G_2 geometry and adiabatic limits

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September 11, 2019

In this lecture we outline a programme studying the adiabatic geometry of G_2-manifolds

Part I Review of the set-up.

Part II Adiabatic assocatives.

Part I.

Start with a (compact) oriented 7-manifold M.

A G_2-structure on M is specified by a positive 3-form ϕ. Such a form defines a Riemannian metric g_ϕ and a dual 4-form $*_\phi \phi$.

A positive 3-form is one which is equivalent at each point to the standard model on

$$\mathbf{R}^7 = \mathbf{R}^4 \times \mathbf{R}^3 = \{(x_0, x_1, x_2, x_3, t_1, t_2, t_3)\},$$

$$\phi_{\mathrm{model}} = dt_1 dt_2 dt_3 - \sum_{i=1}^{3} \omega_i dt_i,$$

where (ω_i) are the standard basis for the self-dual 2-forms on $\mathbf{R}^4$.

$$\omega_i = dx_0 dx_i + dx_j dx_k,$$

(ijk) cyclic.

Some standard questions about G_2 structures.

- Fix a cohomology class $C \in H^3(M; \mathbf{R})$. **Can C be represented by a closed positive 3-form ($d\phi = 0$). ?**
- **If so, can we additionally choose ϕ so that $d *_\phi \phi = 0$?** (The conditions $d\phi = 0, d *_\phi \phi = 0$ mean that ϕ defines a *torsion-free* G_2-structure.)
- **Can we find a torsion-free structure as the limit of the Bryant Laplacian flow**

$$\frac{\partial \phi}{\partial t} = d *_\phi d *_\phi \phi?$$

(This is the gradient flow of the Hitchin volume functional, and a torsion-free structure is a local maximum.)
- If we have a torsion free structure ϕ, what can we say about its geometry: e.g.**can we describe the associative and co-associative submanifolds in (M, ϕ)?**
- **Can we describe the moduli space of torsion-free G_2-structures, the "boundary" of the moduli space and relations to the period mapping $\phi \to [\phi] \in H^3(M)$?**

These questions are largely out of reach at present

The "adiabatic programme" seeks to construct another system of questions in 3-dimensions which one hopes:

- are more tractable than the 7-dimensional questions;
- replicate those questions in a *highly collapsed* regime of G_2-manifolds with *co-associative fibrations* having very *small fibres*. (cf. The talk by Haskins in this meeting.)

$$7 = 3 + 4$$

This programme exploits our detailed knowledge of the K3 4-manifold X and of hyperkähler structures on X.

X is the smooth oriented 4-manifold described as any of:

- a smooth quartic surface in $\mathbf{CP}^3$;
- a "Kummer surface"
$$\widehat{T^4/\pm 1};$$
- a double cover of $\mathbf{CP}^2$ branched over a curve of degree 6;
- ....

The real cohomology $H^2(X)$ has an integer lattice and cup-product form:

$$H^2(X) = \mathbf{R}^{3,19} \supset \Lambda_{K3}.$$

A class $\delta \in \Lambda_{K3}$ with $\delta^2 = -2$ defines a reflection $R_\delta \in O(\Lambda_{K3})$:

$$R_\delta(\alpha) = \alpha + (\delta.\alpha)\delta.$$

We also consider the affine extension of $O(\Lambda_{K3})$

$$(\mathbf{R}^{22}, +) \to \mathrm{Aff}(\Lambda_{K3}) \to O(\Lambda_{K3}),$$

and corresponding reflections in $\mathrm{Aff}(\Lambda_{K3})$.

Definition (for our purposes): A hyperkähler structure on X is given by closed 2-forms $\omega_1, \omega_2, \omega_3$ on X such that

$$\omega_i \wedge \omega_j = a_{ij} \mathrm{vol},$$

for a *constant* matrix $(a_{ij}) > 0$ and volume 4-form vol.

BASIC POINT: The 3-form

$$dt_1 dt_2 dt_3 - \sum \omega_i dt_i$$

defines a torsion-free G_2-structure on $X \times \mathbf{R}^3$.

Torelli theorem for K3

Let $H_+ \subset \mathbf{R}^{3,19}$ be a 3-dimensional subspace such that

1. H_+ is positive with respect to the quadratic form;
2. there are no integer -2 classes in $H_+^\perp$.

Let e_i be an oriented orthonormal basis for H_+.

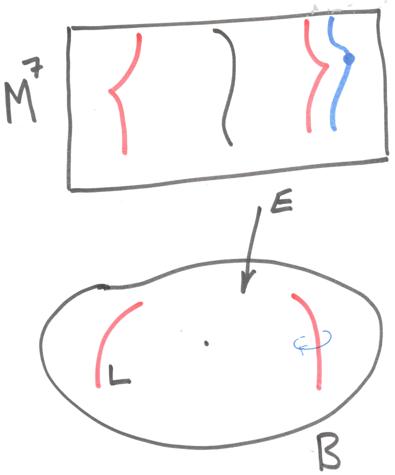
Then there is a hyperkähler structure on X with $[\omega_i] = e_i$ and this is unique up to diffeomorphism. Conversely, any hyperkähler structure defines such a space $H_+ = \mathrm{Span}([\omega_i])$.

