Hermitian Yang Mills connections on reflexive sheaves.

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Introduction

- The main motivation for this project is the construction of $G_2$ instantons.

- Idea is based off of the twisted connected sum construction, pioneered by Kovalev and later extended by Corti-Haskins-Nordström-Pacini.

- One glues together two asymptotically cylindrical Calabi-Yau 3-folds, each equipped with a trivial $S^1$ bundle, in a prescribed fashion.
Can this construction yield a $G_2$ instanton? The main idea is to take a bundle with a Hermitian-Yang-Mills connection on each piece and glue them together.

The starting point for our work is the result of Sá Earp:

**Theorem (Sá Earp)**

Let $(E, \bar{\partial})$ be a holomorphic bundle over an asymptotically cylindrical Calabi-Yau 3-fold. If $E$ is asymptotic to a degree zero stable bundle along the cylindrical end, then there exists a metric $H$ on $E$ satisfying the Hermitian-Yang-Mills equations.
Introduction

From here, Sá Earp-Walpuski outlined a program and developed the perturbation theory needed to construct a $G_2$ instanton from the twisted connected sum.

The perturbation theory places restrictions on the cohomology of the bundle.

Reflexive sheaves are more abundant and bundles, can they be used to construct new examples of singular $G_2$ instantons?
Introduction

Goals of this talk:

▶ Construct a Hermitian-Yang-Mills connection on a reflexive sheaf over an asymptotically cylindrical Kähler manifold

▶ Show the solution is exponentially asymptotically translation invariant

▶ Analyze the connection at singular points under certain assumptions on the complex structure
Background

Let $X$ be a complex manifold.

Fix a Kähler metric $g$ on $T^{1,0}X$, with corresponding Kähler form

$$\omega = \frac{i}{2} g_{j\bar{k}} \, dz^j \wedge d\bar{z}^k.$$

Let $(E, \bar{\partial})$ be a holomorphic vector bundle over $X$.

Given a Hermitian metric $H$ on $E$, one can define the associated Chern connection $d_H$, compatible with $H$ and the holomorphic structure $\bar{\partial}$. We write $d_H = \partial_H + \bar{\partial}$.
The assumption that $\bar{\partial}$ is a holomorphic structure ($\bar{\partial}^2 = 0$) and metric compatibility imply the curvature $F_H$ of $d_H$ is a $(1, 1)$ form.

- Locally, in a holomorphic frame:

$$F_H = \bar{\partial}(H^{-1}\partial H).$$

Let $\Lambda$ denote the adjoint in $g$ of wedging with the Kähler form $\omega$. For any $(1, 1)$ form $\alpha$, $i\Lambda\alpha = g^{jk}\alpha_{j\bar{k}}$.

**Definition**

A metric $H$ satisfying

$$i\Lambda F_H = \mu \text{Id}$$

is called a *Hermitian-Yang-Mills* metric (*Hermitian-Einstein*).
If $H$ solves the Hermitian-Yang-Mills equations, then $d_H$ solves the Yang-Mills equations.

The Kähler identities imply

$$d_H^* F_H = -\partial_H (i \Lambda F_H) + \bar{\partial} (i \Lambda F_H) = 0.$$ 

$d_H$ is a critical point of the Yang-Mills functional

$$YM(d_A) = \| F_A \|_{L^2(X)}^2.$$ 

**Question:** When does $(E, \bar{\partial})$ admit a Hermitian-Yang-Mills metric?
Previous results

If $X$ is compact, solution is given by the following beautiful result:

**Theorem (Donaldson, Uhlenbeck-Yau)**

$E$ admits a Hermitian-Yang-Mills metric if and only if it is stable in the sense of Mumford-Takemoto:

$$\frac{\text{deg}(\mathcal{F})}{\text{rk}(\mathcal{F})} < \frac{\text{deg}(E)}{\text{rk}(E)}$$

for all proper, reflexive subsheaves $\mathcal{F} \subset E$.

The degree is given by

$$\text{deg}(E) = i \int_X \text{tr}(F_H) \wedge \omega^{n-1},$$

and is independent of a choice of metric.
Previous results

Many interesting generalizations of the above Theorem. Most notably for us, the result was extended to the case where $E$ is a reflexive sheaf by Bando-Siu.

