G_2 manifolds and mirror symmetry

Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics – First Annual Meeting, New York, 9/14/2017

Andreas Braun
University of Oxford

based on

- [1602.03521]
- [1701.05202] + [1710.xxxxx] with Michele del Zotto (Stony Brook)
- [1708.07215] with Sakura Schäfer-Nameki (Oxford)

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consequences:

- moduli spaces agree: $\mathcal{M}_1 = \mathcal{M}_2$, i.e. massless states agree
- massive states agree
- dynamics and observables agree
- **However: often need to work in specific limits/truncations to have tractable situation**

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The 'backgrounds' $F_{1/2}$ often include more data than just geometry !
The geometric data is often that of a manifold of special holonomy.

Dualities can be related by fibrations (with singular fibres):

$$F_{1/2} \to X_{1/2} \to B_m$$

theory $\sharp 1$ on background $X_1 \times \mathbb{R}^{1,n-m}$

$\sim$

theory $\sharp 2$ on background $X_2 \times \mathbb{R}^{1,k-m}$

$\Bigg\}$ the same physics

Example: Mirror Symmetry is T-duality

$$\left.\begin{array}{c}\text{type IIA string on } S^1_r \times \mathbb{R}^{1,8} \\ \sim \\ \text{type IIB string on } S^1_{1/r} \times \mathbb{R}^{1,8}\end{array}\right\} \text{` T-duality '}$$

Example: Mirror Symmetry is T-duality

$$\left.\begin{array}{c} \text{type IIA string on } T^3 \times \mathbb{R}^{1,6} \\ \sim \\ \text{type IIB string on } T^3_\vee \times \mathbb{R}^{1,6} \end{array}\right\} \text{3 T-dualities}$$

extra data: B-field !

Example: Mirror Symmetry is T-duality

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from SYZ fibrations [Strominger,Yau,Zaslow] of Calabi-Yau mirror manifolds X and $X^\vee$:

$$\begin{array}{ccccc} T^3 & \to & X & \to & S^3 \\ T^3_\vee & \to & X^\vee & \to & S^3\,. \end{array}$$

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$$\mathcal{M}_{IIA} = \mathcal{M}_{IIB} = \mathcal{M}_V \times \mathcal{M}_H$$

where

$$\mathcal{M}_V = \text{complexified Kähler moduli of } X = \text{complex structure of } X^\vee$$

$$\mathcal{M}_H = \text{complex structure of } X = \text{complexified Kähler moduli of } X^\vee$$

comparing more refined properties leads e.g. to:

- Gromov-Witten invariants of X from periods of the mirror $X^\vee$
- Homological mirror symmetry

Story 1: Mirror Symmetry and G_2 manifolds

For (co)-associative fibrations

$$T^3 \to J \to M_4$$

or

$$T^4 \to J \to M_3$$

on a G_2 manifold J, can use the same logic leading to [Acharya]:

$$\left.\begin{array}{c}\text{type IIA string on } J \times \mathbb{R}^{1,2} \\ \sim \\ \text{type IIB string on } J^\wedge \times \mathbb{R}^{1,2}\end{array}\right\} \text{exploiting } T^3 \text{ fibration}$$

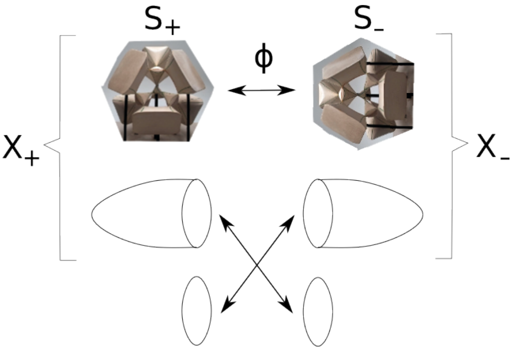
or

$$\left.\begin{array}{c}\text{type IIA/B string on } J \times \mathbb{R}^{1,2} \\ \sim \\ \text{type IIA/B string on } J^\vee \times \mathbb{R}^{1,2}\end{array}\right\} \text{exploiting } T^4 \text{ fibration}$$

a coarse invariant is [Shatashvili, Vafa]

$$\boxed{\dim \mathcal{M}_{\mathbb{C}} = b_2 + b_3}$$

Mirror Symmetry for TCS



G_2 twisted connected sums [Kovalev; Corti,Haskins,Nordström, Pacini] are defined by the data:

$$\left.\begin{array}{c}\text{acyl Calabi-Yau threefolds } X_\pm = Z_\pm \setminus S_{0\pm} \\ + \\ \text{a matching } \phi : S_{0+} \to S_{0-}\end{array}\right\} \text{a } G_2 \text{ manifold } J$$

