Deformations of singular sets and Nash-Moser theory

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PLAN

1. Two deformation problems: description.
2. Context 1.
3. Outline of proof.
4. Context 2.

Related to a lecture in the collaboration meeting of June 2018.

SECTION 1: Description of problems.

Problem I. “multivalued harmonic functions”.

Let (M, g) be a compact (oriented) Riemannian n-manifold and $\Sigma \subset M$ a (co-oriented) codimension-2 submanifold with $[\Sigma]$ even in $H_{n-2}(M)$.

Then there is a flat real line bundle $V \to M \setminus \Sigma$ with holonomy -1 around a small loop locally linking Σ.

There is also a double branched cover $p : \widetilde{M} \to M$, branched over Σ.

We are interested in harmonic 1-forms on $M \setminus \Sigma$ with values in V.
i.e. sections a of $T^*M \otimes V$ with $da = 0, d^*a = 0$.

These pull back to ordinary 1-forms $p^*(a)$ on $\widetilde{M} \setminus p^{-1}(\Sigma)$.

If a is in L^2 one can show that $p^*(a)$ defines a de Rham cohomology class

$$\chi \in H^1(\widetilde{M}, \mathbf{R})^-,$$

the -1 eigenspace of the action of the covering involution on $\widetilde{M}$.

Conversely, given any such class χ a version of the Hodge Theorem shows that there is a unique harmonic representative $p^*(a)$.

We are interested in the question: for which (g, Σ, χ) is the form *a bounded* on $M \setminus \Sigma$.

Fundamental Example $n = 0$, so Σ is a finite set of points.

The harmonic condition is conformally invariant so M should be regarded as a Riemann surface.

We can write $a = \mathrm{Re}(\alpha)$ where α is a holomorphic 1-form on $M \setminus \Sigma$, with values in V.

The square $\alpha^{\otimes 2}$ is a *holomorphic quadratic differential* on $M \setminus \Sigma$.

In a local complex co-ordinate z centred on a point of Σ the condition that $a \in L^2$ means that

$$\alpha = c_0 z^{-1/2} dz + c_1 z^{1/2} dz + c_2 z^{3/2} dz + \dots.$$

Then $\alpha^{\otimes 2} = c_0^2 z^{-1} dz^2 + \dots$.

The condition that the form a is bounded is equivalent to $c_0 = 0$, which is equivalent to the condition that $\alpha^{\otimes 2}$ is a holomorphic quadratic differential on M. In this case

$$\alpha^{\otimes 2} = c_1^2 z \, dz^2,$$

so $\alpha^{\otimes 2}$ has a simple zero provided that $c_1 \neq 0$.

Recall a piece of standard theory. Given a manifold W and a class θ in $H^1(W; \mathbf{R})$ we form a flat bundle F_θ over W with fibre $\mathbf{R}$ and structure group $(\mathbf{R}, +)$ such that sections of F_θ correspond to closed 1-forms on W representing θ in de Rham cohomology.

Similarly, given the class $\chi \in H^1(\widetilde{M}, \mathbf{R})^-$ we can form a bundle $E \to M \setminus \Sigma$ with fibre $\mathbf{R}$ and structure group the isometries of $\mathbf{R}$ (i.e. $y \mapsto \pm y + c$) such that our 1-form a is the derivative of a harmonic section h of E.

When $n = 2$ we have a local description

$$h = \mathrm{Re}\left(Az^{1/2} + Bz^{3/2} + \dots\right),$$

where $A = 2c_0, B = (2/3)c_1$.

The condition that a is bounded is equivalent to $A = 0$.

In general (for any n), the harmonic section h has an asymptotic development with leading term $h = A\zeta^{1/2} + O(r^{3/2})$ where r is the distance to Σ and ζ is a complex normal co-ordinate.

Now A is a section of the complex line bundle $N_\Sigma^{-1/2}$ over Σ, where N_Σ is the normal bundle.