Dictionary item 1.

The adiabatic analogue of M^7 is a compact 3-manifold B, a link $L \subset B$ and a flat bundle E_0 with fibre $\mathbf{R}^{22}$ and structure group $O(\Lambda)$ over $B \setminus L$ such that the monodromy around L is given by reflections.

Explanation

If we have a map $\pi : M \to B$ which is a fibration over $B \setminus L$ with fibre X we get a flat $O(\Lambda)$ bundle over $B \setminus L$ from the cohomology of the fibres. The condition on the monodromy around L corresponds to the assumption that around the singular fibres f has "ordinary double point singularities" modelled on $\{z_1^2 + z^2 + z_3^2 = 0\} \subset \mathbf{C}^3$. The reflection is in the class of the "vanishing cycle" δ.



“Kovalev-Lefschetz fibration”

Dictionary item 2.
The adiabatic analogue of a class $C \in H^3(M^7)$ is a lift of E_0 to an affine bundle E, with structure group $\mathrm{Aff}(\Lambda_{K3})$.
Explanation
We consider $\pi : M \to B$ whose fibres are *co-associative* with respect to a closed form ϕ. This is just the condition that ϕ vanishes on the fibres. Locally over an open set $U \subset B \setminus L$ we can write $\phi = d\sigma$, so σ restricts to a closed 2-form on each fibre and we get a map $h_U : U \to H^2(X)$. The global version of this is a section h of the affine bundle E, which depends only on the cohomology class $[\phi] \in H^3(M)$.
Alternatively, the affine lift E is determined by a class χ in a sheaf cohomology group $H^1(B, E_0)$ and the Leray spectral sequence gives an exact sequence

$$0 \to H^3(B) \to H^3(M) \to H^1(B; E_0) \to 0.$$

Dictionary item 3.
The adiabatic analogue of a closed positive form in the class $C \in H^3(M^7)$ is a "positive" section h of E. Locally, away from L, this is given by a parametrisation of a "spacelike" submanifold in $\mathbf{R}^{3,19}$. We do not take time to explain here the extension of this notion to the singular fibres.

Explanation

At each point of $U \subset B \setminus L$ the image of dh_U is a positive subspace in $\mathbf{R}^{3,19}$ with a basis corresponding to $\frac{\partial}{\partial t_i}$. Let ω_i be the corresponding hyperkähler structure on the fibres of π, these can be extended in the horizontal directions to $\tilde{\omega}_i$ such that

$$\Psi_h = \lambda dt_1 dt_2 dt_3 - \sum \tilde{\omega}_i dt_i$$

is a closed positive 3-form , for suitable $\lambda > 0$.

Dictionary item 4.

The adiabatic analogue of a torsion-free structure in the class $C \in H^3(M^7)$ is a "maximal positive" section h of E_0. Locally, away from L, this is given by a parametrisation of a "maximal spacelike" submanifold in $\mathbf{R}^{3,19}$.

Explanation

- Maximal submanifolds in indefinite spaces like $\mathbf{R}^{3,19}$ are defined just as minimal submanifolds in Euclidean spaces through the Euler-Lagrange equations for the volume functional. In our setting there an an adiabatic analogue of Hitchin's functional, with gradient flow given by a version of mean curvature flow.
- Introduce a parameter $\epsilon > 0$. Given h, we have a positive form

$$\Psi_{h,\epsilon} = \lambda dt_1 dt_2 dt_3 + \epsilon \sum \tilde{\omega}_i dt_i.$$

The condition that h maps to a maximal submanifold is necessary and sufficient for the existence of a (local) formal power series solution to the torsion-free equation

$$\Phi_\epsilon = \Psi_{h,\epsilon} + \epsilon \lambda' dt_1 dt_2 dt_3 + \sum_{k=2}^{\infty} \phi_k \epsilon^k.$$

The fibre diameter is $O(\epsilon^{1/2})$ and the base diameter is $O(1)$.

Adiabatic analogues of the standard questions

- Fix a cohomology class $\chi \in H^1(B; E_0)$ and so affine bundle E. **Is there a positive section of E ?**
- **If so, can we additionally choose the section to be maximal?**
- **Can we find a maximal section as the limit of a mean curvature flow?**
- Adiabatic associatives and co-associatives? See Part II.
- **Can we describe the moduli space of maximal sections, the "boundary" of the moduli space and relations to the periods $\chi \in H^1(B, E_0)$?**

It should be emphasised that many of the foundations of this adiabatic theory are not yet complete. and the adiabatic analogues of the standard questions still seem very difficult (Deformation theory, examples, short term existence for mean curvature flow . . .)

