is a Grassmanian of four-planes $\Sigma_4 \to \Gamma^{4,20} \otimes R$

$$O(\Gamma^{4,20}) \backslash O(4,20)/O(4) \times O(20)$$

'Geometric Interpretation':

$$\Gamma^{4,20} = \Gamma^{3,19} \oplus U_v = H^2(S,\mathbb{Z}) \oplus H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$

$$\text{pick}\, v_0 \in U_v :$$

$$\Sigma_4 = \begin{cases} \hat{\omega}_i & = \omega_i - (\omega_i \cdot B)\, v \\ \hat{B} & = B + v_0 + v(\omega_i^2 - B^2) \end{cases}$$

$$\omega_i \cdot \omega_j = \delta_{ij}$$

The ω_i give the hyper Kähler structure of S and B is the two-form B-field [Aspinwall, Morrison]

Mirror Symmetries

$$\Gamma^{4,20} = \Gamma^{3,19} \oplus U_v = H^2(S,\mathbb{Z}) \oplus H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$

$$\Sigma_4 = \begin{cases} \hat{\omega}_i & = \omega_i - (\omega_i \cdot B)\, v \\ \hat{B} & = B + v_0 + v(\omega_i^2 - B^2) \end{cases}$$

Isometries of $\Gamma^{4,20}$ correspond to identical physics; this involves

- Diffeomorphisms of S
- Mirror Maps: $U_v \leftrightarrow U_w$

Mirror maps can associate smooth with singular geometries !
Physics stays smooth: strings wrapped on vanishing $\mathbb{P}^1$s correspond to massive states (with mass $\sim B$), just as for finite volume !

Mirror maps arise from two T-dualities along a sLag fibration [Strominger, Yau, Zaslow; Gross] !
This is stronger than the equivalence at the level of the CFT and includes states originating from wrapped D-branes; note: we map IIA $\rightarrow$ IIA here

Calabi-Yau threefolds

On a suitably chosen pair of mirror Calabi-Yau threefolds X and $X^\vee$, the worldsheet CFTs associated to IIA and IIB are isomorphic. The Hodge numbers must satisfy

$$h^{1,1}(X) = h^{2,1}(X^\vee)$$
$$h^{2,1}(X) = h^{1,1}(X^\vee)$$

The CFT just sees the unordered set $\{h^{1,1}(X), h^{2,1}(X)\}$, but can't decide which one is which !

The exchange $h^{1,1} \leftrightarrow h^{2,1}$is realized via an automorphism of the symmetry group of the CFT.

This duality has amazing implications [Candelas, de la Ossa, Green, Parkes; ...]

Analyzing states from wrapped branes led to the conjecture of a sLag T^3 fibration for Calabi-Yau threefolds, mirror symmetry in the full string theory $\sim$ three T-dualities aling this T^3 fibre [Strominger, Yau, Zaslow].

how to find $X^\vee$

For some Calabi-Yau threefolds, the exact CFT is known at special point in moduli space, the '*Gepner point*', allowing to construct the mirror geometry [Greene, Plesser]

Example: the quintic:

$$X: \; x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0 \quad \text{in } \mathbb{P}^4$$

The mirror $X^\vee$ is found as a (resolution) of a quotient of X by $\mathbb{Z}_5^3$ acting with weights

$$(1, 0, 0, 0, 4)$$

$$(0, 1, 0, 0, 4)$$

$$(0, 0, 1, 0, 4)$$

Indeed $h^{1,1}(X) = h^{2,1}(X^\vee) = 1$ and $h^{2,1}(X) = h^{1,1}(X^\vee) = 101$.

Batyrev Mirrors

This has a beautiful generalization to toric hypersurfaces [Batyrev]. A pair of lattice polytopes (in lattices $\mathbf{M}$ and $\mathbf{N}$) satisfying

$$\langle \Delta, \Delta^\circ \rangle \geq -1$$

are called reflexive and determine a CY hypersurface as follows:

- Via an appropriate triangulation, Δ° defines a fa$\mathbf{N}$ Σ and a toric variety $\mathbb{P}_\Sigma$.
- Each lattice point ν_i on Δ° except the origin gives rise to a homogeneous coordinate x_i and a divisor D_i.
- Each lattice point m on Δ gives a $\mathbf{M}$onomial and the hypersurface equation is

$$\mathbf{X}_{(\boldsymbol{\Delta}, \boldsymbol{\Delta}^\circ)} : \quad \sum_{m \in \Delta} c_m \prod_{\nu_i \in \Delta^\circ} {x_i}^{\langle m, \nu_i \rangle + 1} = 0$$