If $A = 0$ then $h = B\zeta^{3/2} + O(r^{5/2})$ where B is a section on $N_\Sigma^{-3/2}$ over Σ.

So for any (g, Σ, χ) we get a section $A(g, \Sigma, \chi)$ of $N_\Sigma^{-1/2}$ and if this vanishes we have a section $B(g, \Sigma, \chi)$ of $N_\Sigma^{-3/2}$.

Deformation Theorem
Suppose that $A(g_0, \Sigma_0, \chi_0) = 0$ and $B(g_0, \Sigma_0, \chi_0)$ is nowhere-vanishing on Σ. Then for any (g, χ) close to (g_0, χ_0) there is a unique Σ close to Σ_0 such that $A(g, \Sigma, \chi) = 0$.

A proof of this was outlined in the June 2018 talk, written up in arxiv 1912.08274. The approach we will outline in Section III could be used to give a somewhat different proof.

There are related, earlier, results of Takahashi for harmonic spinors: arxiv 1503.00767.

This Deformation Theorem can be thought of as a generalisation to higher dimensions of the fact that holomorphic quadratic differentials with simple zeros on a Riemann surface M are locally parametrised by the class of the real parts of their square roots in $H^1(\widetilde{M}; \mathbf{R})^-$, as in Ivan Smith's talk.

Problem II “branched maximal sections”.

Let $\mathbf{R}^{n,m}$ be the standard $(n+m)$-dimensional vector space with quadratic form of signature (n,m).
A n-dimensional submanifold of $\mathbf{R}^{n,m}$ is called positive if its tangent spaces are positive subspaces of $\mathbf{R}^{n,m}$.

There is an induced volume form on positive submanifolds which leads to Euler-Lagrange equations for the volume functional, just as in the familiar Euclidean case. The equations are $\text{mean curvature} = 0$.

Solutions of these equations are called *maximal submanifolds*.

Example Consider the graph of a real-valued function f on $\mathbf{R}^n$.

- In Euclidean space $\mathbf{R}^{n+1}$ the minimal submanifold equation is

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial x_i}\right) = 0$$

- In Lorentzian space $\mathbf{R}^{n,1}$ the positive condition is $|\nabla f| < 1$ and the maximal submanifold equation is

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{1-|\nabla f|^2}}\frac{\partial f}{\partial x_i}\right) = 0.$$

Remarks

- If ∇f is small the equations are approximately

$$\Delta f \pm \sum f_{ij} f_i f_j = 0.$$

- The maximal condition can be stated as the condition that the restriction of the coordinate functions to the submanifold are harmonic, with respect to the induced metric.

A vector $v \in \mathbf{R}^{n,m}$ with $v^2 = -1$ defines a *reflection*

$$R_v : \mathbf{R}^{n,m} \to \mathbf{R}^{n,m}$$

$$R_v(w) = w + (v.w)v.$$

Let Γ be the affine extension of $O(n,m)$, so we have an exact sequence of group homomorphisms

$$0 \to (\mathbf{R}^{n,m}, +) \to \Gamma \to O(n,m) \to 1.$$

We have a set of *affine reflections* in Γ, which map to reflections in $O(n,m)$.

Let (M, Σ) be as before but now we do *not* need a Riemannian metric on M. Let $V \to M \setminus \Sigma$ be a flat vector bundle with fibre $\mathbf{R}^{n,m}$ and structure group $O(n,m)$ such that the holonomy around a loop locally linking Σ is a reflection.

We consider lifts of V to a flat affine bundle E with structure group Γ such that the holonomy around a linking loop is an affine reflection. These are parametrised by a certain cohomology group which we denote by H^1_V: a finite dimensional real vector space.

Fix such a bundle E.

A section of E over $M \setminus \Sigma$ is represented in a flat local trivialisation by a map to $\mathbf{R}^{n,m}$.

We say that the section of E over $M \setminus \Sigma$ is maximal if these local maps are embeddings with images maximal submanifolds.

