Invariants of twisted connected sum G_2-manifolds

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These slides available at http://people.bath.ac.uk/jlpn20/InvariantsXTCS.pdf

Topology of examples of G_2-manifolds

What can we say about topology of examples of closed Riemannian 7-manifolds with $Hol = G_2$?

- **Orbifold resolutions (Joyce 1995)**

  Simply connected, but non-trivial π_2 and infinite H^4

  Outside scope of current classification results.

- **Twisted connected sums (Kovalev 2003, Corti-Haskins-N-Pacini 2013)**

  Gluing construction starting from algebraic pieces

  Often 2-connected (*ie* π_1, π_2 both trivial)

  Then we can compute from data of the algebraic pieces all invariants needed to determine diffeomorphism class together with homotopy class of the G_2-structure.

  - G_2-manifolds that are homeomorphic but not diffeomorphic **(Crowley-N 2014)**
  - G_2-manifolds diffeomorphic to total spaces of circle bundles **(Crowley-N 2019)**
  - 7-manifolds with disconnected G_2 moduli space **(Wallis 2018)**

Extra-twisted connected sums

- **Extra-twisted connected sums (Crowley-Goette-N 2015)**

  Divide by finite group before gluing to achieve greater variation in topology

  Even "primary" data like cohomology and Pontrjagin class fully computed only for some classes.

  In addition, a spectral invariant $\overline{\nu}$ helps exhibit

  - 7-manifolds where the G_2 moduli space has more than two components, and where different components belong to the same homotopy class of G_2-structures
  - G_2-manifolds that are non-trivial in the bordism group $\Omega_7^{G_2} \cong \mathbb{Z}/3$

Singular coboundaries

Most recent progress

An avenue to determining all topological classifying invariants for all 2-connected extra-twisted connected sums by explicit coboundaries with singularities.

For some classes it is tractable to compute from an explicit resolution, yielding

- Holonomy G_2 metrics on the same 7-manifold representing different elements of $\Omega_7^{G_2}$
- 7-manifolds where components of G_2 moduli space are distinguished by virtue of quadratic refinement of torsion linking form ruling out orientation-reversing diffeomorphism

Outline

1. Diffeomorphism classification
2. Topological invariants of G_2-structures
3. The spectral invariant
4. Coboundaries of extra-twisted connected sums

1. Diffeomorphism classification
Primary invariants

Let M be a closed 7-manifold, and suppose that M is 2-connected, *ie* $\pi_1(M) = \pi_2(M) = 0$. Then the most obvious remaining topological invariants are

- Cohomology in degree $H^4(M)$
  This determines the rest of $H^*(M)$ and $H_*(M)$ by Poincaré duality
- A spin characteristic class $p_M \in H^4(M)$
  Determines the Pontrjagin class by $p_1(M) = 2p_M$, so p_M determined by $p_1(M)$ if $H^4(M)$ is 2-torsion free.
- The torsion linking for $b_M : TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$

Theorem (Wilkens 1972)

If $H^4(M)$ is 2-torsion free then $(H^4(M), p_M, b_M)$ determines the homeomorphism type of M.
If p_M is not divisible by 7 or 8 then the diffeomorphism type is determined too.
(If $H^4(M)$ is torsion-free then we just need to know $\operatorname{rk} H^4(M) = b_3(M)$ and the greatest integer dividing p_M.)

Primary data of twisted connected sums

Kovalev (2003)

Gluing construction of closed G_2-manifolds, using Fano 3-folds as building blocks

Fano 3-fold: Smooth closed complex manifold Y with positive anticanonical bundle $-K_Y$ ($\Leftrightarrow c_1(Y)$ represented by Kähler form), *eg* $\mathbb{P}^3$.

Corti-Haskins-N-Pacini (2013)

Millions of examples by using "weak" Fano 3-folds

Many are 2-connected, and primary data can be computed from known data of the algebraic building blocks.