The adiabatic analogues of the "standard questions" still seem very difficult

There are more decisive results for **boundary value problems**.

- **Y. Li**: Let Ω be a bounded convex domain in $\mathbf{R}^3$ and $S \subset \mathbf{R}^{3,19}$ the (spacelike) graph of a function $f : \partial\Omega \to \mathbf{R}^{19}$. Then the following are equivalent:
  - S bounds a *maximal* spacelike submanifold in $\mathbf{R}^{3,19}$;
  - S bounds *some* spacelike submanifold in $\mathbf{R}^{3,19}$;
  - $$|f(x) - f(y)| < |x - y|$$

    for all distinct $x, y \in \partial\Omega$.

- **Lambert and Lotay:** long time existence and convergence for mean curvature flow in $\mathbf{R}^{3,19}$.

Part II. Adiabatic associative submanifolds

Joint work with C. Scaduto (Simons Collaboration postdoc 2018-19).

Let (M^7, ϕ) be a G_2-manifold. An *associative submanifold* of M is a 3-dimensional submanifold P such that ϕ vanishes on the normal bundle of P.

This is an elliptic equation of index 0. We expect that for generic ϕ the associative submanifolds are isolated. One is interested in the possibility of enumerative theories, based on counting associative submanifolds in a given homology class.

Connection with moduli

If $P \subset M$ is associative the restriction $\phi|_P$ is the volume form of P and so

$$\int_P \phi > 0.$$

So if we know that a homology class $\pi \in H_3(M)$ is represented by an associative submanifold we get a constraint on the cohomology class $[\phi] \in H^3(M)$:

$$\langle [\phi], \pi \rangle > 0.$$

This gives one kind of "boundary" in the moduli space of torsion-free G_2 structures on M.

The "standard conjecture" (Joyce)

In a generic 1-parameter family ϕ_t there are two reasons why the count of associative submanifolds can change:

- "Joyce-Nordström crossing"
- "Smoothing of Harvey-Lawson cone singularities"

(and possibly another phenomenon arising from multiple covers which we ignore here, cf. the lecture of Doan in this meeting).

J-N crossing

A pair P, Q of associative submanifolds in M do not generically intersect $(3+3<7)$. In a 1-parameter family ϕ_t the submanifolds P_t, Q_t will generically intersect for a discrete set of parameter values $(4+4=7+1)$. If t_0 is such a value then there is an associative created or destroyed, modelled on the connected sum $P_{t_0}\sharp Q_{t_0}$.

Smoothings of H-L cones

There is an explicit singular associative in $\mathbf{C}^3 \subset \mathbf{R}^7 = \mathbf{C}^3 \times \mathbf{R}$ which is a cone on a T^2:

$$\{(z_1, z_2, z_3) : |z_1|^2 = |z_2|^2 = |z_3|^2, \mathrm{Im}(z_1 z_2 z_3) = 0\}.$$

There are three explicit, topologically distinct , smoothings of this, corresponding to three ways of writing $T^2 = \partial(S^1 \times D^2)$.

Suppose that $\Pi \subset (M, \phi_{t_0})$ is a singular associative with this tangent cone. In a 1-parameter family ϕ_t we could encounter a family of associatives P_t for $t \leq t_0$ converging to Π as $t \to t_0$ and *two* such families Q_t, R_t for $t > t_0$.

Topologically, the triple of 3-manifolds P_t, Q_t, R_t are related by *Dehn surgeries*.

The verification of Joyce's "standard conjecture" seems a long way off: it would require a deep understanding of singularity formation for associative submanifolds.

Adiabatic version

Let $\omega_1, \omega_2, \omega_3$ be a hyperkähler structure on X. For $v \in \mathbf{R}^3 \setminus \{0\}$ there is complex structures I_v on X with Kähler form $\omega_v = \sum v_i \omega_i$.
This has a cohomology class $[\omega_v]$ which lies in

$$H_+ = \mathrm{Span}(\omega_1, \omega_2, \omega_3) \subset H^2(X) = \mathbf{R}^{3,19}.$$

With the product G_2 structure on $X \times \mathbf{R}^3$, if $\Sigma \subset X$ is an I_v-complex curve then

$$\Sigma \times \mathbf{R}v \subset X \times \mathbf{R}^3$$

is an associative submanifold.

Let $\sigma \in H^2(X)$ with $\sigma.\sigma \geq -2$.

If $[\omega_v]$ is a positive multiple of the projection of σ to H_+ there is a complex curve in the class σ.

When $\sigma.\sigma = -2$ this is unique and typically (see later) a smooth embedded 2-sphere.