To discuss the behaviour around Σ we consider for simplicity the case when $m = 1$.

Write $\mathbf{R}^{n,1} = \mathbf{R}^n \times \mathbf{R}$ with the reflection R_0 acting as -1 on the $\mathbf{R}$ factor.

Let S be a codimension-2 submanifold of $\mathbf{R}^n$ and choose a diffeomorphism ϕ from $\mathbf{C} \times \mathbf{R}^{n-2}$ to $\mathbf{R}^n$ which takes $\{0\} \times \mathbf{R}^{n-2}$ to S and such that $d\phi$ defines an isometry from the (trivial) normal bundle of $\{0\} \times \mathbf{R}^{n-2}$ to the normal bundle of S.

Let V_0 be the flat $\mathbf{R}^{n,1}$ bundle over $\mathbf{R}^n \setminus S$ with holonomy R_0 around S. We say that a maximal section h_0 of V_0 is *branched* if

$$h_0(x) = (x, f(\phi^{-1}(x))$$

where

$$f(z,t) = \mathrm{Re}(B(t)z^{3/2}) + o(r^{3/2}),$$

with B a non-vanishing complex-valued function of $t \in \mathbf{R}^{n-2}$.

More precisely, everything here is local, defined on suitable small open sets and we should also fix a lift of ϕ to the flat bundles on the complements of the codimension-2 sets $\{0\} \times \mathbf{R}^{n-2}$ and S.

Also, we require that f lies in a certain function space that we will define later.

But all sensible choices of function space should lead to equivalent definitions.

Going back to our set-up (M, Σ, E), we get a corresponding notion of a branched maximal section; locally equivalent to that considered above, for some S.

A small diffeomorphism $\theta : M \to M$ takes a maximal section branched along Σ to one branched along $\theta(\Sigma)$. The only intrinsic continuous data in the triple (M, Σ, E) is the flat Γ-bundle E and in particular the cohomology class $\chi \in H^1_V$.

Deformation Theorem

Suppose that (M, Σ, E) *admits a branched maximal section* h_0 *with* B *nowhere vanishing. Then for any small deformation* E' *of* E *there is a branched maximal section* h' *of* (M, Σ, E') *close to* h_0*, and* h' *is unique up to small diffeomorphisms of* M.

SECTION 2: Context 1.
(This will be quiet vague; apologies for errors or omissions.)

We are particulary interested in the cases $\dim M = n = 2, 3$. The notions above are related to Calabi-Yau 3-folds and G_2-manifolds, respectively, fibred over a base M with hyperkähler 4-manifolds as fibres:

$$X^4 \to Z \to M.$$

In particular to “adiabatic limits” when the fibres are very small.

The deformation statements on M that we have discussed are the counterparts of Torelli-type theorems for deformations of Z.

The harmonic 1-form set-up arises when the generic fibre is the Eguchi-Hanson ALE manifold (i.e. type A_1). The fibres over Σ acquire ordinary double point singularities. This is well-developed in the case when $n = 2$ and everything can be expressed in terms of quadratic differentials, as in the talk of Ivan Smith in this meeting. There are also extensions of the theory with $n = 2$ to other ALE fibres, as mentioned in Ivan Smith's talk.

For $n = 3$, in the case when Σ is empty, constructions on these lines were made by Joyce and Karigiannis, starting with a nowhere vanishing harmonic 1-form on M^3 and producing an approximate G_2-structure on a 7-manifold fibred over M^3 with Eguchi-Hanson fibres. (They used this as a component in a smoothing theory, for compact manifolds.)

Work of Barbosa (presented in his lecture at the October 2020 collaboration meeting) develops this kind of picture for $n = 3$, with non-empty Σ and for other ALE fibres.

The maximal submanifold set-up arises when the generic fibre is the K3 manifold X. The fibres over Σ acquire ordinary double point singularities and the reflections appear from the Picard-Lefschetz formula. We take $m = 19$, so $\mathbf{R}^{3,19}$ is $H^2(X)$ and $\mathbf{R}^{2,19}$ is the orthogonal complement in $H^2(X)$ of a fixed Kähler class.

