Topology of extra-twisted connected sum G_2-manifolds

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Joint work with Diarmuid Crowley and Sebastian Goette
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These slides available at
`http://people.bath.ac.uk/jlpn20/xtcs_cbd.pdf`

Survey of topology of TCS and XTCS

Twisted connected sums generate huge numbers of closed G_2-manifolds, many 2-connected.

Crowley, Corti, Goette, Haskins, N, Pacini, Wallis

By now understand pretty much all there is to know about the topology of the underlying manifolds and the homotopy classes of the G_2-structures in the 2-connected case.

Crowley-Goette-N

Extra-twisted connected sums are less plentiful, many still 2-connected, but topology more complicated—in interesting ways.

One aspect of the G_2-structure well understood analytically.

In some cases we understand basic topological invariants, which sometimes is enough to pin down diffeomorphism type, but still much left.

Work in progress

For a class of extra-twisted connected sums "dual to simply-connected", there seems to be a way to construct a coboundary to compute all invariants (that are relevant in the 2-connected case).

Lennon, Harrison, McCartney, Starkey (1965)
Help! I need somebody
Help! Not just anybody
Help! You know I need someone

...who understands the topology of resolutions of a $\mathbb{C}^3/\mathbb{Z}_k$-bundle over a Riemann surface

Outline

1. Twisted connected sums and primary invariants
2. Coboundary defect invariants
3. Extra-twisted connected sums
4. Non-spin coboundaries of twisted connected sums

1. TCS and primary invariants
Twisted connected sum outline

Kovalev (2003), **Corti-Haskins-N-Pacini (2014)**.
Ingredients:

- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors ($[\Sigma] = c_1(Z)$), normal bundle trivial
- $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ diffeomorphism

Let $V_\pm := Z_\pm \setminus$ tubular neighbourhood $\Sigma_\pm \times \Delta$; so $\partial V_\pm = \Sigma_\pm \times S^1$.
Form simply-connected M^7 by gluing boundaries of $V_+ \times S^1$ to $V_- \times S^1$ by

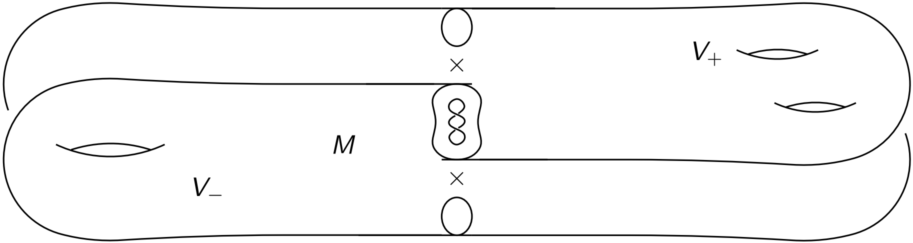
$$\Sigma_+ \times S^1 \times S^1 \to \Sigma_- \times S^1 \times S^1,$$
$$(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$$

Tian-Yau, **Haskins-Hein-N**:
$V_\pm$ admits asymptotically cylindrical Calabi-Yau metrics

$\rightsquigarrow$ metrics on $V_\pm \times S^1$ with holonomy $SU(3) \subset G_2$.

For "hyper-Kähler rotations" $\mathfrak{r}$, these glue to a holonomy G_2 metric on M.

Diagram of twisted connected sum



V_+, V_- asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces.

Truncate ends and glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles.

For large neck length, the G_2-structure on M obtained by gluing has $d\varphi = 0$ and $d^*\varphi$ "small", and can be perturbed to a torsion-free one.

Flipping the circles ensures M simply-connected, so $\mathrm{Hol}(M) = G_2$.

Primary invariants

Many examples of

- building blocks from Fano 3-folds (or weaker)
- hK rotations from deformation theory of Fano 3-folds + Torelli theorem

Compute the cohomology of the resulting TCS by Mayer-Vietoris

Many are 2-connected, *ie* $\pi_1 M$ and $\pi_2 M$ both trivial.

Then we can compute all the "primary" data.

- $H^4(M)$ (with $\mathbb{Z}$ coefficients)
- Spin characteristic class $p_M \in H^4(M)$ ($2p_M =$ Pontrjagin class $p_1(M)$)
- Torsion linking form $TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$

Many examples have torsion-free $H^4(M)$, so primary data reduces to $b_3(M)$ and $d :=$ greatest integer dividing p_M.

Topological classification

Theorem (Wilkens 1971)

Closed smooth 2-connected manifolds M with $H^4(M)$ torsion-free are classified up to homeomorphism by $b_3(M)$ and $d(M)$.
If $d(M)$ is not divisible by 8 or 7 then they also classify up to diffeomorphism.

