Hermitian Yang Mills connections on reflexive sheaves.

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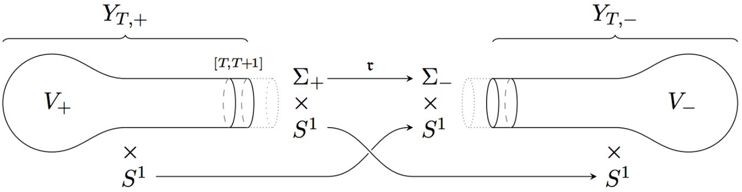
June 9, 2017

Joint work with T. Walpuski and Henrique Sá Earp

Introduction

- The main motivation for this project is the construction of G_2 instantons.
- Idea is based off of the twisted connected sum construction, pioneered by Kovalev and later extended by Corti-Haskins-Nordström-Pacini.
- One glues together two asymptotically cylindrical Calabi-Yau 3-folds, each equipped with a trivial S^1 bundle, in a prescribed fashion.



Introduction

Can this construction yield a G_2 instanton? The main idea is to take a bundle with a Hermitian-Yang-Mills connection on each piece and glue them together.

The starting point for our work is the result of Sá Earp:

Theorem (Sá Earp)

Let $(E, \bar{\partial})$ be a holomorphic bundle over an asymptotically cylindrical Calabi-Yau 3-fold. If E is asymptotic to a degree zero stable bundle along the cylindrical end, then there exists a metric H on E satisfying the Hermitian-Yang-Mills equations.

Introduction

From here, Sá Earp-Walpuski outlined a program and developed the perturbation theory needed to construct a G_2 instanton from the twisted connected sum.

The perturbation theory places restrictions on the cohomology of the bundle.

Reflexive sheaves are more abundant and bundles, can they be used to construct new examples of singular G_2 instantons?

Introduction

Goals of this talk:

- Construct a Hermitian-Yang-Mills connection on a reflexive sheaf over an asymptotically cylindrical Kähler manifold
- Show the solution is exponentially asymptotically translation invariant
- Analyze the connection at singular points under certain assumptions on the complex structure

Background

Let X be a complex manifold.

Fix a Kähler metric g on $T^{1,0}X$, with corresponding Kähler form

$$\omega = \frac{i}{2}\, g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k.$$

Let $(E, \bar{\partial})$ be a holomorphic vector bundle over X.

Given a Hermitian metric H on E, one can define the associated Chern connection d_H, compatible with H and the holomorphic structure $\bar{\partial}$. We write $d_H = \partial_H + \bar{\partial}$.

Background

The assumption that $\bar{\partial}$ is a holomorphic structure ($\bar{\partial}^2 = 0$) and metric compatibility imply the curvature F_H of d_H is a $(1,1)$ form.

- Locally, in a holomoprhic frame:

$$F_H = \bar{\partial}(H^{-1}\partial H).$$

Let Λ denote the adjoint in g of wedging with the Kähler form ω. For any $(1,1)$ form α, $i\Lambda\alpha = g^{j\bar{k}}\alpha_{j\bar{k}}$.

Definition

A metric H satisfying

$$i\Lambda F_H = \mu \, Id$$

is called a *Hermitian-Yang-Mills* metric (*Hermitian-Einstein*).

Background

If H solves the Hermitian-Yang-Mills equations, then d_H solves the Yang-Mills equations.

The Kähler identities imply

$$d_H^* F_H = -\partial_H(i\Lambda F_H) + \bar{\partial}(i\Lambda F_H) = 0.$$

d_H is a critical point of the Yang-Mills functional

$$YM(d_A) = ||F_A||^2_{L^2(X)}.$$

Question: When does $(E, \bar{\partial})$ admit a Hermitian-Yang-Mills metric?