Batyrev Mirrors

More abstract point of view: a polytope Δ defines

- a toric variety $\mathbb{P}_{\Sigma_n(\Delta)}$ via its normal fan $\Sigma_n(\Delta) = \Sigma_f(\Delta^\circ)$
- a line bundle $\mathcal{O}(\Delta)$; Δ is the Newton polytope of a generic section

$$\langle \Delta, \Delta^\circ \rangle \geq -1$$

Combinatorial formulas for Hodge numbers [Danilov,Khovanskii; Batyrev]:

$$h^{1,1}(X_{(\Delta,\Delta^\circ)}) = \ell(\Delta^\circ) - 5 - \sum_{\Theta^{\circ[3]}} \ell^*(\Theta^{\circ[3]}) + \sum_{\Theta^{\circ[2]}} \ell^*(\Theta^{[1]})\ell^*(\Theta^{\circ[2]})$$

$$h^{2,1}(X_{(\Delta,\Delta^\circ)}) = \ell(\Delta) - 5 - \sum_{\Theta^{[3]}} \ell^*(\Theta^{[3]}) + \sum_{\Theta^{[2]}} \ell^*(\Theta^{[2]})\ell^*(\Theta^{\circ[1]})$$

$$h^{1,1}(X_{(\Delta,\Delta^\circ)}) = h^{2,1}(X_{(\Delta^\circ,\Delta)})$$
$$h^{2,1}(X_{(\Delta,\Delta^\circ)}) = h^{1,1}(X_{(\Delta^\circ,\Delta)})$$

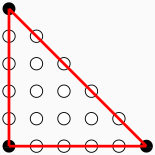
$$\boxed{\mathbf{X}_{(\boldsymbol{\Delta},\boldsymbol{\Delta}^\circ)} = \mathbf{X}^\vee_{(\boldsymbol{\Delta}^\circ,\boldsymbol{\Delta})}}$$

Examples

The Quintic

$$\Delta^\circ \sim \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta \sim \begin{pmatrix} -1 & -1 & -1 & -1 & 4 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \end{pmatrix}$$

For the mirror, $\mathbb{P}_{\Sigma_n(\Delta^\circ)} = \mathbb{P}_{\Sigma_f(\Delta)}$ is $\mathbb{P}^4/(\mathbb{Z}_5)^3$ as in [Greene, Plesser] !
e.g. two-dimensional faces of Δ° look like this:



Extra points $\sim$ refinement $\Sigma \to \Sigma_f$
$\sim$ resolution of orbifold singularities

Algebraic K3 Surfaces: $T(S) = U \oplus \tilde{T}(S)$; mirror symmetry swaps $N \leftrightarrow \tilde{T}$. This is realized by Batyrev's construction using 3D polytopes

Mirror Symmetry: the G_2 Story

We can put IIA or IIB string theory on a manifold of G_2 holonomy to compactify to $10 - 7 = 3$ dimensions.

- The CFT can only detect $b_2 + b_3$ but cannot discriminate [Shatashvili,Vafa].
- Arguments similar to SYZ imply coassociative T^4 fibration for G_2 manifolds. [Acharya]
- Discussed in detail for (few) examples of Joyce [Shatashvili,Vafa; Acharya; Gaberdiel,Kaste]
- And G_2 manifolds of the type $\left[\text{CY} \times S^1\right] / \mathbb{Z}_2$ [Eguchi,Sugawara; Roiban, Romelsberger, Walcher; Pioline, Blumenhagen, V.Braun]

$T^7/\mathbb{Z}_2^3$

Consider $T^7/\mathbb{Z}_2^3$ with action [Joyce]

$$
\begin{aligned}
\alpha : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \\
\beta : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7) \\
\gamma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\mapsto (\tfrac{1}{2} - x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7)
\end{aligned}
$$

