Calabi-Yau Manifolds through Conifold Transitions

Plan: (1) Conifold Transitions

(2) Heterotic Strings

(3) Main Results

(4) Type II strings.

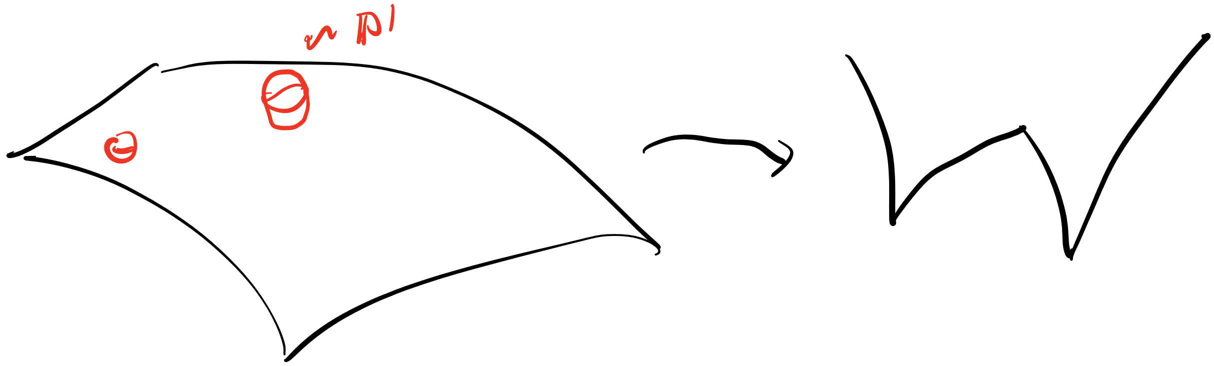
Def'n: A Calabi-Yau 3-fold is a complex manifold X, $\dim_{\mathbb{C}} X = 3$.

(1) X simply connected ($\pi_1(X) < +\infty$)

(2) $K_X \cong \mathcal{O}_X$ (i.e. $\exists$ global hol'c (3,0) form Ω)

Note: Not necessarily Kähler!

Given C_i $\quad 1 \leq i \leq k$ disjoint, $(-1,-1)$ rational curves in X.



There is a contraction $\pi : X \to \underline{X}$

$$\begin{array}{ccc} X & \longrightarrow & \underline{X} \\ \cup\!\!\mid & & \Psi \\ C_i & \longrightarrow & p_i \end{array}$$

where $p_i \in \underline{X}$ are disjoint singular pts.

$$\mathrm{Nbhd}(p_i) \approx \left\{ \sum_{i=1}^{4} z_i^2 = 0 \right\} \cap \left\{ \|z\| < 1 \right\} \subseteq \mathbb{C}^4$$

"Conifold Singularities"

$\underline{X}$ is Gorenstein, $K_{\underline{X}} \cong \mathcal{O}_{\underline{X}}$

Thm (Friedman). (Assume X is Kähler).
if $\exists\ \lambda_i \neq 0 \quad 1 \leq i \leq k \quad \lambda_i \in \mathbb{R}$ s.t.

$$\sum_{i=1}^{k} \lambda_i [C_i] = 0 \text{ in } H_2(X, \mathbb{R}).$$

Then $\exists$ a family $\mu : \mathcal{X} \to \Delta = \{|t| < 1\} \subseteq \mathbb{C}$

s.t. (1) $\mu^{-1}(0) = X_0 = \underline{X}$

(2) $\mu^{-1}(t) = X_t$ are smooth CY.

A Nbhd of $p_i \in \mathcal{X}$ is bihol'c to

$$\left\{ \sum_{i=1}^{4} z_i^2 = t \right\} \subseteq \mathbb{C}^4 \times \Delta.$$

This is a Conifold Transition.

$$\text{"} X \longrightarrow \underline{X} \rightsquigarrow X_t \text{".}$$

Rk: This is a Violent process

Replaces S^2's w/ S^3's.

Example: One can find examples where.

$[C_i]$ generate $H_2(X, \mathbb{R})$

$\Rightarrow X_t$ has $H^2(X_t, \mathbb{R}) = 0$.

So not Kähler.

This process puts all CY into a "web".

$\mathcal{M}_1, \mathcal{M}_2$ deformation families of CY mfds.

$\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ if a general elment of $\mathcal{M}_1$ can be connected to $\mathcal{M}_2$ by conifold transition.

(Rk: Could allow more general contractions)

<u>Reid's Fantasy</u>: The Web of CYs is connected.

<u>One difficulty</u>: Not many tools for studying non-Kähler CYs.

<u>Q</u>: Can we "uniformize" non-Kähler CYs?

<u>Option</u>: Turn to string theory for motivation

The Heterotic String

(Strominger, Candelas-Horowitz-Strominger-Witten in Kähler case)
Hull,

(X, Ω) CY, $E \to X$ hol'c bundle.

