Asymptotic analysis, moment maps and numerical approximations in Kähler geometry

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There are many general *existence theorems* in Kähler geometry (and also important open questions). These include results for

- Metrics with zero Ricci curvature on manifolds with $c_1 = 0$ (Calabi-Yau);
- Hermitian Yang-Mills connections on stable holomorphic vector bundles;
- Constant scalar curvature Kähler (CSCK) metrics and extremal metrics (no completely general existence theorem, but many results linking existence to “K-stability”).

In this talk we will focus on Calabi-Yau (CY) and CSCK metrics. Based on papers:

- *Scalar curvature and projective embeddings II* arxiv 0407534
- *Some numerical results in complex differential geometry* arxiv 0512625
- *Numerical approximations to extremal metrics on toric surfaces*(with R. Bunch) arxiv 0803.0987

Recall that the CY equation can be set up a second order PDE for a potential ϕ. It is a particular case of finding a metric $\omega_\phi = \omega_0 + i\overline{\partial}\partial\phi$ with prescribed volume form

$$\frac{1}{n!}\omega_\phi^n = \mu.$$

The CSCK equation is a fourth order PDE for ϕ.

General existence theorems do not give information about the solution.
For example, a smooth quartic surface $\{P = 0\}$ in $\mathbf{CP}^3$ has, according to Yau, a zero Ricci curvature metric. What is the diameter, the maximum size of the curvature...? Even approximately.

These quantities to depend on the particular polynomial P.

We are interested in finding numerical approximations to the metrics.

Strategic questions:

1. How to approximate general Kähler metrics, in a form which can be put on a computer.
2. How to find approximations to the CY and CSCK metrics, i.e. to the solutions of these nonlinear PDE.

For (1) we use *projective embeddings*, i.e. we consider embeddings $X \longrightarrow \mathbf{CP}^N$ and the restriction of the standard Fubini-Study metric on $\mathbf{CP}^N$.

Background Theory I: asymptotic analysis

Let $L \to X$ be a positive line bundle over a compact complex line bundle. We consider the powers $L^k, k \geq 1$.

An L^2-metric on $H^0(X; L^k)$ is determined by a fibrewise Hermitian metric h on L and a volume form μ on X. Given this data we have Bergman functions $B_k : X \to \mathbf{R}$.

$$B_k = \sum |s_\alpha|^2,$$

where s_α is any orthonormal basis of $H^0(L^k)$.
Alternatively, $\sqrt{B_k}(x)$ is the norm of the evaluation map

$$\mathrm{ev}_x : H^0(L^k) \to L^k_x.$$

Note that

$$\int_X B_k \, \mu = \dim H^0(L^k),$$

which, for large k, is given by the Riemann-Roch formula.

B_k gives the "density" of the holomorphic sections.

The Bergman function and projective embedding

The L^2 metric on $H^0(L^k)$ defines a Fubini Study metric on $\mathbf{P}(H^0(L^k)^*)$. For sufficiently large k there is an intrinsic embedding

$$X \to \mathbf{P}(H^0(L^k)^*).$$

Let $\omega_{FS,k}$ be k^{-1} times the restriction of the Fubini-Study metric to X and ω_h be the i times the curvature of the Hermitian line bundle (L, h). Then

$$\omega_{FS,k} = \omega_h + ik^{-1}\overline{\partial}\partial \log B_k.$$

Asymptotic theory

Take $\mu = \mu_h = \omega_h^n/n!$.

As $k \to \infty$ there is a Yau-Tian-Zelditch-Caitlin-Tian-Lu expansion

$$B_k \sim k^n(1 + a_1 k^{-1} + a_2 k^{-2} + \dots) \qquad (***)$$

where a_i are functions on X given by expressions in the curvature tensor of (X, ω_h) and its derivatives. In particular a_1 is one half the scalar curvature.

Note that the integrals of the a_i are the terms appearing in the Riemann-Roch formula so there is some similarity with the "local index theorem".

Corollary (Tian) The metrics $\omega_{FS,k}$ converge to ω_h as $k \to \infty$. In fact

$$\omega_{FS,k} = \omega_h + O(k^{-2}).$$

So any Kähler metric on X (in the class $c_1(L)$) can be approximated by “algebraic” metrics, induced from the Fubini-Study metric by embeddings of X.