Previous results

If X is compact, solution is given by the following beautiful result:

Theorem (Donaldson, Uhlenbeck-Yau)

E admits a Hermitian-Yang-Mills metric if and only if it is stable in the sense of Mumford-Takemoto:

$$\frac{deg(\mathcal{F})}{rk(\mathcal{F})} < \frac{deg(E)}{rk(E)}$$

for all proper, reflexive subsheaves $\mathcal{F} \subset E$.

The degree is given by

$$deg(E) = i \int_X tr(F_H) \wedge \omega^{n-1},$$

and is independent of a choice of metric.

Previous results

Many interesting generalizations of the above Theorem. Most notably for us, the result was extended to the case where E is a reflexive sheaf by Bando-Siu.

Here, metrics are only defined away from the singular set (of complex codimension at least 3), where E is a holomorphic bundle.

The solution satisfies

$$||i\Lambda F_H||_{L^\infty(X)} \leq C \qquad \text{and} \qquad ||F_H||_{L^2(X)} \leq C.$$

Such metrics are called admissible.

Previous results

Donaldson uses the Yang-Mills flow (and later Simpson and Bando-Siu), while Uhlenbeck-Yau employ the method of continuity.

Let H_0 be a fixed metric and H a metric along either method. Right away one sees $|i\Lambda F_H|$ is controlled.

The key estimate is a uniform C^0 bound for $e^s = H_0^{-1}H$. This is where stability comes into play in the compact setting.

Previous results

Question: What happens if X is complete, non-compact?

In some cases you can make the structure of X work for you.

Theorem (Ni-Ren)

If X admits a spectral gap ($\lambda_1(X) > 0$), and E admits a metric H_0 such that $|i\Lambda F_{H_0} - \mu Id| \in L^p(X)$ for some $p > 1$, then there exists a metric H such that

$$i\Lambda F_H = \mu\, Id.$$

This uses an argument similar to Donaldson's solution of the Dirichlet problem, since we have along the flow

$$\left(\frac{d}{dt} + \Delta\right) |i\Lambda F_H - \mu\, Id|^2 \leq 0.$$

Previous results

(In this talk I use the Geometer's Laplacian, $\Delta = d^*d$ on functions)

Ni also showed that the same conclusion holds, for example, if X satisfies a L^2 Sobolev inequality and $p \in [1, \frac{n}{2})$, or if it is non-parabolic (i.e., admits a positive Greens function) and p $= 1$.

In this case, for a fixed initial metric H_0, one can solve

$$\Delta u = |i\Lambda F_{H_0}|,$$

and use u as a barrier to control s, since

$$\Delta \log \operatorname{tr}(e^s) \leq 4|i\Lambda F_{H_0}|.$$

For asymptotically cylindrical manifolds we have linear volume growth, so the above results can not be used.

Main result

Building off of Sá Earp's result, we prove the following

Theorem (J. - Walpuski)

Let V be an asymptotically cylindrical Kähler manifold with asymptotic cross-section D. Let E_D be a stable vector bundle over D, and E a reflexive sheaf asymptotic to E_D. There exists an asymptotically translation-invariant Hermitian metric H on E which satisfies the projective Hermitian Yang-Mills (PHYM) equation

$$K_H := i\Lambda F_H - \frac{tr(i\Lambda F_H)}{rk(E)} Id = 0.$$

Furthermore $|F_H| \in L^2_{loc}(V)$.

Rmk: Every PHYM metric can be converted to a HYM metric via a conformal change. However, this metric will typically not be asymptotically translation invariant.

ACyl Kähler manifolds

Definition: Let (D, g_D, I_D) be a compact Kähler manifold. A Kähler manifold (V, g, I) is called *asymptotically cylindrical* (ACyl) with asymptotic cross-section (D, I_g, I_D) if there exists a constant $\delta_V > 0$, a compact subset $K \subset V$, and a diffeomorphism $\pi : V \setminus K \to (0, \infty) \times S^1 \times D$, such that

$$|\nabla^k(\pi_* g - g_\infty)| + |\nabla^k(\pi_* I - I_\infty)| = O(e^{-\delta_V \ell})$$

for all $k \geq 0$. Here (ℓ, θ) are the canonical coordinates on $(0, \infty) \times S^1$ and

$$g_\infty := d\ell^2 \oplus d\theta^2 \oplus g_D \qquad I_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_D$$

