Calabi-Yau Manifolds through
Califold Transitions

Plan: (1) Califold Transitions
(2) Heterotic Strings
(3) Main Results
(4) Type II strings.

Defn: A Calabi-Yau 3-fold is a complex manifold \( X \), \( \dim_\mathbb{C} X = 3 \).

(1) \( X \) simply connected \( (\pi_1(X) < \infty) \)
(2) \( K_X \cong \mathcal{O}_X \) \( \iff \exists \text{ global holomorphic form 32} \)

Note: Not necessarily Kähler!
Given \( C_i, 1 \leq i \leq k \) disjoint, \((-1,-1)\) rational curves in \( X \).

There is a contraction \( \Pi: X \to \overline{X} \)

\[ C_i \to p_i \]

where \( p_i \in \overline{X} \) are disjoint singular pts.

\( \text{Nhhd} (p_i) \cong \left\{ \sum_{i=1}^{4} z_i^2 = 0 \right\} \cap \left\{ \|z\| < 1 \right\} \subset \mathbb{C}^4 \)

"Canifolds Singularities"

\( X \) is Gorenstein, \( K_X \cong \mathcal{O}_X \)
**Thm (Friedman)** (Assume $X$ is Kähler)

if $\exists \lambda_i \neq 0$ $1 \leq i \leq k$ $\lambda_i \in \mathbb{R}$ st.

$$\sum_{i=1}^{k} \lambda_i [c_i] = 0 \text{ in } H_2(X, \mathbb{R})$$

Then $\exists$ a family $\mu : X \rightarrow \Delta = \{ |t| < 1 \}$ s.t.

$$(1) \quad \mu^{-1}(0) = X_0 = X$$

$$(2) \quad \mu(t) = X_t \text{ are smooth CY.}$$

A nbhd of $\mu^{-1}(0)$ is biholomorphic to

$$\sum_{i=1}^{q} z_i z_i^* = t \sum_{i=1}^{q} z_i^2 \leq C \times \Delta.$$ 

This is a conifold Transition.

"$X \rightarrow \overline{X} \mapsto X_t$"
This is a Violent process

Replaces $S^2$’s w/ $S^3$’s.

Example: One can find examples where

$[c_i]$ generate $H^2(X_t, \mathbb{R})$

$\Rightarrow X_t$ has $H^2(X_t, \mathbb{R}) = 0$.

So not Kähler.

This process puts all CY into a "web".

$M_1, M_2$ deformation families of CY mfd’s.

$M_1 \longrightarrow M_2$ if a general element of $M_1$ can be connected to $M_2$ by
Coneifold transition.

(Nota: Could allow more general contractions)

Reid's Fantasy: The Web of CYs is Connected

One difficulty: Not many tools for studying non-Kähler CYs.

Q: Can we "uniformize" non-Kähler CYs?

Option: Turn to string theory for motivation.
The Heterotic String.

(Strominger, Candelas-Horowitz-Strominger - Witten) in Kähler case

\((X, \Omega)\) CY, \(E \to X\) hol'c bundle.

Look for \(\int H\) metric on \(E\)

\(\int w\) metric on \(X\).

\[\begin{align*}
\text{S.t.} & \\
& \begin{cases}
(1) & d(\|\Omega\|_w w^2) = 0 \\
& \|\Omega\|_w^2 = \frac{\Omega \cdot \bar{\Omega}}{w^3}
\end{cases} \\
& (2) \quad w^2 \wedge F_H = 0.
\end{align*}\]

\(F_H = \text{curvature of Chern connection}\).
Anomaly (3)
Cancellation

$\kappa^{-1} \bar{\partial} \bar{\partial} \omega = \frac{\alpha'}{4} \left( \text{Tr}(R_m \wedge R_m) - \text{Tr}(F_m \wedge F_m) \right)$

$\alpha' > 0$

NB: (3) $\Rightarrow C_2(x) = C_2(E)$ in Bott-Chern Cohomology.

Example (cHSw)

If $(X, \omega_{CY})$ Kähler CY, $\omega_{CY}$ Ricci-flat

$E = T^{1,0} X \to X$, $H = \omega_{CY}$

Then this solves (1)+(2)+(3)
(1) $\|\sigma\|_w = 1$ since $\Omega \wedge \bar{\Omega} = c \Omega^3$

\[ d(\|\sigma\|_w \wedge \Omega^2) = 0 \text{ since } d\Omega = 0. \]

(2) $\omega^2 \wedge F = \text{Ric}(\omega) = 0$

(3) Holds Trivially.

$\Rightarrow$ Strominger natural extension of Ricci-flatness to non-Kähler CYs.

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Problem: No unique Vacuum. (Tons of Kähler CY!)

Hope: 1. All solutions of the vacuum eqns can be connected by conifold transitions.
(2) The physics is well behaved through these transitions.

Q: Can we pass solutions of the Hull–Strominger system through Calabi–Yau transitions?