Different smoothings $\sim$ 'discrete torsion' in the orbifold CFT [Joyce; Acharya; Gaberdiel,Kaste] give

$$
b_2(Y_l) = 8 + l \qquad b_3(Y_l) = 47 - l
$$

$T^7/\mathbb{Z}_2^3$

Different smoothings ∼ 'discrete torsion' in the orbifold CFT give

$$b_2(Y_l) = 8 + l \qquad b_3(Y_l) = 47 - l$$

Action of various 'mirror maps' ∼ T-dualities:

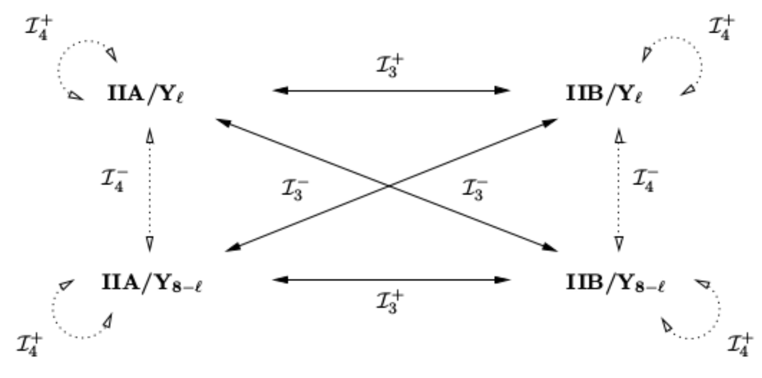


Figure taken from [Gaberdiel, Kaste]

◊

Twisted Connected Sums, Tops

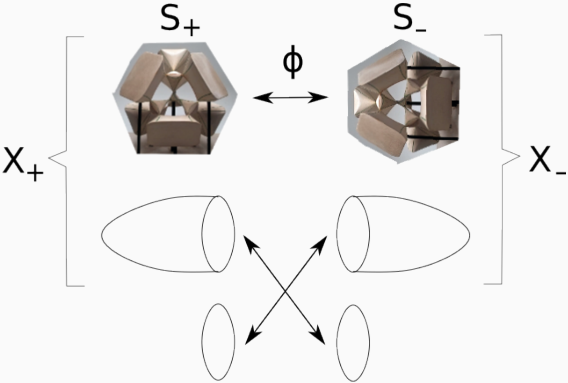
&

Mirror Symmetry

◊

Twisted Connected Sum (TCS) G_2 Manifolds

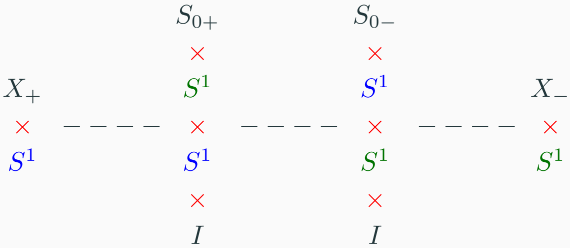
[Kovalev; Corti, Haskins, Nordström, Pacini]



Can we find 'mirror geometries' for a given TCS G_2 manifold ?

Is there an SYZ picture ?

TCS & SYZ



We can exploit the various SYZ fibrations to find a (coassociative) T^4 (at least in the Kovalev limit).

Four T-dualities correspond to

$$X_+ \to X_+^\vee \qquad X_- \to X_-^\vee \qquad S_{0\pm} \to S_{0\pm}^\vee$$

together with T-dualities along the various S^1 factors.
Can we give a construction and check $b_2 + b_3$ is invariant ?

Tops and Building Blocks

The acyl Calabi-Yau manifolds are $X_\pm = Z_\pm / S_{0\pm}$.
$Z_\pm$ are called 'building blocks' [Corti, Haskins, Nordström,Pacini]. In particular, they are $K3$ fibred and satisfy

$$c_1(Z) = [S_0]\,.$$

Can think of X as 'half' a compact K3 fibred Calabi-Yau threefold.
Such Calabi-Yau threefolds can be constructed from 4D reflexive polytopes Δ° with a 3D subpolytope $\Delta^\circ_F = \Delta^\circ \cap F$ cutting it into a pair of 'tops' $\Diamond^\circ_a, \Diamond^\circ_a$
[Candelas, Font; Klemm, Lerche, Mayr; Hosono, Lian, Yau; Avram,Kreuzer,Mandelberg,Skarke]

If $\pi_F(\Diamond) \supseteq \Delta^\circ_F$ we call $\Diamond$ 'projecting'. This implies:
$X_{(\Delta,\Delta^\circ)}$ is fibred by $X_{(\Delta_F,\Delta^\circ_F)}$ and it mirror $X_{(\Delta^\circ,\Delta)}$ is fibred by algebraic mirror family $X_{(\Delta^\circ_F,\Delta_F)}$ of K3 surfaces.