ACyl Kähler manifolds

By a slight abuse of notation denote $\ell : V \to [0, \infty)$. Given $L > 0$, define the truncated manifold

$$V_L := \ell^{-1}([0, L]).$$

Let E be a reflexive sheaf over V. Let S be the singular set of E and *assume* $S \subset V_{L_0}$ for some L_0. Then we have the following:

Definition: Let $(E_D, \bar{\partial}_D)$ be a holomorphic vector bundle over D. Let $(E_\infty, \bar{\partial}_\infty)$ denote the pullback to $(L_0, \infty) \times S^1 \times D$. We say E is *asymptotic* to E_D if there exists a bundle isomorphism $\bar{\pi} : E|_{V \setminus V_{L_0}} \to E_\infty$ and a constant δ_E such that

$$|\nabla^k(\pi_* \bar{\partial} - \bar{\partial}_\infty)| = O(e^{-\delta_E \ell}).$$

Finally, a metric H on E is *asymptotically translation-invariant* if it is asymptotic to a metric H_D on E_D.

Linear analysis

We define the following weighted Hölder spaces. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, define:

$$C^{k,\alpha}_{\delta}(V) := \{f \in C^{k,\alpha}(V) \,|\, ||f||_{C^{k,\alpha}_{\delta}} < \infty\}$$

with

$$||\cdot||_{C^{k,\alpha}_{\delta}} := ||e^{\delta\ell}\cdot||_{C^{k,\alpha}}.$$

Proposition

For $0 < \delta << 1$, the linear map $C^{k+2,\alpha}_{\delta}(V) \oplus \mathbb{R} \to C^{k,\alpha}_{\delta}(V)$ defined by

$$(f, A) \mapsto \Delta f - A\Delta\ell$$

is an isomorphism.

Proof

As a first step, we prove the result when E is smooth. Construct a background Hermitian metric H_0 on E which is asymptotically translation-invariant and satisfies

$$K_{H_0} \in C^\infty_\delta(V, isu(E, H_0)).$$

Given such an H_0, we define a map

$$\mathcal{L} : C^\infty_\delta(V, isu(E, H_0)) \times [0,1] \to C^\infty_\delta(V, isu(E, H_0))$$

by

$$\mathcal{L}(s,t) : Ad(e^{s/2})K_{H_0 e^s} + t \cdot s.$$

A solution s to the equation $\mathcal{L}(s,0) = 0$ proves the theorem.

Proof

For estimates, it is helpful to think of the the equation $\mathcal{L}(s,t) = 0$ as

$$\left(\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0} + t\right)s + B(\nabla_{H_0}s \otimes \nabla_{H_0}s) = C(K_{H_0}),$$

where B and C are linear with coefficients depending on s, but not on its derivatives.

We now follow the method of continuity. Set

$$I := \{t \in [0,1] : \mathcal{L}(s,t) = 0 \text{ for some } s\}.$$

We prove I is open, closed, and nonempty.

Proof

- First, to show $1 \in I$, we use a trick discovered by Lübke-Teleman.
- To show I is open, we demonstrate that the Linearization of $\mathcal{L}$ is invertible.
- Key step is to show that $|s|$ is bounded uniformly, from which all other estimates follow.
- Use barriers to show that if $|s|$ is large, it must be large far down the tube, where we can take advantage of the stability assumption.

C^0 bound

Fix L_0 large, and denote $N := ||s||_{L^\infty(V)}$ and $M := ||s||_{L^\infty(V \setminus V_{L_0})}$.