Here, metrics are only defined away from the singular set (of complex codimension at least 3), where $E$ is a holomorphic bundle.

The solution satisfies

$$\|i \Lambda F_H\|_{L^\infty(X)} \leq C \quad \text{and} \quad \|F_H\|_{L^2(X)} \leq C.$$ 

Such metrics are called admissible.
Donaldson uses the Yang-Mills flow (and later Simpson and Bando-Siu), while Uhlenbeck-Yau employ the method of continuity.

Let $H_0$ be a fixed metric and $H$ a metric along either method. Right away one sees $|i\Lambda F_H|$ is controlled.

The key estimate is a uniform $C^0$ bound for $e^s = H_0^{-1}H$. This is where stability comes into play in the compact setting.
Previous results

**Question:** What happens if $X$ is complete, non-compact?

In some cases you can make the structure of $X$ work for you.

**Theorem (Ni-Ren)**

If $X$ admits a spectral gap ($\lambda_1(X) > 0$), and $E$ admits a metric $H_0$ such that $|i\Lambda F_{H_0} - \mu Id| \in L^p(X)$ for some $p > 1$, then there exists a metric $H$ such that

$$i\Lambda F_H = \mu Id.$$  

This uses an argument similar to Donaldson’s solution of the Dirichlet problem, since we have along the flow

$$\left( \frac{d}{dt} + \Delta \right) |i\Lambda F_H - \mu Id|^2 \leq 0.$$
Previous results

(In this talk I use the Geometer’s Laplacian, $\Delta = d^*d$ on functions)

Ni also showed that the same conclusion holds, for example, if $X$ satisfies a $L^2$ Sobolev inequality and $p \in [1, \frac{n}{2})$, or if it is non-parabolic (i.e., admits a positive Greens function) and $p = 1$.

In this case, for a fixed initial metric $H_0$, one can solve

$$\Delta u = |i \Lambda F_{H_0}|,$$

and use $u$ as a barrier to control $s$, since

$$\Delta \log \text{tr}(e^s) \leq 4|i \Lambda F_{H_0}|.$$

For asymptotically cylindrical manifolds we have linear volume growth, so the above results can not be used.
Main result

Building off of Sá Earp’s result, we prove the following

Theorem (J. - Walpuski)

Let $V$ be an asymptotically cylindrical Kähler manifold with asymptotic cross-section $D$. Let $E_D$ be a stable vector bundle over $D$, and $E$ a reflexive sheaf asymptotic to $E_D$. There exists an asymptotically translation-invariant Hermitian metric $H$ on $E$ which satisfies the projective Hermitian Yang-Mills (PHYM) equation

$$K_H := i \Lambda F_H - \frac{\text{tr}(i \Lambda F_H)}{\text{rk}(E)} \text{Id} = 0.$$ 

Furthermore $|F_H| \in L^2_{\text{loc}}(V)$.

Rmk: Every PHYM metric can be converted to a HYM metric via a conformal change. However, this metric will typically not be asymptotically translation invariant.
Definition: Let \((D, g_D, I_D)\) be a compact Kähler manifold. A Kähler manifold \((V, g, I)\) is called asymptotically cylindrical (ACyl) with asymptotic cross-section \((D, I_g, I_D)\) if there exists a constant \(\delta_V > 0\), a compact subset \(K \subset V\), and a diffeomorphism \(\pi : V \setminus K \to (0, \infty) \times S^1 \times D\), such that

\[
|\nabla^k (\pi_* g - g_\infty)| + |\nabla^k (\pi_* I - I_\infty)| = O(e^{-\delta_V \ell})
\]

for all \(k \geq 0\). Here \((\ell, \theta)\) are the canonical coordinates on \((0, \infty) \times S^1\) and

\[
g_\infty := d\ell^2 \oplus d\theta^2 \oplus g_D \\
I_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_D
\]
ACyl Kähler manifolds