Tops and Building Blocks

There are stable degeneration limits into K3 fibred threefolds

$$X_{(\Delta,\Delta^\circ)} \to Z_{(\Diamond_a,\Diamond_a^\circ)} \cup Z_{(\Diamond_b,\Diamond_b^\circ)}$$
$$X_{(\Delta^\circ,\Delta)} \to Z_{(\Diamond_a^\circ,\Diamond_a)} \cup Z_{(\Diamond_b^\circ,\Diamond_b)}\,.$$

- $Z_{(\Diamond_a,\Diamond_a^\circ)}$ and $Z_{(\Diamond_b,\Diamond_b^\circ)}$ each capture half of the 'twisting' in the K3 fibration; Singular fibres of X (over pts p_i) are distributed into two halfs such that $\prod_{Z_{(\Diamond_a,\Diamond_a^\circ)}} \mu_i = \prod_{Z_{(\Diamond_b,\Diamond_b^\circ)}} \mu_i = 1$.
- $X = Z_{(\Diamond,\Diamond^\circ)}/S_0$ and $X^\vee = Z_{(\Diamond^\circ,\Diamond)}/S_0^\vee$ are an open mirror pair (at least in the SYZ sense);

Tops and Building Blocks

This motivates: a pair of dual projecting tops is a pair of lattice polytopes which satisfy

$$\langle \Diamond, \Diamond^\circ \rangle \geq -1$$

$$\langle \Diamond, \nu_0 \rangle \geq 0 \qquad \langle m_0, \Diamond^\circ \rangle \geq 0$$

with ν_0 and $m_0 \perp F$, $\langle m_0, \nu_0 \rangle = -1$ and $\pi_F(\Diamond) \supseteq \Delta^\circ \cap F$. In fact, starting from $\Diamond$, $Z_{(\Diamond,\Diamond^\circ)}$ is constructed as a hypersurface $\sim \mathcal{O}(\Diamond)$ in $\mathbb{P}_\Sigma$, $\Sigma \to \Sigma_n(\Diamond)$ as in Batyrev's construction:

$$Z_{(\Diamond,\Diamond^\circ)} : \sum_{m \in \Diamond} c_m x_e^{\langle \nu_0, m \rangle} \prod_{\nu_i \in \Diamond^\circ} x_i^{\langle \nu_i, m \rangle + 1} = 0 \quad \text{with } [x_e] \sim [S_0]$$

This allows a combinatorial computation of Hodge numbers [AB]:

$$h^{1,1} = -4 + \sum_{\Theta^{[3]} \in \Diamond} 1 + \sum_{\Theta^{[2]} \in \Diamond} \ell^*(\sigma_n(\Theta^{[2]})) + \sum_{\Theta^{[1]} \in \Diamond} (\ell^*(\Theta^{[1]}) + 1)(\ell^*(\sigma_n(\Theta^{[1]})))$$

$$h^{2,1} = \ell(\Diamond) - \ell(\Delta_F) + \sum_{\Theta^{[2]} < \Diamond} \ell^*(\Theta^{[2]}) \cdot \ell^*(\sigma_n(\Theta^{[2]})) - \sum_{\Theta^{[3]} < \Diamond} \ell^*(\Theta^{[3]})$$

comments

Blowing up the intersection of two anticanonical divisors $P = 0$ and $P' = 0$ in a semi-Fano toric threefold $\mathbb{P}_F$ with rays $\sim \Delta_F^\circ$ gives a threefold equation

$$z_1 P = z_2 P' \in \mathbb{P}^1 \times \mathbb{P}_F \,,$$

which precisely corresponds to a 'trivial' top, i.e. $\Diamond^\circ$ is the convex hull of

$$(\Delta_F^\circ, 0) \qquad\qquad (0,0,0,1)\,.$$

Note: the normal fan of $\Diamond$, which is the convex hull of

$$(\Delta_F, 0) \qquad\qquad (\Delta_F, -1)\,.$$

includes the ray $(0,0,0,-1)$ giving $\mathbb{P}^1 \times \mathbb{P}_F$ as the ambient space.

comments

Strenght of using polytopes is in resolving and analysing situations which degenerate K3 fibres. Can have large

$$|K| = |ker(H^2(Z,\mathbb{Z}) \to H^2(S,\mathbb{Z}))/[S]|$$

giving large $b_2(J)$ for resulting TCS G_2 manifolds.