Two points of view:

- Fix the embedding $X \to \mathbf{CP}^N$ and vary the Hermitian metric on $\mathbf{C}^{N+1}$ defining the FS metric on $\mathbf{CP}^N$
- Fix the metric on $\mathbf{C}^{N+1}$ and vary the embedding by the action of $GL(N+1, \mathbf{C})$.

One can show that there are algebraic metrics ω_k of degree k, converging to all orders as $k \to \infty$. That is, $\omega_k = \omega_h + o(k^{-r})$ for all r, analogous to the approximation of smooth functions by truncated Fourier series.

A variant: let μ be an arbitrary volume form. Then

$$B_k \sim \frac{\mu_h}{\mu} k^n + O(k^{n-1}). \qquad (***)_\mu$$

Idea of proof of the asymptotic expansion (***).
Let $x \in X$. From the definition, $B_k(x) = |\sigma_x(x)|^2$ where $\sigma_x \in H^0(L^k$ is the section of L^2 norm 1 which is L^2-orthogonal to the sections vanishing at x.
Let $\Lambda \to \mathbf{C}^n$ be the Hermitian holomorphic line bundle with curvature the standard metric on $\mathbf{C}^n$. The same definitions make sense and the section σ_0 has rapid decay :

$$|\sigma_0(z)| = \exp(-|z|^2/2).$$

Replacing L by L^k we replace the Kähler form ω_h by $k\omega_h$. That is, we scale lengths by a factor $k^{1/2}$. As $k \to \infty$ the rescaled geometry around x converges to the flat model. For large k the section σ_x is a small perturbation of σ_0.

General principle

The sections of L^k can "see" the metric geometry of X on length scales down to $O(k^{-1/2})$

Now turn to item (2): how to find good approximations to CY and CSCK metrics.

The CSCK metrics are minima of the *Mabuchi functional*. One approach would be to seek minima of the functional restricted to the algebraic metrics of degree k. That is, we minimise a function on a finite dimensional space M_k of $N_k \times N_k$ Hermitian matrices where $N_k = \mathrm{dim} H^0(L^k) = O(k^n)$. Such minima should converge rapidly to the CSCK metric, to all orders in k.

There is a similar discussion in the CY case.

Another, related, approach uses explicit iterative procedures.

Background Theory II: moment maps and Geometric Invariant Theory

Let $p_1, \dots p_d$ be distinct points in $S^2 \subset \mathbf{R}^3$.
If $d > 1$ it is a fact that there is a Möbius map $g : S^2 \to S^2$ so that $p_i' = g(p_i)$ is a "balanced" configuration with

$$\sum p_i' = 0 \in \mathbf{R}^3.$$

Writing $S^2 = \mathbf{P}(\mathbf{C}^2)$, we choose vectors $P_i \in \mathbf{C}^2$ representing p_i and define

$$F(P_1, \dots, P_d) = \sum \log |P_i|^2.$$

Then $SL(2, \mathbf{C})$ acts on the set of d-tuples $(P_1, \dots P_d)$ and the balanced configuration corresponds to a minimum of the function F on the $SL(2, \mathbf{C})$-orbit.

More generally, we have an embedding $\mathbf{CP}^N$ in the Lie algebra of $SU(N+1)$. If ν is a measure on $\mathbf{CP}^N$ we say ν is balanced if its centre of mass in $\mathrm{Lie}(SU(N+1))$ is zero.
Under very mild hypotheses on ν we can find a $g \in SL(N+1, \mathbf{C})$ such that $g(\nu)$ is balanced. This minimises a function similar to F above.

Suppose that X has $c_1 = 0$ and trivial canonical bundle, so there is a measure $\mu = \mathrm{const.}\Theta \wedge \overline{\Theta}$ on X, for a trivialising section Θ. Let $L \to X$ be a positive line bundle as before, giving embeddings $X \to \mathbf{CP}^{N_k}$.
Regarding μ as a measure on $\mathbf{CP}^{N_k}$ (supported on X), the theory above allows us to move X to a balanced embedding. Let $\tilde{\omega}_k$ be k^{-1} times the restriction of the Fubini-Study metric to this balanced embedding.

Then it can be shown that the $\tilde{\omega}_k$ converge to the Calabi-Yau metric on X in the class $c_1(L)$ as $k \to \infty$.

In one direction, it follows from the definitions that $\tilde{\omega}_k$ is characterised by the property that B_k is constant, so if we know that the $\tilde{\omega}_k$ converge then the fact that the limit is the CY metric follows from $(***)_{CY}$.