When $n = 2$ there is a Weierstrasse representation of maximal positive submanifolds. This leads to a holomorphic description of branched maximal sections in terms of a complex structure on M and a holomorphic 1-form with values in $V \otimes \mathbf{C}$, which is the usual complex geometry point of view.

Remarks

- One could also take $m = 3$ and consider 4-torus fibres, as in the work of Baraglia (who was the first to introduce the maximal submanifold equation in G_2 geometry). But then one would need some other class of singular fibres.
- The metric geometry around the double points of the singular fibres should be modelled on the Calabi-Yau metric on $\mathbf{C}^3$ discovered by Yang Li.
- There should be a theory with $n = 4$, for fibred $\mathrm{Spin}(7)$-manifolds with hyperkähler fibres.

Enumerative geometry
When $n = 2$, with a quadratic differential on a Riemann surface, there are theories counting configurations of trajectories on the surface, as described in Ivan Smith's talk, and these are related to (special) Lagrangian submanifolds in the corresponding complex 3-fold X and hence the Fukaya category of Z.

When $n = 3$ there are similar (partly conjectural) theories counting configurations of local gradient flow lines and level sets, related to associative and coassociative submanifolds in the G_2 setting. For the case of ALE fibres these were considered by Pantev and Wijnholt and in the K3 case by the speaker and Scaduto, as presented in the June 2019 collaboration meeting and arxiv 2004.07314.

SECTION 3.

3.1 **Nash-Moser Theory**

(**Sketch**: following X.Saint-Raymond, *A simple Nash-Moser Implicit function theorem* L'Enseignement Math. 1989)

Let V, W be Fréchet spaces, with scales of norms $\| \|_k$: for example C^∞ functions with the C^k norms.

Suppose that there are *smoothing operators* $S_\epsilon : V \to V$ with, for $r \geq 0$:

$$\|S_\epsilon u\|_{k+r} \leq C\epsilon^{-r} \|u\|_k$$

and

$$\|u - S_\epsilon u\|_{k-r} \leq C\epsilon^{r} \|u\|_k .$$

And suppose that the norms satisfy interpolation inequalities.

Let $\mathcal{U} \subset V$ be an open subset and $\mathcal{F} : \mathcal{U} \to W$ be a smooth map—for example a differential operator—satisfying certain natural estimates.
Suppose that for each $u \in \mathcal{U}$ there is a right inverse Q_u to the derivative of $\mathcal{F}$ at u which obeys an estimate, for some d:

$$\|Q_u \sigma\|_k \leq C \left(\|\sigma\|_{k+d} + \|u\|_{k+d} \|\sigma\|_{2d} \right)$$

Suppose that $\mathcal{F}(u_0) = 0$ for some $u_0 \in \mathcal{U}$. Then for ρ in a neighbourhood of 0 in W there is solution near to u_0 of the equation $\mathcal{F}(u) = \rho$.

ALSO (the case we need):

if $\mathcal{F} = \mathcal{F}_0$ is one of a smooth family of maps $\mathcal{F}_t$, where $t \in \mathbf{R}^N$ say, then for small t there is a solution u_t near u_0 of the equation $\mathcal{F}_t(u_t) = 0$.

The point is that , compared to the implicit function theorem in Banach spaces, the inverse Q of the linearisation is allowed to “lose” any fixed number of derivatives.

The proof involves introducing judicious smoothings of the sequence of approximations obtained by Newton’s method.

3.2 **The differential-geometric set-up (for the branched maximal section problem).**

First, imagine that Σ is empty. The orthogonal complement of the image of the derivative of h_0 is a line subbundle $\nu \subset V$. We consider a normal variation $h(x) = h_0(x) + f(x)$ where f is a section of ν. We let $\mathcal{F}(f)$ be the projection of the mean curvature of the section h to ν. So $\mathcal{F}$ maps (small) sections of ν to sections of ν. The derivative at 0 is given by a “second variation’ formula:

$$\mathcal{L}(f) = \nabla^* \nabla f + |S|^2 f,$$

where S is the second fundamental form of the section. This is a positive operator, hence invertible.