Reducible K3 fibres can easily be found from Δ° [Davis et al; AB,Watari]; similar to theory by [Kulikov], but can have multiplicities > 1 for fibre components.

Caveat: for arbitrary Δ_F°, need to make sure moduli space of algebraic K3 surfaces is 'large enough' to tune in order to find gluings. Guaranteed at least for 1009 semi-Fano out of 4319 weak Fano options [Corti,Haskins,Nordström,Pacini].

Mirror Building Blocks

Inverting the roles of $\Diamond$ and $\Diamond^\circ$ gives us 'mirror building blocks' Z and $Z^\vee$ with

$$h^{2,1}(Z) = |K(Z^\vee)|$$

where $K = ker\left[H^2(Z,\mathbb{Z}) \to H^2(S_0,\mathbb{Z})\right]/[S]$.

Recall that (for orthogonal gluings):

$$b_2 + b_3 = 23 + 2\left[h^{2,1}(Z_+) + h^{2,1}(Z_-)\right] + 2\left[|K(Z_+)| + |K(Z_-)|\right] .$$

We are in business !

G2 Mirrors

In fact, we can do a lot better: From our discussion of SYZ, we should trade *both* building blocks for their mirrors and use the mirrors of $S_{0\pm}$.

Hence: for a G_2 manifold J constructed from

$$X_+ = Z_{(\Diamond_+,\Diamond_+^\circ)}/S_+ \quad \text{and} \quad X_- = Z_{(\Diamond_-,\Diamond_-^\circ)}/S_-$$

the mirror $J^\vee$ is found using

$$X_+^\vee = Z_{(\Diamond_+^\circ,\Diamond_+)}/S_+^\vee \quad \text{and} \quad X_-^\vee = Z_{(\Diamond_-^\circ,\Diamond_-}/S_-^\vee$$

The hyper Kähler rotation used to glue $S_{0\pm}^\vee$ is found from that for $S_{0\pm}$.

If $T_+ \cap T_- \supset U$:

$$H^2(J,\mathbb{Z}) \oplus H^4(J,\mathbb{Z}) = H^2(J^\vee,\mathbb{Z}) \oplus H^4(J^\vee,\mathbb{Z})$$
$$H^3(J,\mathbb{Z}) \oplus H^5(J,\mathbb{Z}) = H^3(J^\vee,\mathbb{Z}) \oplus H^5(J^\vee,\mathbb{Z})$$

for any matching.

G2 Mirrors

The asymptotic K3 fibres $S_{0\pm}$ of $Z_\pm$ are mapped to $S^\vee_{0\pm}$.
This exchanges $N \leftrightarrow \tilde{T}$ where $N = \mathrm{im}(\rho)$ and $T = U \oplus \tilde{T} = N^\perp \in \Gamma^{3,19}$

Depending on the choice of ω and Ω, $S^\vee_{0\pm}$ can have ADE singularities (so TCS construction does not apply ... yet ? T^7/Γ gives singular examples of this type !)
In physics, we should include a B-field $B_\pm$ in $N_+ \cap N_-$; matching condition becomes

$$\begin{aligned} \omega_+ &= \mathrm{Re}(\Omega_-) \qquad & \mathrm{Im}(\Omega_+) &= -\,\mathrm{Im}(\Omega_-) \\ \omega_- &= \mathrm{Re}(\Omega_+) \qquad & B_+ &= B_- \end{aligned}$$

preserved by mirror map on K3 surfaces with SYZ fibre calibrated by $\mathrm{Im}(\Omega_+)$ and $-\mathrm{Im}(\Omega_-)$.