By a slight abuse of notation denote $\ell : V \to [0, \infty)$. Given $L > 0$, define the truncated manifold

$$V_L := \ell^{-1}([0, L]).$$

Let $E$ be a reflexive sheaf over $V$. Let $S$ be the singular set of $E$ and assume $S \subset V_{L_0}$ for some $L_0$. Then we have the following:

**Definition:** Let $(E_D, \bar{\partial}_D)$ be a holomorphic vector bundle over $D$. Let $(E_\infty, \bar{\partial}_\infty)$ denote the pullback to $(L_0, \infty) \times S^1 \times D$. We say $E$ is *asymptotic* to $E_D$ if there exists a bundle isomorphism $\bar{\pi} : E|_{V \setminus V_{L_0}} \to E_\infty$ and a constant $\delta_E$ such that

$$|\nabla^k (\pi_* \bar{\partial} - \bar{\partial}_\infty)| = O(e^{-\delta_E \ell}).$$

Finally, a metric $H$ on $E$ is *asymptotically translation-invariant* if it is asymptotic to a metric $H_D$ on $E_D$. 
We define the following weighted Hölder spaces. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, define:

$$C_{\delta}^{k,\alpha}(V) := \{ f \in C^{k,\alpha}(V) \mid \|f\|_{C_{\delta}^{k,\alpha}} < \infty \}$$

with

$$\|\cdot\|_{C_{\delta}^{k,\alpha}} := \|e^{\delta \ell} \cdot \|_{C^{k,\alpha}}.$$

**Proposition**

*For $0 < \delta << 1$, the linear map $C_{\delta}^{k+2,\alpha}(V) \oplus \mathbb{R} \to C_{\delta}^{k,\alpha}(V)$ defined by*

$$(f, A) \mapsto \Delta f - A\Delta \ell$$

*is an isomorphism.*
Proof

As a first step, we prove the result when $E$ is smooth. Construct a background Hermitian metric $H_0$ on $E$ which is asymptotically translation-invariant and satisfies

$$K_{H_0} \in C^\infty_\delta(V, isu(E, H_0)).$$

Given such an $H_0$, we define a map

$$\mathcal{L} : C^\infty_\delta(V, isu(E, H_0)) \times [0, 1] \to C^\infty_\delta(V, isu(E, H_0))$$

by

$$\mathcal{L}(s, t) : \text{Ad}(e^{s/2})K_{H_0}e^s + t \cdot s.$$

A solution $s$ to the equation $\mathcal{L}(s, 0) = 0$ proves the theorem.
Proof

For estimates, it is helpful to think of the the equation $\mathcal{L}(s, t) = 0$ as

$$\left(\frac{1}{2} \nabla^*_H \nabla_H + t\right) s + B(\nabla_H s \otimes \nabla_H s) = C(K_H),$$

where $B$ and $C$ are linear with coefficients depending on $s$, but not on its derivatives.

We now follow the method of continuity. Set

$$I := \{ t \in [0, 1] : \mathcal{L}(s, t) = 0 \text{ for some } s \}.$$  

We prove $I$ is open, closed, and nonempty.
Proof

- First, to show $1 \in I$, we use a trick discovered by Lübke-Teleman.

- To show $I$ is open, we demonstrate that the Linearization of $\mathcal{L}$ is invertible.

- Key step is to show that $|s|$ is bounded uniformly, from which all other estimates follow.

- Use barriers to show that if $|s|$ is large, it must be large far down the tube, where we can take advantage of the stability assumption.
**C^0 bound**

Fix $L_0$ large, and denote $N := \|s\|_{L^\infty(V)}$ and $M := \|s\|_{L^\infty(V \setminus V_{L_0})}$. Using the equation $\mathcal{L}(s, t) = 0$, derive the inequality

$$\Delta |s|^2 \leq 4N|K_{H_0}|.$$  

Let $f \in C^\infty_\delta(V)$ and $A > 0$ be the unique solution to

$$\Delta (f - A\ell) = 4|K_{H_0}|.$$  

Apply the maximum principle to $|s|^2 - N(f - A\ell)$ on $V_{L_0}$ to conclude

$$N^2 \leq M^2 + N(AL_0 + 2\|f\|_{L^\infty})$$

so

$$N \leq M + C(L_0 + 1).$$
$C^0$ bound

Thus if $N$ approaches infinity so does $M$. As a result it suffices to bound the supremum of $|s|$ on the tubular end.