Most have $H^4(M)$ torsion-free and Wilkens theorem $\rightsquigarrow$

M homeomorphic to a connected sum of 3-sphere bundles over S^4
(and smooth structure usually determined too)

Some are homeomorphic to circle bundles

Secondary invariants

Completing the diffeomorphism classification of 2-connected 7-manifolds requires

- "Family of quadratic refinements" q_M of the torsion linking form b_M to eliminate ambiguity in homeomorphism class when $H^4(M)$ has 2-torsion
- Generalisation μ_M of Eells-Kuiper invariant to distinguish different smooth structures on same topological manifold.

Theorem (Crowley 200X)

The homeomorphism type of a closed 2-connected 7-manifold M is determined by the isomorphism class of $(H^4(M), p_M, q_M)$

Theorem (Crowley-N)

The diffeomorphism type of a closed 2-connected 7-manifold M is determined by the isomorphism class of $(H^4(M), p_M, q_M, \mu_M)$

Coboundary defect invariants

For a closed spin 8-manifold X

$$y^2 = p_X y \mod 2 \quad \text{for all } y \in H^4(X)$$
$$\frac{p_W^2 - \sigma(X)}{8} = 0 \mod 28$$

where $\sigma(X)$ is the signature of the intersection form.
q_M and μ_M measure the failure of these relations to hold on a compact spin 8-manifold with boundary M.

If $p_M = 0$ then

- $q_M : TH^4(M) \to \mathbb{Q}/\mathbb{Z}$ such that $q(x+y) = q(x) + q(y) + b(x,y)$ and $q(x) = q(-x)$.
- μ_M is the classical Eells-Kuiper invariant, a constant in $\mathbb{Q}/28\mathbb{Z}$
  Its image in $\mathbb{Q}/\mathbb{Z}$ is determined by q_M, so can distinguish 28 different smooth structures

Exotic G_2-manifolds

If $H^4(M)$ is torsion-free then

- q_M is vacuous
- $\mu_M \in \mathbb{Z}/\gcd\left(28, \mathrm{Numerator}(\frac{d}{4})\right)$, where d is the greatest integer dividing p_M.

For any twisted connected sum d divides $p_{K3} = 24$, so μ_M can only be non-trivial in small minority of examples where d is divisible by 8.

Example (Crowley-N 2014)

There are 2-connected twisted connected sums with $H^4(M) \cong \mathbb{Z}^{89}$ and $d = 8$ and $\mu_M = 0, 1$.

$\rightsquigarrow$ pair of closed G_2-manifolds that are homeomorphic but not diffeomorphic.

We computed μ_M from an explicit spin^c coboundary.

2. Invariants of G_2-structures
The homotopy invariant $\nu \in \mathbb{Z}/48$

On a spin 8-manifold X, the spinor bundle S_X is real of rank 8.

If X is closed and $s_+ \in \Gamma(S_X)$ has transverse zeros, then $\# s_+^{-1}(0)$ (counted with signs) does not depend on s_+. It equals the Euler class $e(S_X)$, related to Euler characteristic $\chi(X)$ by

$$-3\sigma(X) + \chi(X) - 2\,\# s_+^{-1}(0) = -48 \operatorname{ind} D_X$$

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For W compact spin 8-manifold with boundary M, $\# s_+^{-1}(0)$ of $s_+ \in \Gamma(S_W)$ depends only on W and $s := s_{+|M} \in \Gamma(S_M)$. Therefore

$$\nu(M, s) := 3\sigma(X) + \chi(X) - 2\,\# s_+^{-1}(0) \in \mathbb{Z}/4$$

8 is a well-defined diffeomorphism invariant of (M, s), *ie* of M equipped with a G_2-structure.

Also clear that ν is invariant under continuous deformation of a G_2-structure.

Classification of G_2-structures

Further relation on closed spin 8-manifolds X

$$\frac{3p_1(X)^2 - 180\sigma(X)}{8} + 7\chi(X) - 14\,\#s_+^{-1}(0) = 0$$

"causes" a second invariant $\xi(M, s)$. Like μ_M its type is complicated in general, but for $H^4(M)$ torsion-free it is a constant in $\mathbb{Z}/3\,\mathrm{lcm}(d, 4))$.