Look for $\begin{cases} H \text{ metric on } E \\ \omega \text{ metric on } X. \end{cases}$

s.t.

$$\begin{cases} (1) \quad d\left(\|\Omega\|_\omega \, \omega^2\right) = 0 \\ \qquad \|\Omega\|_\omega^2 = \dfrac{\Omega \wedge \bar{\Omega}}{\omega^3} \\ (2) \quad \omega^2 \wedge F_H = 0 \\ \qquad F_H = \text{curvature of Chern connection.} \end{cases}$$

Anomaly Cancellation (3)

$$\sqrt{-1}\,\partial\bar{\partial}\omega = \frac{\alpha'}{4}\left(\mathrm{Tr}(Rm\wedge Rm) - \mathrm{Tr}(F_H\wedge F_H)\right)$$

$$\alpha' > 0$$

<u>NB</u>: (3) $\Rightarrow$ $c_2(X) = c_2(E)$ in Bott-Chern Cohomology.

<u>Example</u> (CHSW)

if (X, ω_{CY}) Kähler CY, ω_{CY} Ricci-flat

$E = T^{1,0}X \to X$, $\quad H = \omega_{CY}$

Then this solves (1)+(2)+(3)

(1) $\|\Omega\|_\omega = 1$ since $\Omega \wedge \bar{\Omega} = c\,\omega^3$

$d\left(\|\Omega\|_\omega \omega^2\right) = 0$ since $d\omega = 0$.

(2) $\omega_{CY}^2 \wedge F_H = Ric(\omega_{CY}) = 0$

(3) Holds Trivially.

$\Rightarrow$ Strominger natural extension of Ricci-flatness to non-Kähler CYs.

Problem: No unique Vacuum. (Tons of Kähler CY!)

Hope: (1) All solutions of the Vaccum eq'ns can be connected by conifold Transitons (Reid!)

② The physics is well behaved through these transitions

Q: Can we pass solutions of the Hull-Strominger System through Conifold transitions?

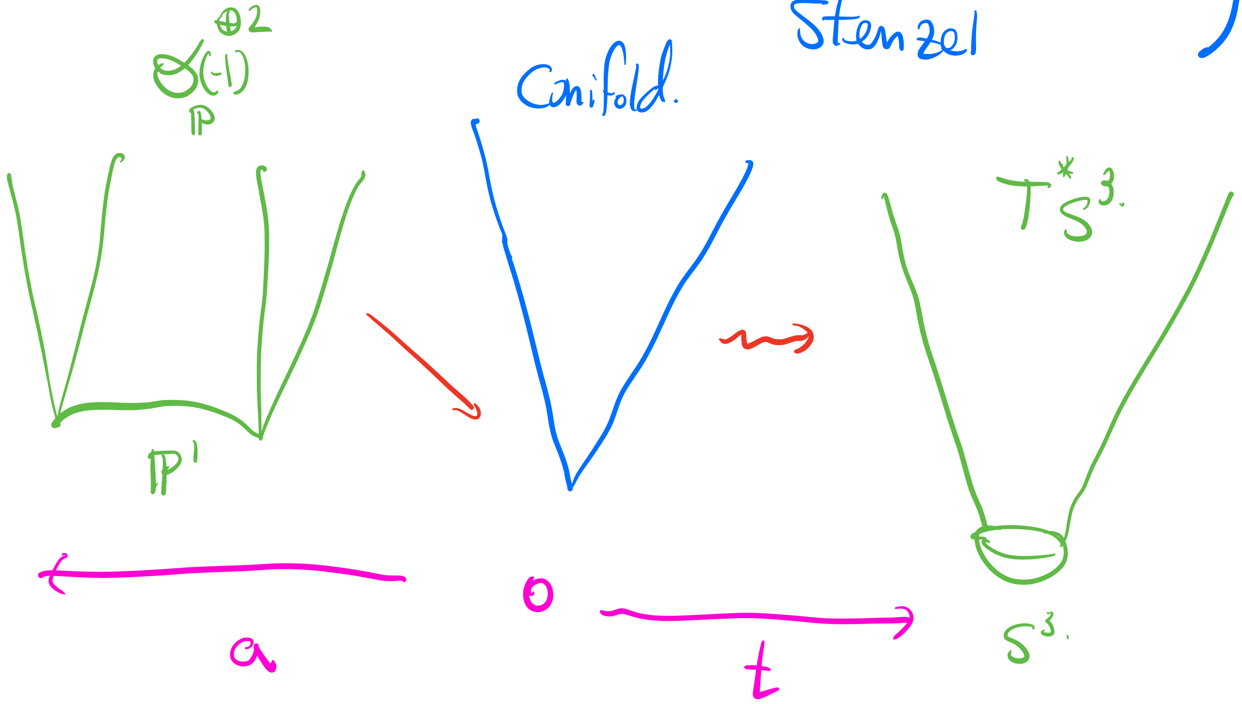
Theorem if X CY and $X \longrightarrow \underline{X} \rightsquigarrow X_t$ conifold transition then X_t admits solutions of (1)+(2) in the HS system for $E = T^{1,0}X_t$.

NB: Chuan proved (2) when E comes from X, and is hol'c trivial near C_i

(1) Conformally Balanced
due to Fu-Li-Yau

(2) C. - Picard-Yau.