Using the barrier function, we show that if $|s|$ gets large at some point down the tube, it must be large on a portion of the tube with length proportional to $M$. Integrating along this portion of tube and applying bounds from stability yields the result.
A few more details. Let \( x_0 \in \overline{V \setminus V_{L_0}} \) be such that \(|s|(x_0) = M\). Apply maximum principle to

\[
|s|^2 - N(f - A\ell)
\]
on \( V_L \), for \( L \geq \ell(x_0) \), to see

\[
M \leq C(\|s\|_{L^\infty(\partial V_L)} + L - \ell(x_0)).
\]

Thus, for a length of tube \( L - \ell(x_0) = M/2C \), we have \( \|s\|_{L^\infty} \) is larger than \( M/2C \) on transverse slices.

Now we use our stability assumption.
\( C^0 \) bound

On a transversal slice \( D_z \), by a Theorem of Donaldson

\[
\|s\|_{L^\infty(D_z)} \leq C(\mathcal{M}(H_0, H_0 e^s|_{D_z}) - 1).
\]

This relays on stability

In fact, one can argue further that

\[
\|s\|_{L^\infty(D_z)} \leq C(\mathcal{M} \int_{D_z} |K_{H_0 e^s|_{D_z}}| - 1).
\]

An energy bound shows we can control the curvature term, even when integrated along the tube of length \( L - \ell(x_0) \). Thus integrating we achieve our desired bound:

\[
M^2 \leq CM.
\]
Asymptotic decay

This establishes uniform $C^0$ control of $|s|$. By openness along the method of continuity, it follows that $|s| \in C_\delta^\infty$ for each time $t$, although this bound may depend on $t$.

We need a uniform bound of the form

$$|s| \leq C e^{-\delta \ell}.$$  

To accomplish this, we use the following inequality derived from $\mathcal{L}(s, t) = 0$ and our $C^0$ bound:

$$|\nabla^H_0 s|^2 \leq C (|K_{H_0}| - \Delta |s|^2).$$
Asymptotic decay

Integrating the above inequality over \( V \) gives:

\[
\int_{V} |\nabla^{H_0} s|^2 \leq C.
\]

This is not good enough to give us decay. Instead we integrate over \( V \setminus V_L \) to see

\[
\int_{V \setminus V_L} |\nabla^{H_0} s|^2 \leq C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s||s|).
\]

Because the bundle \( E_D \) is stable, it follows that \( \nabla^{H_0} \) has trivial kernel on trace free endomorphisms. This yields

\[
\int_{\partial V_L} |s|^2 \leq C \int_{\partial V_L} |\nabla^{H_0} s|^2.
\]
Asymptotic decay

As a result we conclude

\[ g(L) := \int_{V \setminus V_L} |\nabla^{H_0} s|^2 \leq C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s|^2). \]

We can now follow an ODE argument.

**Proposition**

If \( g : [0, \infty) \to [0, \infty) \) satisfies \( g(L) \leq Ae^{-\delta L} - Bg'(L) \), with \( A, B > 0 \), then

\[ g(L) \leq (2A + g(0))e^{-\epsilon L} \]

with \( \epsilon := \min\{\delta, \frac{1}{B}\} \).

This gives the correct decay for \( g \), and by elliptic regularity we can bootstrap this up to exponential control of \(|s|\) and all its derivatives.
The singular case

To prove our main Theorem for reflexive sheaves $E$ we use a regularization scheme based on ideas of Bando and Siu.

Blow up $\hat{V}$ along $S := \text{sing}(E)$

$$\pi : \tilde{V} \to V,$$

and equip $\tilde{V}$ with a family of Kähler metrics $\omega_\epsilon$ that degenerate to $\pi^* \omega$ as $\epsilon \to 0$.