To treat the CSCK case we use a variant of these ideas. We call $X \subset \mathbf{CP}^N$ balanced if the centre of mass of the measure induced by the restriction of the Fubini-Study metric is zero. A variety can be moved to a balanced one by a projective transformation if and only if it is "Chow stable". Assuming this is the case we get metrics $\tilde{\omega}_k$ as above and it can be shown that *if* X admits a CSCK metric ω_{CSCK} then the $\tilde{\omega}_k$ converge to ω_{CSCK}.

(More precisely, this is known to hold if X has no holomorphic automorphisms.)

As before, if we know that the $\tilde{\omega}_k$ converge the fact that the limit is CSCK follows from the appearance of the scalar curvature in (***).

Iteration to find the $\tilde{\omega}_k$ for fixed k. (CSCK case).
Let $\mathcal{H}$ be the infinite dimensional space of metrics on L and M_k be the finite dimensional space of metrics on $H^0(L^k$.
We have two constructions: $M_k \to \mathcal{H}$ and $\mathcal{H} \to M_k$.
The first assigns to a metric on the vector space the metric on the line bundle corresponding to the restriction of the Fubini-Study metric. The second assigns to a metric on the line bundle the L^2 metric on the space of sections defined using the volume form μ_h. Composing these we get a map $T : M_k \to M_k$.
It can be shown that for any starting point $m \in M_k$ the iterates $T^j m$ converge to a fixed point as $j \to \infty$, and this fixed point corresponds to a balanced embedding.

The same holds in the Calabi-Yau case, except that we use the given volume form μ to define the L^2 metric.

Putting this together we have explicit schemes to find CY metrics and CSCK metrics, when they exist.

Remark The relation between the sequence of finite dimensional questions in M_k and the infinite dimensional question in $\mathcal{H}$ can be developed a long way and is connected to Geometric Quantisation.

To see the iteration explicitly in the CY case, let s_α be any basis of $H^0(L^k)$ and G be a Hermitian metric defined by a matric $G_{\alpha\beta}$.

1. Form the inverse matrix $G^{\alpha\beta}$.
2. For each γ, δ define a function $f_{\gamma\delta}$ on X by

$$f_{\gamma\delta} = \frac{s_\gamma \overline{s}_\delta}{D}, \qquad (****)$$

   where $D = \sum G^{\alpha\beta} s_\alpha \overline{s}_\beta$. (While the s_α are not functions, the formula(****) makes invariant sense.)
3. Now set

$$\tilde{G}_{\gamma\delta} = c \int_X f_{\gamma\delta}\ \mu$$

   where c is a normalising constant chosen so that

$$\sum G^{\gamma\delta} \tilde{G}_{\gamma\delta} = N + 1.$$

The iteration is to pass from G to $\tilde{G}$.

The CSCK case is the same except that we need to compute a volume from the Fubini-Study metric at each stage. This extra complication reflects the extra complication of the fourth order PDE.

LIMITATION A

The $\tilde{\omega}_k$ will usually only converge slowly to the CSCK or CY metric, as $O(k^{-1})$. The rate of convergence will be related to the size of the next terms in the asymptotic expansion.

However If the $\tilde{\omega}_k$ are reasonably close to the actual solution we can use different techniques (Newton method) to find better approximations like ω_k, converging to all orders.

LIMITATION B

In a region of X where the actual solution has $|\mathrm{Riem}| \sim C$ the natural length scale is $O(C^{-1/2})$ so to get good approximations one needs to take $k \sim C$. If the there is a small region where the curvature is large we need to take an impractically large value of k to study it. But on the bulk of the manifold a much smaller value of k suffices.

In the context of this Simons Collaboration, the method is well-adapted to studying points “deep inside moduli space”, opposite to “gluing techniques”.

A possible approach to get around Limitation B is to combine different values of k on different parts of the manifold. (cf. Other talks in this meeting.)

EXAMPLES

1. *Toric surfaces*

A toric manifold is associated to a polytope $P \subset \mathbf{R}^n$ and a basis of sections of L^k to lattice points $k^{-1}\mathbf{Z}^n \cap P$. The metric matrices are diagonal.