Pay-off: Many different TCS realise the same smooth 7-manifold.

In general, classification also requires some "secondary" invariants defined via coboundaries (obstructions to improving a bordism to an h-cobordism)

- If one drops torsion-free hypothesis, then one needs to add a "quadratic refinement" of the torsion-linking form (Crowley)
- If $d(M)$ is divisible by 7 or 8 one needs a generalisation of the Eells-Kuiper invariant to detect smooth structure

2. Coboundary defect invariants
The Eells-Kuiper invariant

For a closed spin 8-manifold X, combining the Atiyah singer index theorem and the Hirzebruch signature theorem to eliminate $p_2(X)$ gives

$$\frac{p_1(X)^2 - 4\sigma(X)}{32} = 28 \operatorname{ind} \not{D}_X .$$

For compact spin 8-manifolds W with boundary M such that $p_1(M) = 0$, $\sigma(W)$ and $p_1(W)^2$ are both

- well-defined integers (use that $p_1(W)$ has a pre-image in $H^4(W, M)$)
- additive under gluing coboundaries.

Therefore

$$\mu(M) = \frac{p_1(W)^2 - 4\sigma(W)}{32} \in \mathbb{Z}/28$$

depends only on the boundary M of W, and not on W itself.

$\mu(M)$ distinguishes all 28 classes of smooth structures on S^7.

$d \neq 0$ and $H^4(M)$ torsion-free $\rightsquigarrow$ $\mu(M)$ well-defined in $\mathbb{Z}/\gcd(28, \tilde{d}/4)$,
for $\tilde{d} = \operatorname{lcm}(4, d)$.

G_2-structures and spinors

We found some spin coboundaries W of twisted connected sums, but they were too complicated to compute $p_1(W)^2$...
but they did have some explicit spinor fields (sections of real rank 8 bundle)

$$G_2\text{-structure on } M^7 \leftrightarrow$$
metric g + spin structure + nowhere vanishing spinor field s modulo scale

If W is a compact spin 8-manifold with boundary M, and s_+ is a positive spinor restricting to s, then $\# s_+^{-1}(0)$ depends only on W and s.

On a closed spin 8-manifold X, Hirzebruch signature theorem + Aityah-Singer + relation between Euler classes of spinor and tangent bundles $\Rightarrow$ can eliminate *both* $p_1(X)^2$ and $p_2(X)$ in favour of $\# s_+^{-1}(0)$ and $\chi(X)$:

$$\chi(X) - 3\sigma(X) - 2\,\# s_+^{-1}(0) = -48 \operatorname{ind} D_X$$

$\rightsquigarrow$ well-defined invariant $\nu \in \mathbb{Z}/48$ of (M, s), *ie* of M with G_2-structure.

ν of twisted connected sums

$$\nu(\varphi) = \chi(W) - 3\sigma(W) - 2\,\#s_+^{-1}(0) \in \mathbb{Z}/48$$

Also clear that ν is invariant under continuous deformation of a G_2-structure (fibre-homotopy).

Theorem

If M is closed 2-connected, $H^4(M)$ torsion-free and greatest divisor d of p_M divides $16 \cdot 7$, then there are exactly 24 equivalence classes of G_2-structures on M up to diffeomorphism and homotopy, distinguished by ν.

Can we use ν to distinguish G_2-structures of different twisted connected sums on the same underlying manifold? No.

Theorem (Crowley-N)

Any twisted connected sum has $\nu = 24$.

Twisted connected sums always have d dividing 24, so unless d is divisible by 3, there is no chance at all to use the homotopy class of the G_2-structure to distinguish components of the G_2 moduli space.

Analytic refinement

One can use eta invariants to define an analytic invariant $\bar{\nu} \in \mathbb{Z}$, refining ν in the sense that $\nu = \bar{\nu} + 24 \mod 48$.

$\bar{\nu}$ can jump by 48 if one deforms through non-torsion-free G_2-structures, so can distinguish components of the G_2 moduli space even when the G_2-structures are homotopic.

Can that distinguish twisted connected sums? No.

Theorem (Crowley-Goette-N)

Any twisted connected connected sum has $\bar{\nu} = 0$.

Proof.

Boundary conditions $b_\pm \rightsquigarrow$ well-defined $\bar{\nu}$ for each half $M_\pm = V_\pm \times S^1$.

$$\bar{\nu}(M) = \bar{\nu}(M_+, b_+) + \bar{\nu}(M_-, b_-) + G(b_+, b_-).$$

For obvious choices of $b_\pm$, $\bar{\nu}(M_\pm, b_\pm) = 0$ by spectral symmetry. Then $G(b_+, b_-) = 0$ because the external circle direction in the cross-sections $K3 \times T^2$ are aligned at right angle in the gluing. □

3. Extra-twisted connected sums
Tori

Recall:

From a building block (Z, Σ) we get an ACyl Calabi-Yau 3-fold $V := Z \setminus \Sigma$ with cylindrical end $\mathbb{R} \times S^1 \times \Sigma$. Think of this circle factor as "internal".