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Theorem (Crowley-N 2015)

Let M_i be closed 2-connected 7-manifolds with torsion-free $H^4(M_i)$, and G_2-structures φ_i. Then there is a diffeomorphism $f : M_1 \to M_2$ such that $f^*\varphi_2$ is homotopic to φ_1 if and only if $(H^4(M_1), p_{M_1}, q_{M_1}, \nu(\varphi_1), \xi(\varphi_1)) \cong (H^4(M_2), p_{M_2}, q_{M_2}, \nu(\varphi_2), \xi(\varphi_2))$

ν and ξ always determine μ_M via

$$12\mu = \xi - 7\nu \mod \gcd(12 \cdot 28, 3\,\mathrm{lcm}(d, 4)).$$

If d divides 112 this also determines ξ from μ_M and ν.

Invariants of twisted connected sums

ν fails to distinguish components of the G_2 moduli space reached by twisted connected sums.

Theorem (Crowley-N 2015)

Any twisted connected sum G_2-manifold has $\nu = 24$.

Because the greatest divisor d of p_M divides $p_{K3} = 24$ for any twisted connected sum, ξ is determined by ν and μ_M unless d is divisible by 3.

For a 2-connected 7-manifold with d not divisible by 3, this means that any G_2-structures realised from twisted connected sums *must* be homotopic.

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Example (Wallis 2019)

$\exists$ 2-connected twisted connected sums with $H^4(M) \cong \mathbb{Z}^{71}$, $d = 6$ and $\xi = 0, 12 \in \mathbb{Z}/36$.

$\rightsquigarrow$ underlying 7-manifolds are diffeomorphic, but the G_2-structures are not homotopic, so the G_2 moduli space is disconnected.

Invariants of $U(3)$-structures

For a closed 8-manifold X with almost complex structure

$$-3\sigma(X) + \chi(X) - c_1(X)c_3(X) = -48\,\mathrm{ind}(\partial + \bar{\partial}),$$

$$\frac{-(c_1(X)^2)^2 + 3c_2(X)^2 - 45\sigma(X)}{2} + 7\chi(X) - 7c_1(X)c_3(X) = 0.$$

$\rightsquigarrow$ invariants of $U(3)$-structures on closed 7-manifolds, analogous to ν and ξ for G_2-structures.

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For a compact W with $SU(4)$-structure, the formulas for the invariants of the $SU(3)$-structure on ∂W and of the induced G_2-structureare the same.

Therefore ν and ξ of any G_2-structure can be computed by coboundaries with $U(4)$-structure instead of spin structure!

Twisted connected sums

Ingredients:

- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors ($[\Sigma_\pm] = c_1(Z_\pm)$) with trivial normal bundle
- $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ diffeomorphism

Let $V_\pm := Z_\pm \setminus$ tubular neighbourhood $\Sigma_\pm \times \Delta$; so $\partial V_\pm = \Sigma_\pm \times S^1$.
Form simply-connected M^7 by gluing boundaries of $V_+ \times S^1$ to $V_- \times S^1$ by

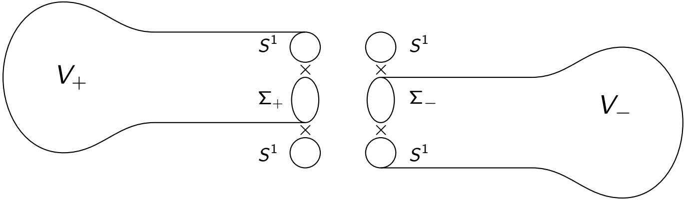
$$\Sigma_+ \times S^1 \times S^1 \to \Sigma_- \times S^1 \times S^1,$$
$$(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$$

Tian-Yau, **Haskins-Hein-N**: $V_\pm$ admits asymptotically cylindrical Calabi-Yau metrics.
$\rightsquigarrow$ metric on $V_\pm \times S^1$ with holonomy $SU(3) \subset G_2$.

For "hyper-Kähler rotation" $\mathfrak{r}$, these metrics glue to a holonomy G_2 metric on M.

Coboundary of twisted connected sum

$V_\pm := Z_\pm \setminus \Sigma_\pm \times \Delta$. Glue the $\partial(V_+ \times S^1) = \Sigma_+ \times S^1 \times S^1$ by $(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$.