$\tilde{V}$ carries a holomorphic vector bundle $\tilde{E}$, which agrees with the reflexive sheaf $E$ outside $S$, and to which the smooth case can be applied to construct a PHYM metric $H_\epsilon$. 
The singular case

Proposition

There is a complex manifold $\tilde{V}$, a holomorphic map $\pi : \tilde{V} \to V$ which induces a biholomorphic map to $V \setminus S$, and a holomorphic vector bundle $\tilde{E}$ over $\tilde{V}$ such that

$$\tilde{E}|_{\tilde{V} \setminus \pi^{-1}(S)} \cong \pi^*(E|_{V \setminus S}).$$

Moreover, there exists a one-parameter family of Kähler metrics $\omega_\epsilon$ on $\tilde{V}$ such that

- on $\pi^{-1}(V \setminus B_\epsilon(S))$, we have $\omega_\epsilon = \pi^* \omega$.

- for $L \geq L_0$, the Neumann-Poincaré constant of $(\pi^{-1}(V_L), g_\epsilon)$ is bounded above by a constant independent of $\epsilon$.

- $\pi^{-1}(V_{L_0})$ is contained in a geodesic ball whose radius does not depend on $\epsilon$. 
The singular case

We can not solve the equation with the metric $\pi^* \omega$ on $\tilde{V}$ directly, since this metric is singular and the associated operators fail to be uniformly elliptic.

Instead, for each $\epsilon \in (0, 1]$, we can construct a PHYM metric $\tilde{H}_\epsilon$ on $\tilde{E}$. Define

$$\tilde{s}_\epsilon := \log \tilde{H}_1^{-1} \tilde{H}_\epsilon.$$

The desired PHYM metric on $E$ will be constructed by taking the limit as $\epsilon$ tends to zero.

Fix and arbitrary neighborhood $U$ of $S \subset V$. Need estimates independent of $\epsilon$, specifically

$$\| \tilde{s}_\epsilon \|_{C^k_\delta (\tilde{V}_\epsilon \setminus \tilde{U})} \leq C_{k, U}.$$
The singular case

Set

\[ K_\epsilon := i \wedge_\epsilon F_{\tilde{H}_1} - \frac{\text{tr}(i \wedge_\epsilon F_{\tilde{H}_1})}{\text{rk}(E)} \text{Id} \]

and let \( f_\epsilon \in C^0_\delta(\tilde{V}_\epsilon) \) and \( A_\epsilon > 0 \) be the unique solutions to

\[ \Delta_\epsilon (f_\epsilon - A_\epsilon \ell) = 4|K_\epsilon|. \]

The key is to show that these barriers are independent of \( \epsilon \), i.e.

\[ \|f_\epsilon\|_{L^\infty(\tilde{V}_\epsilon \setminus \tilde{U})} \leq c_U, \quad A_\epsilon < c, \quad \|F_{\tilde{H}_1}\|_{L^2(\tilde{V}_\epsilon, L_0)} \leq c. \]

If so we can use the same argument as before to achieve convergence.
The singular case

Control of $\| F_{\tilde{H}_1} \|_{L^2(\tilde{V}_\epsilon, L_0)}$ follows by scaling

$$| F_{\tilde{H}_1} |^{p} \omega_\epsilon vol_\epsilon \leq C(1 + \epsilon^{2n-2p}) | F_{\tilde{H}_1} |^{p} \omega vol_1.$$ 

Also note that $\omega_\epsilon$ is independent of $\epsilon$ on the tubular end, so

$$\| K_\epsilon \|_{C^k(\mathcal{V}\setminus\mathcal{V}_L_0)} \leq c_k.$$ 

Both of these estimates yield control of $A_\epsilon$, since

$$A_\epsilon \leq C \| K_\epsilon \|_{L^1(\tilde{V}_\epsilon)}.$$
The singular case

Thus, all that is left is control for $f_\epsilon$. Here we need our proposition which gave a uniform weighted Poincaré inequality

$$\| e^{-\frac{\delta \ell}{2}} (f_\epsilon - \bar{f}_\epsilon) \|_{L^2(\tilde{V}_\epsilon)}^2 \leq C \| \nabla f_\epsilon \|_{L^2(\tilde{V}_\epsilon)}^2.$$