Mirror Symmetry for TCS: T^4

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expectation: there exist coassocitive T^4 fibrations restricting to the SYZ fibrations of $X_\pm$ (see [Donaldson]). The corresponding mirror $J^\vee$ is constructed from

$$\left.\begin{array}{c}\text{acyl Calabi-Yau threefolds } X^\vee_\pm = Z^\vee_\pm \setminus S^\vee_{0\pm} \\ + \\ \text{a matching } \phi^\vee : S^\vee_{0+} \to S^\vee_{0-}\end{array}\right\} \text{the } G_2 \text{ manifold } J^\vee$$

Found an explicit construction for $Z_\pm$ = toric hypersurface [AB,del Zotto]. In general:

- $b_2(J) + b_3(J) = b_2(J^\vee) + b_3(J^\vee)$
- $H^\bullet(J, \mathbb{Z}) = H^\bullet(J^\vee, \mathbb{Z})$

Mirror Symmetry for TCS: T^3

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expectation: if X_+ (and S_+) is elliptic, there exist an associative T^3 fibration restricting to the SYZ fibration of X_- and the elliptic fibration on X_+. The corresponding mirror $J^\wedge$ is constructed from

$$\left.\begin{array}{c}\text{acyl Calabi-Yau threefolds } X_-^\vee \text{ and } X_+ \\ + \\ \text{a matching } \phi^\wedge : S_{0+}^\vee \to S_{0-}\end{array}\right\} \text{the } G_2 \text{ manifold } J^\wedge$$

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For [Joyce] orbifolds, our mirror map is consistent with CFT realization [Acharya; Gaberdiel,Kaste; AB, del Zotto] .

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This is similar to mirror symmetry for $K3$ surfaces; every $K3$ fibre of $X_\pm$ and hence of J

$$S \to J \to S^3$$

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has ADE singularities. These are perfectly well-behaved due to the B-field, so we

conjecture: There exist metrics of G_2 holonomy on twisted connected sum G_2 manifolds J in which every K3 fibre of $X_\pm$ (and J) has ADE singularities.

Story 2: M-Theory/Heterotic Duality

Starting point [Witten] (see also talks by [Acharya; Morrison]):

$$\left.\begin{array}{c}\text{M-Theory on } K3\times\mathbb{R}^{1,6}\\ \sim \\ \text{Heterotic String Theory on } T^3\times\mathbb{R}^{1,6}\\ \text{with a flat } E_8\times E_8 \text{ bundle } W \text{ on } T^3\end{array}\right\}\ \text{the same physics}$$

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Let's fiber this data over S^3 (see [Donaldson]):

$$\left.\begin{array}{c}\text{M-Theory on } J\times\mathbb{R}^{1,3}\\ \sim\\ \text{Heterotic String Theory on } X\times\mathbb{R}^{1,3}\\ \text{with a holomorphic } E_8\times E_8 \text{ bundle } V \text{ on } X\end{array}\right\}\ \text{the same physics}$$

$T^3 \to X \to S^3$ with X a Calabi-Yau manifold

$K3 \to J \to S^3$ with J a G_2 manifold

expectation: $\exists$ coassocitive $K3$ fibration for J a TCS G_2 manifold

Moduli space

The moduli spaces are

$$\mathcal{M}_M = \left\{ \int_L C_3 + i\Phi_3 \mid L \in H^3(J) \right\} \rightarrow \text{dim}_{\mathbb{C}} \mathcal{M}_M = b_3(J)$$

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One can work out X and V explicitly for TCS G_2 manifolds, assuming $S_\pm$ are elliptic:

$$\begin{aligned} J &= \{X_+ \times S^1\} & \cup & \quad \{X_- \times S^1\} \\ X &= \{(dP_9 \setminus T^2) \times T^2\} & \cup & \quad \{(dP_9 \setminus T^2) \times T^2\} \\ V &= \{V_+\} & \cup & \quad \{V_-\} \end{aligned}$$

X is the 'Schoen' Calabi-Yau threefold with $h^{1,1} = h^{2,1} = 19$, [Kovalev,AB,Schäfer-Nameki].

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We were able to prove:

$$\text{dim}\mathcal{M}_{het} = \text{dim}\mathcal{M}_M$$

Lessons from heterotic/M-Theory duality

- If the structure group G of V has non-abelian commutant $G^{\perp}$ in $E_8 \times E_8$, there are ADE singularities in every K3 fibre of J.

> **conjecture** (again) : There exist metrics of G_2 holonomy on twisted connected sum G_2 manifolds J in which every K3 fibre of $X_{\pm}$ has ADE singularities.

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- **Question:** There are quantum corrections $\sim$ couting curves on X on heterotic side which sum to a E_8 Θ-function [Donagi,Grassi,Witten] ... what do these become on J ? Do they count something interesting (such as associatives) there ? What is the relation to recent work of [Joyce] ?

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$\rightarrow$ **Thank you !** $\leftarrow$