Using the equation $\mathcal{L}(s,t) = 0$, derive the inequality

$$\Delta |s|^2 \leq 4N|K_{H_0}|.$$

Let $f \in C^\infty_\delta(V)$ and $A > 0$ be the unique solution to

$$\Delta(f - A\ell) = 4|K_{H_0}|.$$

Apply the maximum principle to $|s|^2 - N(f - A\ell)$ on V_{L_0} to conclude

$$N^2 \leq M^2 + N(AL_0 + 2||f||_{L^\infty})$$

so

$$N \leq M + C(L_0 + 1).$$

C^0 bound

Thus if N approaches infinity so does M. As a result it suffices to bound the supremum of $|s|$ on the tubular end.

Using the barrier function, we show that if $|s|$ gets large at some point down the tube, it must be large on a portion of the tube with length proportional to M. Integrating along this portion of tube and applying bounds from stability yields the result.

C^0 bound

A few more details. Let $x_0 \in \overline{V \backslash V_{L_0}}$ be such that $|s|(x_0) = M$.

Apply maximum principle to

$$|s|^2 - N(f - A\ell)$$

on V_L, for $L \geq \ell(x_0)$, to see

$$M \leq C(||s||_{L^\infty(\partial V_L)} + L - \ell(x_0)).$$

Thus, for a length of tube $L - \ell(x_0) = M/2C$, we have $||s||_{L^\infty}$ is larger than $M/2C$ on transverse slices.

Now we use our stability assumption.

C^0 bound

On a transversal slice D_z, by a Theorem of Donaldson

$$||s||_{L^\infty(D_z)} \leq C(\mathcal{M}(H_0, H_0 e^s|_{D_z}) - 1).$$

This relays on stability

In fact, one can argue further that

$$||s||_{L^\infty(D_z)} \leq C(M \int_{D_z} |K_{H_0 e^s|_{D_z}}| - 1).$$

An energy bound shows we can control the curvature term, even when integrated along the tube of length $L - \ell(x_0)$. Thus integrating we achieve our desired bound:

$$M^2 \leq CM.$$

Asymptotic decay

This establishes uniform C^0 control of $|s|$. By openess along the method of continuity, it follows that $|s| \in C^\infty_\delta$ for each time t, although this bound may depend on t.

We need a uniform bound of the form

$$|s| \leq Ce^{-\delta\ell}.$$

To accomplish this, we use the following inequality derived from $\mathcal{L}(s,t) = 0$ and our C^0 bound:

$$|\nabla^{H_0} s|^2 \leq C(|K_{H_0}| - \Delta |s|^2).$$

Asymptotic decay

Integrating the above inequality over V gives:

$$\int_V |\nabla^{H_0} s|^2 \leq C.$$

This is not good enough to give us decay. Instead we integrate over $V \backslash V_L$ to see

$$\int_{V \backslash V_L} |\nabla^{H_0} s|^2 \leq C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s||s|).$$

Because the bundle E_D is stable, it follows that ∇^{H_0} has trivial kernel on trace free endomorphisms. This yields

$$\int_{\partial V_L} |s|^2 \leq C \int_{\partial V_L} |\nabla^{H_0} s|^2.$$

Asymptotic decay

As a result we conclude

$$g(L) := \int_{V \setminus V_L} |\nabla^{H_0} s|^2 \leq C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s|^2).$$

We can now follow an ODE argument.

Proposition

If $g : [0, \infty) \to [0, \infty)$ *satisfies* $g(L) \leq Ae^{-\delta L} - Bg'(L)$, *with* $A, B > 0$, *then*

$$g(L) \leq (2A + g(0))e^{-\epsilon L}$$

with $\epsilon := \mathit{min}\{\delta, \frac{1}{B}\}$.

This gives the correct decay for g, and by elliptic regularity we can bootstrap this up to exponential control of $|s|$ and all its derivatives.

The singular case

To prove our main Theorem for reflexive sheaves E we use a regularization scheme based on ideas of Bando and Siu.

Blow up V along S :=sing(E)

$$\pi : \tilde{V} \to V,$$

and equip $\tilde{V}$ with a family of Kähler metrics ω_ϵ that degenerate to $\pi^*\omega$ as $\epsilon \to 0$.