Using control of $A_\epsilon$ and that $f_\epsilon$ solves the Poisson equation, we have

$$\| \nabla f_\epsilon \|_{L^2(\tilde{V}_\epsilon)}^2 = \int_{\tilde{V}_\epsilon} \langle \Delta f_\epsilon - \bar{f}_\epsilon, f_\epsilon - \bar{f}_\epsilon \rangle$$

$$\leq \| e^{\frac{\delta \ell}{2}} (K_\epsilon + A_\epsilon \Delta_\epsilon) \|_{L^2(\tilde{V}_\epsilon)} \| e^{-\frac{\delta \ell}{2}} (f_\epsilon - \bar{f}_\epsilon) \|_{L^2(\tilde{V}_\epsilon)}$$

Combining with the Poincaré inequality gives $L^2$ control of both $e^{-\frac{\delta \ell}{2}} (f_\epsilon - \bar{f}_\epsilon)$ and $\nabla f_\epsilon$. 
The singular case

Now, working on the tube, set

\[ F(L) := \int_{V \setminus V_{L_0}} |\nabla_\epsilon f_\epsilon|^2. \]

We just saw that \( F(L) \leq c \).

As before, an integration by parts estimate shows

\[ F(L) \leq C(e^{-\frac{\delta L}{2}} - F'(L)), \]

which implies on \( V \setminus V_{L_0} \)

\[ F(L) \leq Ce^{-\gamma L} \]

Control of \( f_\epsilon \) everywhere on \( \tilde{V}_\epsilon \) follows from interior estimates.
This completes the existence theorem. However, it says nothing about the behavior of the connection near the singular set (where $E$ is not locally free).

In the case of constructing $G_2$ instantons, for the perturbation theory to work we need to know the structure of the singularity. Recall the construction uses Calabi-Yau 3-folds as building blocks.

In complex dimension three, reflexive sheaves only have point singularities.
We consider the case of a reflexive sheaf $E$ with an isolated singularity at the origin in $B_1(0) \subset \mathbb{C}^n$. Equip $B_1(0)$ with a metric
\[
\omega = \frac{i}{2} \partial \bar{\partial} |z|^2 + O(|z|^2).
\]

Let $H$ solve $i \Lambda F_H = 0$ on $B_1(0)$, and denote by $A$ the Chern connection.

For $\lambda_i \to 0$, let $\tau_i : B_1(0) \to B_{\lambda_i}(0)$ be defined by $z \mapsto \lambda_i z$. Set $A_i := \tau_i^* A$. 

**Tangent cones**
Tangent cones

\( A_i \) is a sequence of Yang-Mills connections, using Price monotonicity \( A_i \) converges (away from a singular set) to a limit connection \( A_\infty \), satisfying \( \nu \frac{\partial}{\partial r} A_\infty = 0 \). (Tian)

Questions:

- Does \( A_\infty \) depend on the choice of sequence \( \lambda_i \)?
- Can \( A_\infty \) be identified?

In general this type of question is very hard. B. Yang demonstrated an affirmative answer to the first question assuming that \( |F_A| \leq \frac{C}{r^2} \). Can we use the complex structure to our advantage?
Tangent cones

Let \( \pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1} \) be the natural projection, and \( \iota : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \) the inclusion. Let \( \Theta = \sum_{j=1}^{n} \frac{\bar{z}^j dz^j - z^j d\bar{z}^j}{2i |z|^2} \) be the pullback of the standard contact structure on \( S^{2n-1} \).

We assume that \( E|_B \cong \iota_\ast \pi_\ast \mathcal{F} \), where \( \mathcal{F} = \bigoplus \mathcal{F}_p \) is a direct sum of stable bundles on \( \mathbb{P}^{n-1} \).