Some differential geometry on a toric surface X.
A 4-fold cover of the polygon P is embedded as totally geodesic submanifold Σ in X. The curvature tensor of X has 4 components

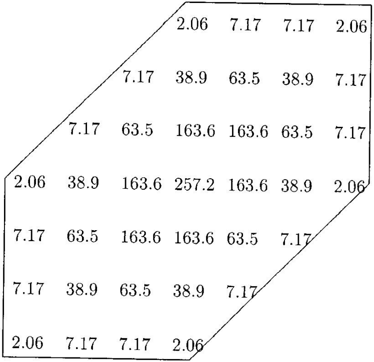
- The Scalar curvature S,
- The Gauss curvature K of Σ.
- A quadratic differential on Σ which corresponds to Ric_0.
- A quartic differential on Σ which corresponds to the Weyl curvature W.

To compare different manifolds we scale so that the average value of the scalar curvature is 1.
Then for example $\mathbf{CP}^2$ has $K = 1/6, S = 1$ as the only non vanishing components while for $S^2 \times S^2$ the non-vanishing components are $|W|^2 = 1/8, S = 1$

The "hexagon manifold" is the Kähler-Einstein surface of Tian-Yau and Siu: the blow-up of $\mathbf{CP}^2$ in three general points. We find it has

Max $|\mathrm{Riem}|$	2.48
Min $|\mathrm{Riem}|$	0.61
Max K	0
Min K	-0.82
Max$|W|$	2.1
Ric_0	0

These are obtained using $k = 20$ but one gets a reasonable approximation using $k = 6$ in which case there are just 6 independent matrix entries to find.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The "heptagon manifold" is obtaining by blowing up 4 points and has an extremal metric with:

Max $|\mathrm{Riem}|$	13.9
Min $|\mathrm{Riem}|$	0.54
Max K	.25
Min K	-5.5
Max$|W|$	11.7
Max$|\mathrm{Ric}_0|$	4.3

from an approximation obtained using $k = 45$.

curve.

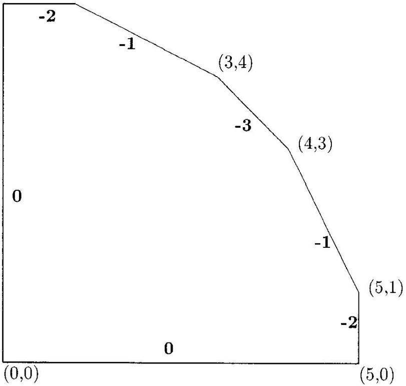
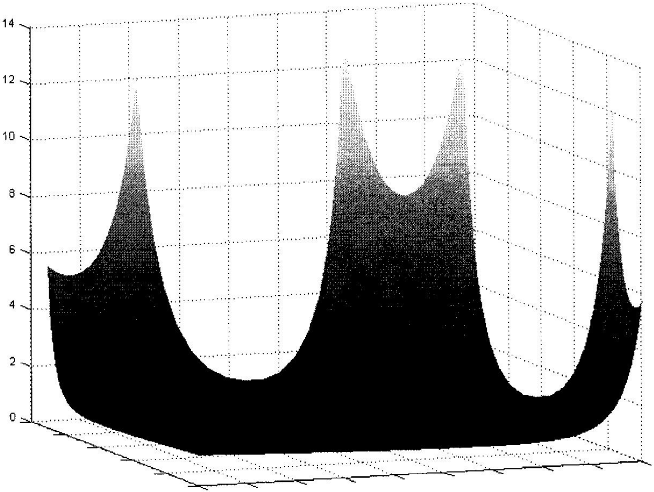


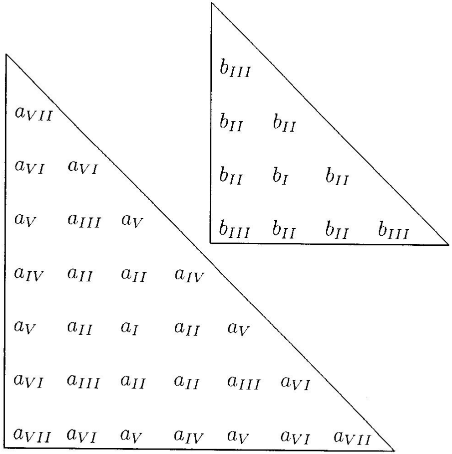
Figure 13: |Riem| for the heptagon



2.**A K3 surface**

We consider the double cover of the plane branched over the curve $D = 0$ where $D = x^6 + y^6 + z^6 = 0$. Holomorphic sections can be expressed as $p + q\sqrt{D}$ where p is a polynomial of degree k and q of degree $k - 3$.

One gets a reasonable approximation to the Calabi-Yau metric using $k = 6$. Taking account of the symmetries there are just 11 matrix entries to find.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