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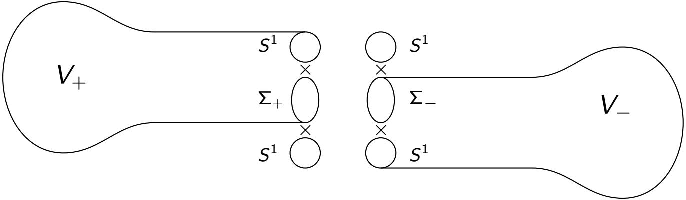
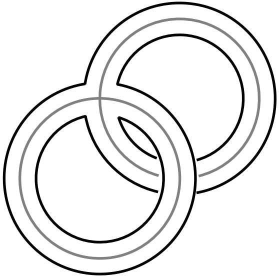


Form an 8-manifold W by gluing $Z_+ \times \Delta$ to $Z_- \times \Delta$ along open subsets

$$\Sigma_+ \times \Delta \times \Delta \to \Sigma_- \times \Delta \times \Delta,$$
$$(x, z, w) \mapsto (\mathfrak{r}(x), w, z)$$



Then ∂W is the twisted connected sum M.
W has a $U(4)$-structure compatible with the G_2-structure on M.

3. Analytic invariant and extra-twisted connected sums
Definition of the analytic invariant

Given a metric g on a closed spin M^7, define

$$\overline{\nu}(g) := -24\eta(D) + 3\eta(B) \in \mathbb{R}$$

where $D =$ Dirac operator, $B : \Omega^{ev} \to \Omega^{ev} =$ odd signature operator,
$\eta(D) := \eta(D, 0) \in \mathbb{R}$ is defined by analytic continuation from

$$\eta(D, s) := \sum_{\lambda \in \mathrm{Spec} D \setminus \{0\}} (\mathrm{sign}\ \lambda)|\lambda|^{-s} \quad \text{for } \mathrm{Re}\, s \gg 0.$$

Theorem (Crowley-Goette-N)

$\overline{\nu}(g) \in \mathbb{Z}$ for any metric with holonomy G_2, and the associated G_2-structure s has

$$\nu(s) = \overline{\nu}(g) + 24 \mod 48.$$

$\overline{\nu}(g)$ is invariant under deformations through holonomy G_2 metrics, so can distinguish components of the G_2 moduli space even when the associated G_2-structures are homotopic

Gluing formula for analytic invariant

Theorem (Crowley-Goette-N)

Any twisted connected sum has $\overline{\nu} = 0$.

Proof.

$\overline{\nu}$ constant under stretching neck, with limit

$$\overline{\nu}(M_+) + \overline{\nu}(M_-) - 72\frac{\rho}{\pi} + 3m$$

for

- $\rho := \pi - 2\vartheta$, where ϑ is the angle between the "external" circle factors under the gluing map $S^1 \times S^1 \times \Sigma_+ \to S^1 \times S^1 \times \Sigma_-$
- $m \in \mathbb{Z}$ depending on ρ and action of hyper-Kähler rotation $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ on H^2

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Spectral symmetry for $M_\pm \Rightarrow \overline{\nu}(M_\pm) = 0$.

But ϑ is always $\frac{\pi}{2}$, so $\rho = 0$, and that makes $m = 0$ too! □

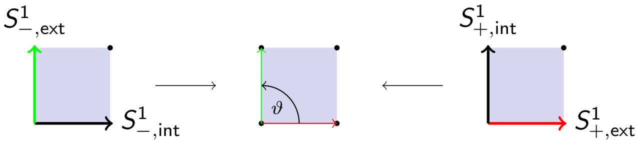
Inflexibility of the gluing angle for twisted connected sums

Recall twisted connected sum set-up:

ACyl Calabi-Yau 3-fold V with cylindrical end $\mathbb{R} \times S^1 \times \Sigma$. Think of this circle as "internal".
Consider $S^1_{\text{ext}} \times V$ with asymptotic cross-section $S^1_{\text{ext}} \times S^1_{\text{int}} \times \Sigma$.
Identify $S^1_{+,ext} \times S^1_{+,int} \times \Sigma_+$ and $S^1_{-,ext} \times S^1_{-,int} \times \Sigma_-$ by the product of an isometry $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ and the "flip" isometry

$$\mathfrak{t} : S^1_{+,ext} \times S^1_{+,int} \to S^1_{-,ext} \times S^1_{-,int}, \quad (u, v) \mapsto (v, u).$$

We can choose the circumferences of $S^1_{+,ext} = S^1_{-,int}$, and $S^1_{-,ext} \cong S^1_{+,int}$, but the angle ϑ between the external circle direction will *always* be $\frac{\pi}{2}$.