If \( B_p \) is the unique HYM connection on \( \mathcal{F}_p \) with HYM constant \( \mu_p \), then

\[
A_0 := \bigoplus \pi_\ast B_p + i \mu_p \operatorname{Id}_{\mathcal{F}_p} \Theta
\]

is the unique tangent cone for any HYM connection on \( E \).
Tangent cones

Theorem (J. - Sá Earp-Walpuski)

In $B_1(0)$, assume $E \cong \iota_* \pi^* \mathcal{F}$, where $\mathcal{F}$ is a direct sum of stable bundles on $\mathbb{P}^{n-1}$. Let $A$ be a HYM connection on $E$. Then there exists a unique connection $A_0$ satisfying

$$|z|^{k+1} |\nabla^k_0 (A - A_0)| \leq \frac{C_k}{(-\log |z|)^{1/2}}.$$

The constants $C_k$ depend on $\omega, \mathcal{F}, A|_{B_R(0) \setminus B_{\frac{R}{2}}(0)}$ and $\|FA\|_{L^2(B_R(0))}^2$.

Moreover, if $\mathcal{F}$ is stable, then there exists constants $C'_k, \alpha$, such that

$$|z|^{k+1} |\nabla^k_0 (A - A_0)| \leq C'_k |z|^\alpha.$$
Tangent cones

Why do we assume $\mathcal{F} = \bigoplus \mathcal{F}_p$ is a direct sum of stable bundles?

Our tangent cone $A_\infty$ satisfies $\nu_r \frac{\partial}{\partial r} F_{A_\infty} = 0$. In the complex setting if $J(r \frac{\partial}{\partial r}) = \xi$, then $\nu_\xi F_{A_\infty} = 0$ as well.

Proposition

Let $M$ be a manifold with a free $S^1$ action. Denote by $\xi$ the killing field and $q : M \to M/S^1$ the projection. Let $\eta \in \Omega^1(M)$ satisfy $\eta(\xi) = 1$ and $\mathcal{L}_\xi \eta = 0$.

If $A$ is a unitary connection on a Hermitian bundle $E$ over $M$ such that $\nu_\xi F_A = 0$, then $E$ decomposes orthogonally as $E = \bigoplus E_p$, with $E_p := q^* V_p$. Furthermore there exists unitary connections $B_p$ on $V_p$ and $\mu_p \in \mathbb{R}$ so that

$$A := \bigoplus q^* B_p + i \mu_p \text{Id}_{V_p} \eta$$
Examples

This is an example of Okonek-Schneider-Spindler.

Recall the Euler sequence over $\mathbb{P}^3$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \overset{p}{\longrightarrow} \mathcal{T}_{\mathbb{P}^3}(-1) \longrightarrow 0.$$ 

Let $\nu \in \mathbb{C}^4\setminus\{0\}$, and let $s_{\nu} \in H^0(\mathcal{T}_{\mathbb{P}^3}(-1))$ to be the section defined by

$$s_{\nu}(x) := p(x, \nu).$$

This section vanishes at the point $[\nu] \in \mathbb{P}^3$.

This leads to an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \overset{s_{\nu}}{\longrightarrow} \mathcal{T}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{E}_{\nu} \longrightarrow 0,$$

where the sheaf $\mathcal{E}_{\nu}$ is locally free except at $[\nu]$. 
Examples

Okonek-Schneider-Spindler prove that $E_v$ is stable and reflexive, and thus it admits a HYM connection $A$.

Let $U \subset \mathbb{P}^3$ be an affine coordinate chart so $[v] = [1 : 0 : 0 : 0]$ is at the origin. Then on $U$ the Euler sequence becomes

$$
0 \longrightarrow \mathcal{O}_U \xrightarrow{(1,x_1,x_2,x_3)} \mathcal{O}_U^\oplus 4 \xrightarrow{p} T_{\mathbb{P}^3}(-1)|_U \longrightarrow 0.
$$

Since on $U$ the tangent bundle is trivial we conclude

$$
0 \longrightarrow \mathcal{O}_U \xrightarrow{s_v=(x_1,x_2,x_3)} \mathcal{O}_U^\oplus 3 \longrightarrow E_v|_U,
$$

which we recognize as the pullback of the Euler sequence from $\mathbb{P}^2$. Thus

$$
E_v|_U \cong \iota_* \pi^* T_{\mathbb{P}^2}
$$
Thank You!