Singularities come from -2 curves in $N_+ \cap N_-$ which will receive a stringy volume from B.

example

Consider two identical K3 fibred building blocks $Z_\pm = Z$ with $N_+ = N_- = U \oplus (-E_8)$

$$
\begin{array}{llll}
h^{1,1}(Z) = 11 & h^{2,1}(Z) = 240 & |N(Z)| = 10 & |K(Z)| = 0 \\
h^{1,1}(Z^\vee) = 251 & h^{2,1}(Z^\vee_+) = 0 & |N(Z^\vee)| = 10 & |K(Z^\vee)| = 240
\end{array}
$$

We can glue such that $N_+ \cap N_- = 0 = \tilde{T}_+ \cap \tilde{T}_-$ to find smooth mirrors with

$$b_2 + b_3 = 983$$

$$
\begin{array}{ll}
b_2(J) = 0 & b_2(J^\vee) = 480 \\
b_3(J) = 983 & b_3(J^\vee) = 503
\end{array}
$$

However, perpendicularly gluing to building blocks with quartic K3 fibre $N_\pm = (4)$ gives

$$N^\vee_+ \cap N^\vee_- \subset (-E_8)^{\oplus 2}$$

Comparing to an example of Joyce

Consider again the smoothings Y_l of the orbifold $T^7/\mathbb{Z}_2^3$.
They can be realized as a TCS [Nordström, Kovalev] with building blocks

$$((T^4/\mathbb{Z}_2) \times S^1 \times \mathbb{R}_+)/\mathbb{Z}_2$$

We can now compare our construction to the mirror maps $I_3^{\pm}$ and $I_4^{\pm}$ of [Gaberdiel,Kaste].

Our mirror map is $I_4^+ : Y_l \to Y_l$ which is trivial in this case and acts as $Z_+ \leftrightarrow Z_-$.

As an aside : the discrete torsion in the CFT description precisely corresponds to $H^3(Y_l, \mathbb{Z}) = \mathbb{Z}_2^{8-l}$.

The Curious Case of three T-dualities

What about other mirror maps in TCS picture ?

$I_3^- : Y_l \to Y_{8-l}$ corresponds to performing three T-dualities along T^3 SYZ fibre of Z_+ and elliptic fibre of Z_- !

$$
\begin{array}{ccccccc}
 & & S_{0+} & & S_{0-} & & \\
 & & \times & & \times & & \\
X_+ & & S^1 & & S^1 & & X_- \\
\times & ----- & \times & ----- & \times & ----- & \times \\
S^1 & & S^1 & & S^1 & & S^1 \\
 & & \times & & \times & & \\
 & & I & & I & &
\end{array}
$$

Hence: for a TCS J built from Z_+ and Z_-, we construct $J^{\vee 3}$ from $Z_+^\vee$ and Z_-. If $T_+ \cap N_- \supset U$:

$$H^\bullet(J, \mathbb{Z}) = H^\bullet(J^{\vee 3}, \mathbb{Z})$$

For orthogonal gluings $b_2 + b_3$ trivially invariant as

$$b_2 + b_3 = 23 + 2\left[h^{2,1}(Z_+) + h^{2,1}(Z_-)\right] + 2\left[|K(Z_+)| + |K(Z_-)|\right] .$$

Summary

We have motivated our construction by a ‘physics picture’ of SYZ fibrations and their generalization to G_2 as proposed by [Acharya], exploiting the structure of TCS G_2 manifolds in the Kovalev limit.

Our construction stands on its own and gives many pairs of G_2 manifolds with the same $b_2 + b_3$, as expected for G_2 mirrors from a CFT analysis [Shatashvili, Vafa]

Interestingly, mirrors can be singular; TCS G_2 manifolds are (real) K3 fibrations over S^3 and every K3 fibre has an ADE singularity. Mathematically rigorous treatment of such solutions ?

Is ‘mirror symmetry’ the wrong name because we have more than a $\mathbb{Z}_2$? Are all G_2 manifolds with the same $b_2 + b_3$ (or $H^\bullet(J, \mathbb{Z})$) dual as suggested by Shatashvili,Vafa ?

Finally

$\rightarrow$ **Want to exploit physics to learn about** G_2 $\leftarrow$

Thank You !

comments

Need better understanding of CFT picture ! B-field and geometric singularities ?
We know the CFT for some examples of the type $\left[\text{CY} \times S^1\right]/\mathbb{Z}_2$, compare to TCS examples ?

Categories of D-branes ?

TCS G_2's vs. CY $\times S^1/\mathbb{Z}_2$; Gepner models ? S^1 fibrations and M-Theory - IIA duality ?

Topological G_2 strings [de Boer,Naqvi,Shomer] ?