$\tilde{V}$ carries a holomorphic vector bundle $\tilde{E}$, which agrees with the reflexive sheaf E outside S, and to which the smooth case can be applied to construct a PHYM metric H_ϵ.

The singular case

Proposition

There is a complex manifold $\tilde{V}$, a holomorphic map $\pi : \tilde{V} \to V$ which induces a biholomorphic map to $V \backslash S$, and a holomorphic vector bundle $\tilde{E}$ over $\tilde{V}$ such that

$$\tilde{E}|_{\tilde{V} \backslash \pi^{-1}(S)} \cong \pi^*(E|_{V \backslash S}).$$

Moreover, there exists a one-parameter family of Kähler metrics ω_ϵ on $\tilde{V}$ such that

- *on $\pi^{-1}(V \backslash B_\epsilon(S))$, we have $\omega_\epsilon = \pi^*\omega$.*
- *for $L \geq L_0$, the Neumann-Poincaré constant of $(\pi^{-1}(V_L), g_\epsilon)$ is bounded above by a constant independent of ϵ.*
- *$\pi^{-1}(V_{L_0})$ is contained in a geodesic ball whose radius does not depend on ϵ.*

The singular case

We can not solve the equation with the metric $\pi^*\omega$ on $\tilde{V}$ directly, since this metric is singular and the associated operators fail to be uniformly elliptic.

Instead, for each $\epsilon \in (0, 1]$, we can construct a PHYM metric $\tilde{H}_\epsilon$ on $\tilde{E}$. Define

$$\tilde{s}_\epsilon := \log \tilde{H}_1^{-1} \tilde{H}_\epsilon.$$

The desired PHYM metric on E will be constructed by taking the limit as ϵ tends to zero.

Fix and arbitrary neighborhood U of $S \subset V$. Need estimates independent of ϵ, specifically

$$||\tilde{s}_\epsilon||_{C^k_\delta(\tilde{V}_\epsilon \backslash \tilde{U})} \leq C_{k,U}.$$

The singular case

Set

$$K_\epsilon := i\Lambda_\epsilon F_{\tilde{H}_1} - \frac{\mathrm{tr}(i\Lambda_\epsilon F_{\tilde{H}_1})}{rk(E)} Id$$

and let $f_\epsilon \in C^0_\delta(\tilde{V}_\epsilon)$ and $A_\epsilon > 0$ b the unique solutions to

$$\Delta_\epsilon(f_\epsilon - A_\epsilon \ell) = 4|K_\epsilon|.$$

The key is to show that these barriers are independent of ϵ, i.e.

$$||f_\epsilon||_{L^\infty(\tilde{V}_\epsilon \setminus \tilde{U})} \leq c_U, \qquad A_\epsilon < c, \qquad ||F_{\tilde{H}_1}||_{L^2(\tilde{V}_\epsilon, L_0)} \leq c.$$

If so we can use the same argument as before to achieve convergence.

The singular case

Control of $||F_{\tilde{H}_1}||_{L^2(\tilde{V}_\epsilon, L_0)}$ follows by scaling

$$|F_{\tilde{H}_1}|^p_{\omega_\epsilon} vol_\epsilon \leq C(1 + \epsilon^{2n-2p})|F_{\tilde{H}_1}|^p_{\omega_1} vol_1.$$

Also note that ω_ϵ is independent of ϵ on the tubular end, so

$$||K_\epsilon||_{C^k_\delta(V \setminus V_{L_0})} \leq c_k.$$

Both of these estimates yield control of A_ϵ, since

$$A_\epsilon \leq C||K_\epsilon||_{L^1(\tilde{V}_\epsilon)}.$$

The singular case

Thus, all that is left is control for f_ϵ. Here we need our proposition which gave a uniform weighted Poincaré inequality

$$||e^{-\frac{\delta\ell}{2}}(f_\epsilon - \bar{f}_\epsilon)||^2_{L^2(\tilde{V}_\epsilon)} \leq C||\nabla_\epsilon f_\epsilon||^2_{L^2(\tilde{V}_\epsilon)}.$$