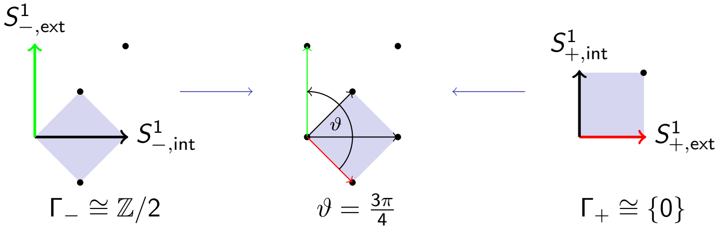
Tori

Now suppose V has a cyclic automorphism group Γ acts trivially on the Σ factor in the asymptotic end while rotating the S^1_{int} factor.

Next choose a free action of Γ on "external" circle S^1_{ext}.

Then $(S^1_{\text{ext}} \times V)/\Gamma$ is a smooth ACyl G_2-manifold. Its asymptotic end is still of the form $\mathbb{R} \times T^2 \times \Sigma$, but the torus $T^2 := (S^1_{\text{ext}} \times S^1_{\text{int}})/\Gamma$ need *not* be a metric product of two circles.

For a pair k_+, k_-, there can be different choices of circumferences for $S^1_{\pm,\text{int}}$ and $S^1_{\pm,\text{ext}}$ for which there exists an isometry $\mathrm{t} : (S^1_{+,\text{ext}} \times S^1_{+,\text{int}})/(\mathbb{Z}/k_+) \to (S^1_{-,\text{ext}} \times S^1_{-,\text{int}})/(\mathbb{Z}/k_-)$



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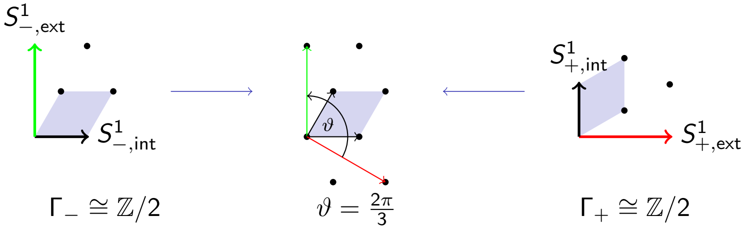

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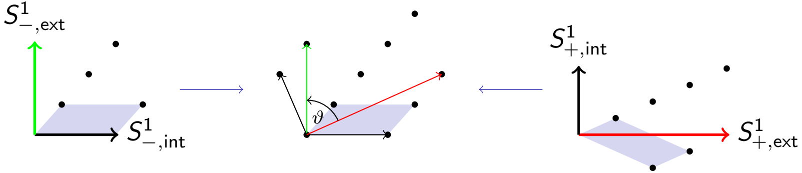

$\Gamma_- \cong \mathbb{Z}/3$ $\qquad$ $\vartheta = \arccos\left(\frac{1}{\sqrt{6}}\right)$ $\qquad$ $\Gamma_+ \cong \mathbb{Z}/4$

Extra-twisted connected sums

One can find V with the required $\mathbb{Z}/k$ action *eg* from branched covers of Fano 3-folds Y where $-K_Y$ is divisible by k.

Given torus isometry $\mathrm{t} : (S^1_{+,\mathrm{ext}} \times S^1_{+,\mathrm{int}})/(\mathbb{Z}/k_+) \to (S^1_{-,\mathrm{ext}} \times S^1_{-,\mathrm{int}})/(\mathbb{Z}/k_-)$, it is often possible to find isometries $\mathrm{r} : \Sigma_+ \to \Sigma_-$ so that $\mathrm{t} \times \mathrm{r}$ matches up ACyl G_2-structures of $M_+ = (V_+ \times S^1_{+,\mathrm{ext}})/(\mathbb{Z}/k_+)$ and $M_- = (V_- \times S^1_{-,\mathrm{ext}})/(\mathbb{Z}/k_-)$

$\rightsquigarrow$ closed G_2-manifold M.