Now suppose the building block (Z, Σ) has a cyclic automorphism group Γ that fixes Σ pointwise.
Then the action of Γ on V acts trivially on the Σ factor in the asymptotic end while rotating the S^1_{int} factor.

Next choose a free action of Γ on "external" circle S^1_{ext}.
Then $(S^1_{ext} \times V)/\Gamma$ is a smooth ACyl G_2-manifold. Its asymptotic end is of the form $\mathbb{R} \times T^2 \times \Sigma$, but the torus $T^2 := (S^1_{ext} \times S^1_{int})/\Gamma$ need *not* be a metric product of two circles.
The geometry of T^2 depends on the circumferences of S^1_{ext} and S^1_{int}, which can be chosen freely.

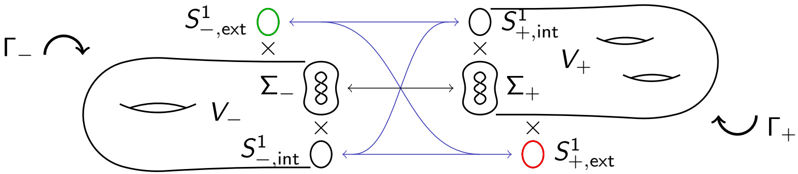
Adding the extra twist

To make an extra-twisted connected sum

- Find some building blocks $(Z_\pm, \Sigma_\pm)$ with automorphism groups $\Gamma_\pm$
- Choose circumferences so that there is an isometry $\mathfrak{t} : T^2_+ \to T^2_-$
- Find ACyl Calabi-Yau metrics so that there is $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ that makes

$$(-1) \times \mathfrak{t} \times \mathfrak{r} : \mathbb{R} \times T^2_+ \times \Sigma_+ \to \mathbb{R} \times T^2_- \times \Sigma_-$$

an isomorphism of the asymptotic limits of the G_2-structures.

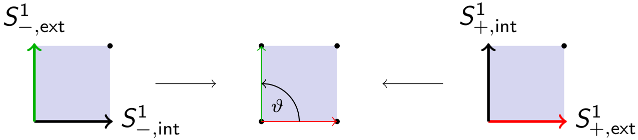


Inflexibility of the gluing angle for TCS

In the twisted connected sum construction we identify the asymptotic cross-sections $S^1_{+,ext} \times S^1_{+,int} \times \Sigma_+$ and $S^1_{-,ext} \times S^1_{-,int} \times \Sigma_-$ by the product of an isometry $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ and the "flip" isometry

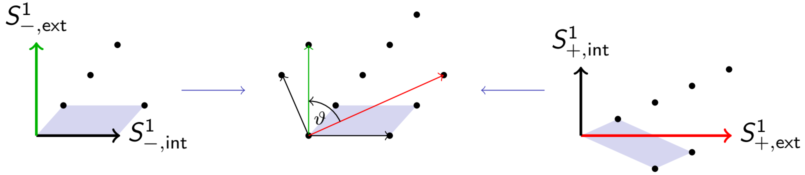
$$S^1_{+,ext} \times S^1_{+,int} \to S^1_{-,ext} \times S^1_{-,int}, \quad (u, v) \mapsto (v, u).$$

We can choose the circumferences of $S^1_{+,ext} = S^1_{-,int}$, and $S^1_{-,ext} \cong S^1_{+,int}$, but the angle ϑ between the external circle direction will *always* be $\frac{\pi}{2}$.



More exciting torus isometries

As soon as at least one of the tori T^2_+ and T^2_- is not simply an isometric product $S^1_{ext} \times S^1_{int}$, there are other possibilities for the gluing angle ϑ.



$\Gamma_- \cong \mathbb{Z}/3$ $\qquad$ $\Gamma_+ \cong \mathbb{Z}/4$

eg $\qquad \vartheta = \dfrac{3\pi}{4},\ \dfrac{2\pi}{3}$ or $\arccos\left(\dfrac{1}{\sqrt{6}}\right)$.

Using $\bar{\nu}$ to disconnect the G_2 moduli space

$$\bar{\nu}(M) = \bar{\nu}(M_+, b_+) + \bar{\nu}(M_-, b_-) + G(b_+, b_-).$$

As soon as the "gluing angle" $\vartheta \neq \frac{\pi}{2}$, the gluing term $G(b_+, b_-)$ can be non-zero.