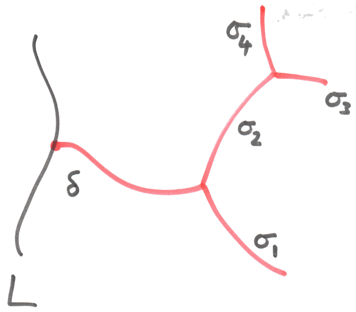
The condition on the line $\mathbf{R}[\omega_v]$ is that it is the *gradient line* of the restriction of the linear function on $\mathbf{R}^{3,19}$ defined by σ to the subspace $H_+ \subset \mathbf{R}^{3,19}$.

Now consider an adiabatic set-up (B, L, E, h). If this comes from a fibration $\pi : M \to B$ there is an exact sequence

$$0 \to H_1(E_0) \to H_3(M) \to \mathbf{R} \to 0,$$

for a certain homology group $H_1(E_0)$. This can be described by cycles which are embedded (directed) graphs in B, with edges labelled by locally constant sections of E_0 and:

- For an edge which terminates on a vertex on L the label is multiple of the corresponding vanishing cycle.
- At a vertex in $B \setminus L$ the sum of the labels of the incoming edges is 0.



σ4
σ3
σ2
δ
σ1
L

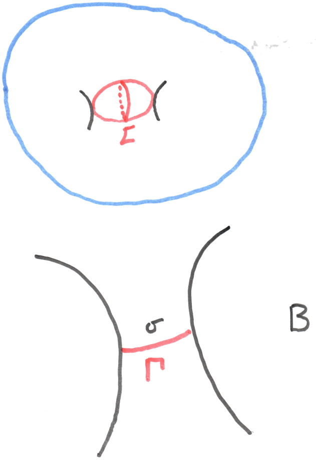
We restrict attention here to the case when the labels are integer classes of square -2.

The label of an edge defines a gradient vector field, as in the product case above.

We define a *calibrated cycle* to be one where each edge is a segment of an integral curve of the gradient vector field.

In simple situations, at least, it can be shown that such a calibrated cycle defines an associative submanifold in (M, Φ_ϵ), for small ϵ.

Connection with moduli problem. One kind of boundary of the moduli space of positive maximal sections occurs when two components on L which are joined by a gradient curve come together. which implies that $\langle [\Gamma], \chi \rangle \to 0$. This corresponds to associative 3-spheres in M shrinking to a point.



One would like to "count" calibrated cycles in a given class in $H_1(E_0)$ and understand how this count can change in 1-parameter families.

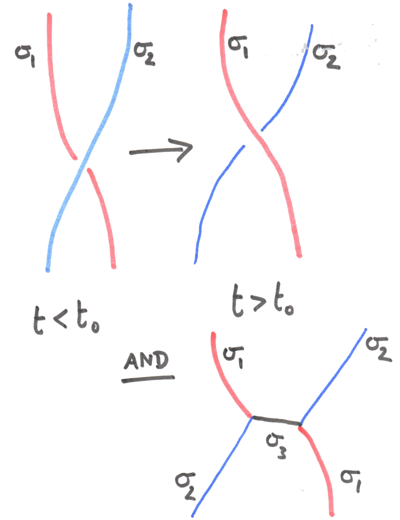
There are adiabatic analogues of JN crossing and HL smoothings. These arise from the fact that a complex curve in X could be *reducible*, with several components.

If $\sigma_1^2 = \sigma_2^2 = -2$ and $\sigma_1.\sigma_2 = 1$ then $\sigma_3 = -(\sigma_1 + \sigma_2)$ has

$$\sigma_3^2 = -2 + 2 - 2 = -2.$$

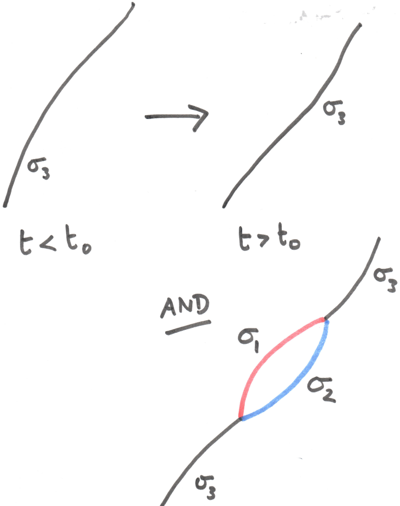
JN crossing

Gradient flow lines in B do not generically intersect ($1+1<3$) but in a 1-parameter family they do. So we can expect to see:



HL smoothings

The gradient vectors defined by two classes σ_1, σ_2 are generically linearly independent, but in a 1-parameter family we can expect to see:



The analysis of calibrated cycles is relatively straightforward, so one can hope to prove an adiabatic analogue of the "standard conjecture"