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Two features of the torus isometry $\mathfrak{t}$ are reflected in primary topological data of M.

$p :=$ index of lattice generated by internal directions $= |\pi_1(M)|$

$n :=$ index of lattice generated by external directions "causes" n-torsion in $H^4(M)$

If we set $p = 1$ then usually get 2-connected examples.

If also $n = 1$ we have a fair chance to compute the primary data.

Analytic invariant of extra-twisted connected sums

Crowley-Goette-N For $k_\pm = 2$

- $\overline{\nu}(M_\pm) = 0$ due to spectral symmetry,
- boundary term in gluing formula for $\overline{\nu}$ can now be non-zero (but always divisible by 3)
- computed primary topological invariants, which usually suffice to classify

$\rightsquigarrow$ examples with disconnected G_2 moduli space

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Goette-N For $k_\pm \geq 3$

- $\overline{\nu}(M_\pm)$ non-zero
- $\overline{\nu}(M)$ need not be divisible by 3

$\rightsquigarrow$ $[M] \neq 0 \in \Omega_7^{G_2}$ (M not the boundary of any 8-manifold with G_2-structure)

4. Coboundaries of extra-twisted connected sums
Singular coboundary

Naively fill in S^1_{ext} by unit disc Δ and consider

$$W_\pm := (V_\pm \times \Delta)/(\mathbb{Z}/k_\pm)$$

It has

- two boundary pieces $V_\pm \times S^1_{\pm,\text{ext}}$ and $S^1_{\text{int}} \times \Delta \times \Sigma$, meeting along
- corner $S^1_{\text{ext}} \times S^1_{\text{int}} \times \Sigma$
- Singularities of codimension 3 and possibly also 4 in interior from fixed set

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Stitch W_+ to W_- along corner $\rightsquigarrow$ singular 8-manifold with

- "outside" boundary the extra-twisted connected sum M
- "inside" boundary $(S^1_{+,\text{int}} \times \Delta_+ \cup S^1_{-,\text{int}} \times \Delta_-) \times \Sigma \cong S^3/(\mathbb{Z}/n) \times \Sigma$.

Fill in inside boundary with $(\Delta \times \Delta)/(\mathbb{Z}/n) \times \Sigma \rightsquigarrow$ singular W with $\partial W = M$.

Resolving

If $n > 1$, the codimension 2 singularities from $(\Delta \times \Delta)/(\mathbb{Z}/n)$ have a well understood resolution.

Enough examples lack isolated fixed points to happily assume there are none.

A generator of $\mathbb{Z}/k$ acts

- on normal bundle of fixed curve $C \subset V$ with weights $(1, -1)$
- on Δ factor with some weight ϵ coprime to k.

For $k = 2, 3, 4, 6$ only possibility is $\epsilon = \pm 1$, so singularities have type $\frac{1}{k}(1, -1, -1)$.

Weighted blow-up $\rightsquigarrow$ codimension 2 singularities of type $\frac{1}{k-1}(1, -1) = A_{k-2}$,
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The resulting computation for ν recovers the value implied by the gluing formula
$\overline{\nu}(M) = \overline{\nu}(M_+) + \overline{\nu}(M_-) - 72\frac{\rho}{\pi} + 3m$. E.g.

- The signature of the complement to the singuarities in W equals the integer term m.
- The resolution of the codimension 2 singularities contributes a Dedekind sum that **Zagier** identified as a term in $\overline{\nu}(M_+) + \overline{\nu}(M_-) - 72\frac{\rho}{\pi}$

Torsion data

If we can compute the intersection form on $H^4(W, M)$ with integer coefficients then we can also determine $TH^4(M)$, torsion linking form b_M and its quadratic refinement q_M.