Local Geometric Picture (Candelas - de la Ossa
Stenzel)



$\exists$ Ricci-flat Kähler metrics

$$\begin{cases} \bullet\ \omega_{co,a} \text{ on } \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \quad a>0 \text{ s.t.} \\ \qquad \mathrm{Vol}(\mathbb{P}^1, \omega_{co,a}) \sim a \to 0 \\ \bullet\ \omega_{co,0} \text{ Conical CY metric on Conifold.} \end{cases}$$

- $\omega_{co,t}$ on X_t (smooth Conifold).
  asymptotically conical.

$$\begin{array}{ccc} \left(\mathcal{O}_{\mathbb{P}}(-1)^{\oplus 2}, \omega_{co,a}\right) & \longrightarrow & (\text{conifold}, \omega_{co,0}) \\ & & \Big\uparrow \text{(rescaling)} \\ & & (X_t, \omega_{co,t}) \end{array}$$

Sketch Fu-Li-Yau:

(1) Fix ω_{CY} Kähler, Ricci-flat on X.

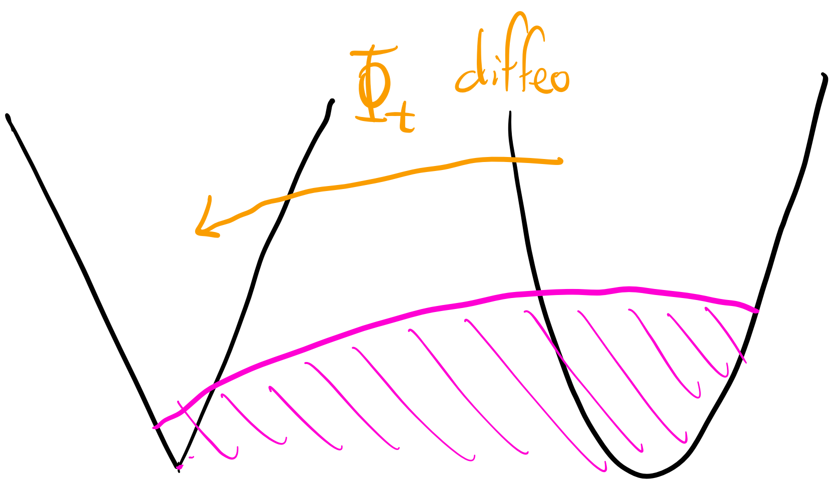
(2) glue (by partition of unity)

ω_{CY} to $\omega_{co,a}$ near C_i

to get $\omega_a > 0$

$d(\omega_a^2) = 0$, $[\omega_a^2] = [\omega_{CY}^2]$

$\omega_a = \omega_{co,a}$ near C_i.

(3) 

$$\left\{ \begin{array}{l} \left(\Phi_t^* \omega_{CY}\right)^{1,1} \text{ glue (by partition of unity)} \\ \text{to } \omega_{co,t} \text{ to get } \omega_t. \end{array} \right.$$

(4) [Perturb ω_t to satisfy $d(\omega_t^2) = 0$.

(This is where the Hard work happens).

Result is $\omega_{FLY,t}$

$$\omega_{FLY,t} \sim \omega_{co,t} \quad \text{near } P_i$$

Sketch of (2): have $\omega_{FLY,t}$

$$\omega_{FLY,t} \sim \omega_t$$

Suffices to solve (2) $\omega_t^2 \wedge F_{H_t} = 0$.

and then apply perturbation.

Step 1:

X Kähler CY, simply connected.

$\Rightarrow$ $T^{1,0}X$ is stable

wrt. $[\omega_{CY}^2] = [\omega_a^2]$

Li-Yau $\Rightarrow$ $\exists$ HYM metrics H_a

wrt. ω_a on $T^{1,0}X$.

<u>Step2</u>: $H_a \longrightarrow H_0$ smoothly on cpct sets of $\underline{X} \setminus \{p_i\}$.

as $a \to 0$.

$\Rightarrow H_0$ HYM on $\underline{X}_0$ wrt ω_0.

$\Rightarrow$ near p_i, H_0 HYM wrt $\omega_{co,0}$.

<u>Step3</u> Quantitative Convergence of H_0 to $\lambda_i \omega_{co,0}$.

<u>Key</u>: $T^{1,0}$ Conifold $= \pi^* E$.

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to E \to T^{1,0}\mathbb{P}^1 \times \mathbb{P}^1 \to 0.$$

E is stable and indecomposable.

(uses estimates of
Jacob - Sá Earp - Walpuski)

Step 4 glue $\left(\Phi_t^* H_0\right)^{1,1}$ to $\omega_{co,t}$.

Run singular perturbation argument to correct to HYM. ☒

IIB: $d\Omega = 0$ (complex integrable)

$d(\omega^2) = 0$ PD dual.

$2i\partial\bar\partial(e^{-2f}\omega) = \rho_B$ hol'c curve

IIA $d\omega = 0$.

$d\,\mathrm{Re}\,\Omega = 0$ P.D to

$\partial_+\partial_- * (e^{-2f}\mathrm{Re}\,\Omega) = \rho_A$ sLag.