**Theorem** if $X$ CY and $X \rightarrow X_t$ in a Calabi–Yau transition then $X_t$ admits solutions of (1) + (2) in the HS system for $E = T^{1,0} X_t$.

NB: Chen proved (2) when $E$ comes from $X$, and is holomorphic near $C_i$. 
(1) Conformally Balanced
due to Fu-Li-Yau

→ (2) C.-Picard-Yau.

Local Geometric Picture (Candelas-de la Ossa)

\[ \mathbb{C}^2 \]
\[ \mathbb{P}^1 \]

\[ \mathcal{O}(-1) \]

Canifold.

\[ \mathbb{T}^3 \]
\[ S^3 \]

J Ricci-flat Kähler metrics
\[
\left\{
\begin{aligned}
&\text{\(w_{c, a}\) on } \mathcal{F}_{\mathbb{P}}^{(-1)} \text{, } a > 0 \text{ s.t.} \\
&\text{Vol}(\mathbb{P}_t, w_{c, a}) \sim a \to 0
\end{aligned}
\right.
\]

- \(w_{c, 0}\) conical CY metric on conifold.
- \(w_{c, t}\) on \(X_t\) (smooth conifold) asymptotically conical.

\[
\left(\mathcal{F}_{\mathbb{P}}^{(-1)}, w_{c, a}\right) \rightarrow \left(\text{conifold, } w_{c, 0}\right)
\]

(rescaling)

\[
(X_t, w_{c, t})
\]
Sketch Fu-Li-Yau:

1. Fix $\omega_{CY}$ Kähler, Ricci-flat on $X$.
2. Glue (by partition of unity) $\omega_{CY}$ to $\omega_{co,a}$ near $C_i$ to get $\omega_a > 0$

$$d(\omega_a^2) = 0, \quad [\omega_a^2] = [\omega_{CY}^2]$$

$\omega_a = \omega_{co,a}$ near $C_i$.

(3) $\Phi_t$ diffeo
\[
\left\{ \left( \Phi_t^{*} \mathcal{W}_{c,y} \right)^{y_1} \right\} \text{ glue (by partition of unity)}
\]

\[
\text{to } \mathcal{W}_{c_0,t} \text{ to get } \mathcal{W}_t.
\]

(4) Perturb \( \mathcal{W}_t \) to satisfy \( d(\mathcal{W}_t^2) = 0 \).

(This is where the hard work happens.)

Result is \( \mathcal{W}_{fly,t} \)

\[
\mathcal{W}_{fly,t} \sim \mathcal{W}_{c_0,t} \text{ near } P_i.
\]
Sketch of (2): have \( W_{FLY,t} \)

\[ W_{FLY,t} \sim W_t \]

Suffices to solve (2) \( W_t^2 \wedge F_{H_t} = 0 \) and then apply perturbation.

**Step 1:**

\( X \) Kähler CY, simply connected.

\( \Rightarrow T^{1,0} X \) is stable

\( \text{wrt. } [w^2_{CY}] = [w^2_\omega] \)

Li-Yau \( \Rightarrow \) 3 HYM metrics \( H_\omega \)

\( \text{wrt. } w_\omega \) on \( T^{1,0} X \).
Step 2: $H_a \rightarrow H_0$ smoothly on compact sets of $\mathbb{X} \setminus \mathbb{F}_{p_i, b}$ as $a \rightarrow 0$.

$\Rightarrow H_0$ HYM on $\mathbb{X}$ wrt $\mathbb{W}_0$.

$\Rightarrow$ near $p_i$, $H_0$ HYM wrt $\mathbb{W}_{co, 0}$.

Step 3: Quantitative Convergence of $H_0$ to $\chi_{\mathbb{W}_{co, 0}}$.

Key: $T^{1,0}$ Conifold = $\mathbb{N}^*$.

$\Theta \rightarrow \Theta_{\mathbb{P} \times \mathbb{P}^1} \rightarrow \mathcal{E} \rightarrow T^{1,0} \mathbb{P} \times \mathbb{P}^1 \rightarrow 0$.

$\mathcal{E}$ is stable and indecomposable.
(uses estimates of Jacob - Sa Forp - Walpuski)

Step 4: glue \((\Phi_t^{-1} \Omega_0)^{1/4}\) to \(\omega_{co,t}\).

Run singular perturbation argument to correct to HYM.

\[
\begin{align*}
\{ \Pi_B \} & : \\
\frac{\partial S}{\partial t} &= 0 \quad \text{(complex integrable)} \\
\frac{\partial (\omega^2)}{\partial t} &= 0 \quad \text{(PD dual)} \\
2\pi i \bar{\partial} \partial (e^{2f} \omega) &= \rho_B \quad \text{holc curve}
\end{align*}
\]

\[
\{ \Pi_A \} : \\
\frac{\partial w}{\partial t} &= 0 \\
\frac{\partial \text{Re} S}{\partial t} &= 0 \quad \sqrt{\text{Slat}} \\
\frac{\partial}{\partial t} e_\omega \times (e^{-2f} \text{Re} S) &= \rho_A
\]