Using control of A_ϵ and that f_ϵ solves the Poisson equation, we have

$$||\nabla_\epsilon f_\epsilon||^2_{L^2(\tilde{V}_\epsilon)} = \int_{\tilde{V}_\epsilon} \langle \Delta_\epsilon (f_\epsilon - \bar{f}_\epsilon), f_\epsilon - \bar{f}_\epsilon \rangle$$

$$\leq ||e^{\frac{\delta\ell}{2}}(K_\epsilon + A_\epsilon \Delta_\epsilon \ell)||_{L^2(\tilde{V}_\epsilon)} ||e^{-\frac{\delta\ell}{2}}(f_\epsilon - \bar{f}_\epsilon)||_{L^2(\tilde{V}_\epsilon)}$$

Combining with the Poincaré inequality gives L^2 control of both $e^{-\frac{\delta\ell}{2}}(f_\epsilon - \bar{f}_\epsilon)$ and $\nabla_\epsilon f_\epsilon$.

The singular case

Now, working on the tube, set

$$F(L) := \int_{V \setminus V_{L_0}} |\nabla_\epsilon f_\epsilon|^2.$$

We just saw that $F(L) \leq c$.

As before, an integration by parts estimate shows

$$F(L) \leq C(e^{-\frac{\delta L}{2}} - F'(L)),$$

which implies on $V \setminus V_{L_0}$

$$F(L) \leq Ce^{-\gamma L}$$

Control of f_ϵ everywhere on $\tilde{V}_\epsilon$ follows from interior estimates.

Tangent cones

This completes the existence theorem. However, it says nothing about the behavior of the connection near the singular set (where E is not locally free).

In the case of constructing G_2 instantons, for the perturbation theory to work we need to know the structure of the singularity. Recall the construction uses Calabi-Yau 3-folds as building blocks.

In complex dimension three, reflexive sheaves only have point singularities.

Tangent cones

We consider the case of a reflexive sheaf E with an isolated singularity at the origin in $B_1(0) \subset \mathbb{C}^n$. Equip $B_1(0)$ with a metric

$$\omega = \frac{i}{2}\partial\bar{\partial}|z|^2 + O(|z|^2).$$

Let H solve $i\Lambda F_H = 0$ on $B_1(0)$, and denote by A the Chern connection.

For $\lambda_i \to 0$, let $\tau_i : B_1(0) \to B_{\lambda_i}(0)$ be defined by $z \mapsto \lambda_i z$. Set $A_i := \tau_i^* A$.

Tangent cones

A_i is a sequence of Yang-Mills connections, using Price monotonicity A_i converges (away from a singular set) to a limit connection A_∞, satisfying $\iota_{\frac{\partial}{\partial r}} A_\infty = 0$. (Tian)

Questions:

- Does A_∞ depend on the choice of sequence λ_i?
- Can A_∞ be identified?

In general this type of question is very hard. B. Yang demonstrated an affirmative answer to the first question assuming that $|F_A| \leq \frac{C}{r^2}$. Can we use the complex structure to our advantage?

Tangent cones

Let $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{P}^{n-1}$ be the natural projection, and $\iota : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n$ the inclusion. Let $\Theta = \sum_{j=1}^{n} \frac{\bar{z}^j \, dz^j - z^j \, d\bar{z}^j}{2i \, |z|^2}$ be the pullback of the standard contact structure on S^{2n-1}.

We assume that $E|_B \cong \iota_* \pi^* \mathcal{F}$, where $\mathcal{F} = \oplus \mathcal{F}_p$ is a direct sum of stable bundles on $\mathbb{P}^{n-1}$.

If B_p is the unique HYM connection on $\mathcal{F}_p$ with HYM constant μ_p, then

$$A_0 := \bigoplus \pi^* B_p + i \, \mu_p \, Id_{\mathcal{F}_p} \, \Theta$$

is the unique tangent cone for any HYM connection on E.