It now helps to assume

- $n = 1$, to avoid additional n-torsion
  Then W is a union of two pieces $B_\pm$, resolutions of quotients of total spaces of line bundles on the closed complex 3-manifolds $Z_\pm$ such that $V_\pm = Z_\pm \setminus \Sigma$.
- That $Z_\pm$ are the result of blowing up k-fold cover $X \to Y$ over Σ, where Y is a semi-Fano 3-fold with $-K_Y$ divisible by k.
  Then $B_\pm$ can be obtained from a line bundle over Y by a sequence of smooth blow-ups.

Then $H^4(W, M) \cong H^2(Y_+) \oplus H^2(Y_-) \oplus$ unimodular piece from exceptional divisors.

Intersection form on $H^2(Y_+) \oplus H^2(Y_-)$ is determined by combinatorics of $\mathfrak{t}$ and intersection form on sum of images of $H^2(Y_\pm)$ in $H^2(\Sigma)$ ($\Leftrightarrow$ action $\mathfrak{r}^* : H^2(\Sigma_-) \to H^2(\Sigma_+)$).

An ACyl Calabi-Yau with order 3 automorphism

Let

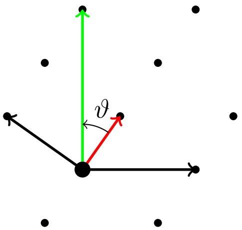
- Y_+ small resolution of the quadric cone $\{x_0 x_1 = x_2 x_3\} \subset \mathbb{P}^4$.
- $\Sigma \subset Y_+$ a section by a cubic.
- $C \subset \Sigma$ section by a plane
- $X_+ \to Y_+$ 3-fold cover branched over Σ
- $Z_+ \to X_+$ blow-up in C
- $V_+ = Z_+ \setminus \Sigma$

Then V_+ has action of desired form by $\mathbb{Z}/3$, with fixed set a copy of C.
Intersection form on $\mathrm{Im}(H^2(Y_+) \to H^2(\Sigma))$ is

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

A matching

Obtain V_- (without automorphisms) in a similar way from $\mathbb{P}^3$.



Glue $(V_+ \times S^1)/(\mathbb{Z}/3)$ to $V_- \times S^1$ using torus isometry with circumferences

$$\begin{aligned} S^1_{+,\mathrm{ext}} &: \sqrt{3} \\ S^1_{+,\mathrm{int}} &: \sqrt{6} \\ S^1_{-,\mathrm{ext}} &: 1 \\ S^1_{-,\mathrm{int}} &: \sqrt{2} \end{aligned}$$

and $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ encoded by the matrix

$$\begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & 4 \end{pmatrix};$$

the angle between $(1,1) \in H^2(X_+)$ and generator of $H^2(\mathbb{P}^3)$ equals $\vartheta = \arccos\left(\sqrt{\frac{2}{3}}\right)$.

Non-bordant torsion-free G_2-structures on same 7-manifold

Intersection form on $H^2(Y_+) \oplus H^2(\mathbb{P}^3) \subset H^4(W, M)$ is obtained by rescaling diagonal blocks

$$\begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 8 \end{pmatrix}.$$

This is unimodular, so $H^4(M)$ is torsion-free.

$H^4(M) \cong \mathbb{Z}^{95}$, $d = 2$ determine diffeomorphism type.

There are many rectangular twisted connected sums diffeomorphic to it.

$\overline{\nu} = -56$ while the twisted connected sums have $\overline{\nu} = 0$, so they differ mod 3
$\Rightarrow$ different classes in $\Omega_7^{G_2}$.

Quadratic refinement example

Use V_+ from small resolution of determinantal cubic

V_- from small resolution of cubic containing a plane

$\vartheta = \frac{\pi}{4}$, configurations

$$\begin{pmatrix} 6 & 6 & 7 & 1 \\ 6 & 2 & 5 & 1 \\ 7 & 5 & 6 & 4 \\ 1 & 1 & 4 & 0 \end{pmatrix} \qquad \begin{pmatrix} 6 & 6 & 5 & 2 \\ 6 & 2 & 3 & 2 \\ 5 & 3 & 6 & 4 \\ 2 & 2 & 4 & 0 \end{pmatrix}$$

lead to examples with equal primary data.

$TH^4 = \mathbb{Z}/2$, so they are diffeomorphic.

Quadratic refinement of linking form shows that diffeomorphism is orientation-reversing.