If $M_\pm = (V_\pm \times S^1)/\mathbb{Z}_2$ then

- we can work out primary invariants
- still have spectral symmetry so that $\bar{\nu}(M_\pm, b_\pm) = 0$

Pay-off: Can find diffeomorphic TCS and XTCS, with components of G_2 moduli space distinguished by $\bar{\nu}$.
In some cases, the G_2-structures are nevertheless homotopic.

If $M_\pm = (V_\pm \times S^1)/\mathbb{Z}_k$ for $k \geq 3$ then $\bar{\nu}(M_\pm, b_\pm)$ can be non-zero
$\rightsquigarrow$ examples with $\bar{\nu}(M)$ not divisible by 3.

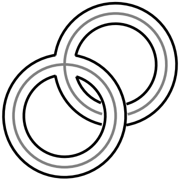
4. Non-spin coboundaries of TCS
Plumbing coboundary of TCS

Recall: Twisted connected sum was defined from

- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors with trivial normal bundle
- Hyper-Kähler rotation $\mathfrak{r} : \Sigma_+ \to \Sigma_-$

$\Sigma_\pm$ has tubular neighbourhood of the form $\Sigma_\pm \times \Delta$ (for disc $\Delta \subset \mathbb{C}$).
Form an 8-manifold W by "parametric plumbing along the $K3$":
glue $Z_+ \times \Delta$ to $Z_- \times \Delta$ along open subsets

$$\Sigma_+ \times \Delta \times \Delta \to \Sigma_- \times \Delta \times \Delta,$$
$$(x, z, w) \mapsto (\mathfrak{r}(x), w, z)$$



Then ∂W is the twisted connected sum M.
W not spin, but spin^c good enough to compute generalised Eells-Kuiper.

Pay-off: Examples of TCSs that are homeomorphic but not diffeomorphic.
(Needs some work to find the right hyper-Kähler rotations)

Almost complex coboundaries

When computing Eells-Kuiper, we can use spin^c coboundaries, applying Atiyah-Singer index theorem to twisted Dirac operator.

But apparently little prospect of using spin^c coboundaries to compute ν, since the twisted spinor bundle is complex of rank 8 and thus does not have useful Euler class.

Wallis (2018)
For $U(3)$-structures on closed 7-manifolds, one can define an invariant by evaluating

$$\chi(W) - 3\sigma(W) - c_1(W)c_3(W) \in \mathbb{Z}/48$$

for any $U(4)$-boundary W.
For any $SU(3)$ reduction of a G_2-structure φ, this coincides with $\nu(\varphi)$.

The plumbing coboundaries of TCS have $U(4)$-structure
$\rightsquigarrow$ another (third!) computation $\nu = 24$.

Final ingredient for 2-connected classification

Another way to combine Hirzebruch signature theorem, Atiyah-SInger index theorem and relation between Euler classes $\rightsquigarrow$

$$\xi(M,s) := \frac{3p_1(W)^2 - 180\sigma(W)}{8} + 7\chi(W) - 14\,\#s_+^{-1}(0) \in \mathbb{Z}/3\tilde{d}$$

is a well-defined invariant of G_2-structures on closed 7-manifold M with torsion-free $H^4(M)$.
(ν and ξ together determine Eells-Kuiper by $\mu = \frac{\xi - 7\nu}{12}$.)

Theorem (Crowley-N)

Closed 2-connected 7-manifolds M with G_2-structures, with $H^4(M)$ torsion-free, are classified up to diffeomorphism + homotopy of G_2-structure by $b_3(M), d, \nu$ and ξ.

Wallis: ξ can also be computed by $U(4)$-coboundaries
$\rightsquigarrow$ pairs of TCS that are diffeomorphic, but G_2-structures are not homotopic

Coboundaries of some XTCS

Let $B_\pm \to Z_\pm$ disc bundle with c_1 a multiple of $\Sigma_\pm$.

Lift action of cyclic group $\Gamma_\pm$ to $B_\pm$, act trivially on fibres over $\Sigma_\pm$

$\rightsquigarrow$ this fixed component yields smoothable singularities in $W_\pm := B_\pm/\Gamma_\pm$

$\rightsquigarrow$ $W_\pm$ contains region $\Sigma_+ \times \Delta \times \Delta$

Gluing by map that swaps Δ factors
$\rightsquigarrow$ compact orbifold W^s whose boundary is an XTCS
(but only some XTCS can possibly be realised this way)

Γ also has a fixed component in $Z_\pm$ that is a curve C
$\rightsquigarrow$ singularities in W^s modelled on $\mathbb{C}^3/\Gamma$ bundle over C.

For $\Gamma = \mathbb{Z}/2$ we can simply blow up

For a special case of $\Gamma = \mathbb{Z}/3$ I could carry out computations by working out an explicit resolution