Tangent cones

Theorem (J. - Sá Earp-Walpuski)

In $B_1(0)$, assume $E \cong \iota_* \pi^* \mathcal{F}$, where $\mathcal{F}$ is a direct sum of stable bundles on $\mathbb{P}^{n-1}$. Let A be a HYM connection on E. Then there exists a unique connection A_0 satisfying

$$|z|^{k+1} |\nabla_0^k (A - A_0)| \leq \frac{C_k}{(-log|z|)^{1/2}}.$$

The constants C_k depend on $\omega, \mathcal{F}, A|_{B_R(0) \setminus B_{\frac{R}{2}}(0)}$ and $||F_A||^2_{L^2(B_R(0))}$.

Moreover, if $\mathcal{F}$ is stable, then there exists constants C'_k, α, such that

$$|z|^{k+1} |\nabla_0^k (A - A_0)| \leq C'_k |z|^{\alpha}.$$

Tangent cones

Why do we assume $\mathcal{F} = \oplus \mathcal{F}_p$ is a direct sum of stable bundles?

Our tangent cone A_∞ satisfies $\iota_{r\frac{\partial}{\partial r}} F_{A_\infty} = 0$. In the complex setting if $J(r\frac{\partial}{\partial r}) = \xi$, then $\iota_\xi F_{A_\infty} = 0$ as well.

Proposition

Let M be a manifold with a free S^1 action. Denote by ξ the killing field and $q : M \to M/S^1$ the projection. Let $\eta \in \Omega^1(M)$ satisfy $\eta(\xi) = 1$ and $\mathcal{L}_\xi \eta = 0$.

If A is a unitary connection on a Hermitian bundle E over M such that $\iota_\xi F_A = 0$, then E decomposes orthogonally as $E = \oplus E_p$, with $E_p := q^* V_p$. Furthermore there exists unitary connections B_p on V_p and $\mu_p \in \mathbb{R}$ so that

$$A := \bigoplus q^* B_p + i\, \mu_p\, Id_{V_p}\, \eta$$

Examples

This is an example of Okonek-Schneider-Spindler.

Recall the Euler sequence over $\mathbb{P}^3$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{p} \mathcal{T}_{\mathbb{P}^3}(-1) \longrightarrow 0.$$

Let $v \in \mathbb{C}^4 \setminus \{0\}$, and let $s_v \in H^0(\mathcal{T}_{\mathbb{P}^3}(-1))$ to be the section defined by

$$s_v(x) := p(x, v).$$

This section vanishes at the point $[v] \in \mathbb{P}^3$.

This leads to an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s_v} \mathcal{T}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{E}_v \longrightarrow 0,$$

where the sheaf $\mathcal{E}_v$ is locally free except at $[v]$.

Examples

Okonek-Schneider-Spindler prove that $\mathcal{E}_v$ is stable and reflexive, and thus it admits a HYM connection A.

Let $U \subset \mathbb{P}^3$ be an affine coordinate chart so $[v] = [1:0:0:0]$ is at the origin. Then on U the Euler sequence becomes

$$0 \longrightarrow \mathcal{O}_U \xrightarrow{(1,x_1,x_2,x_3)} \mathcal{O}_U^{\oplus 4} \xrightarrow{p} \mathcal{T}_{\mathbb{P}^3}(-1)|_U \longrightarrow 0.$$

Since on U the tangent bundle is trivial we conclude

$$0 \longrightarrow \mathcal{O}_U \xrightarrow{s_v=(x_1,x_2,x_3)} \mathcal{O}_U^{\oplus 3} \longrightarrow \mathcal{E}_v|_U,$$

which we recognize as the pullback of the Euler sequence from $\mathbb{P}^2$.

Thus

$$\mathcal{E}_v|_U \cong \iota_* \pi^* \mathcal{T}_{\mathbb{P}^2}$$

Thank You!