On a neighbourhood Ω of $\Sigma \subset M$ we can write $E = E_+ \times_M E_-$ where $E_\pm$ are flat bundles.

Choose the line subbundle ν to be given by E_- on Ω and by the normals, as above, away from Ω.

Suppose for simplicity that the flat bundle E_+ is trivial over Ω and fix a trivialisation. Then we can identify $\Omega \subset M$ with an open set $\Omega' \subset \mathbf{R}^n$ so that Σ maps to a submanifold $S \subset \Omega' \subset \mathbf{R}^n$ and we have a flat line bundle E_- over the complement of S.

Under this identification, the section h_0 is $h_0(x) = (x, u_0(x))$ for $x \in \Omega'$ and u_0 a section of E_-.

Let $\psi : \Omega' \to \Omega'$ be a small compactly-supported diffeomorphism and f be a section of E_- (with suitable behaviour at the singular set S). Then we can define a new section h over Ω' by

$$h(x) = (\psi(x), u_0(x) + f).$$

We require that $d\psi$ is an isometry on the normal bundle of S.

In this way we parametrise variations of h_0 by pairs (f, ψ) where ψ is a suitable diffeomorphism of a neighbourhood of S (or equivalently Σ) and f is a section of the line bundle ν.

We define $\mathcal{F}(\psi, f)$ to be the projection of the mean curvature of this section to ν.
Near Σ, in the co-ordinates given by the identification with $\Omega' \subset \mathbf{R}^n$, we have

$$\mathcal{F} = \Delta_g(u),$$

where $u = u_0 + f$ and g is the metric

$$g_{ij} = \langle \psi_i, \psi_j \rangle - u_i u_j.$$

The point here is that the singularity of u is at the *fixed* set S.

Our condition on ψ means that the metric is standard on the normal bundle of S.

The derivative of $\mathcal{F}$ with respect to f is a Laplace-type operator

$$\mathcal{L} : \Gamma(\nu) \to \Gamma(\nu).$$

The derivative of $\mathcal{F}$ with respect to ψ is a linear operator P taking vector fields $\widetilde{v}$ on M to $\Gamma(\nu)$. One finds the formula

$$P(v) = \nabla_{\widetilde{v}}\mathcal{F} - \mathcal{L}(\nabla_{\widetilde{v}}u).$$

III.3 **Function spaces and estimates**

The flat model is the Laplace operator Δ acting on sections of the line bundle with holonomy -1 over $\mathbf{C}^* \times \mathbf{R}^{n-2}$. The Greens function can be expressed by a formula involving Bessel functions. It has homogeneity $G(\lambda x, \lambda y) = \lambda^{2-n} G(x, y)$.

Fix $\alpha \in (0, 1/2)$. We work with Hölder spaces $C^{,\alpha}$. Note that in this context if f is a section in $C^{,\alpha}$ then $|f| = O(r^{\alpha})$ where $r = |z|$.

By estimating integrals one finds that if $\rho \in C^{,\alpha}$ then $f = G\rho$ has an asymptotic expansion

$$f \sim \mathrm{Re}\left(A_\rho z^{1/2} + B_\rho z^{3/2}\right),$$

for complex valued functions A_ρ, B_ρ on $\mathbf{R}^{n-2}$ with

$$A_\rho \in C^{1,\alpha+1/2} \quad , \quad B_\rho \in C^{,\alpha+1/2}$$

Let

$$\mathcal{D}^k = \{\rho : D\rho \in C^\alpha\},$$

for all operators D which are the product of at most k *tangential vector fields*. For example, $r\partial_r, \partial_\theta$ and any derivative in the $\mathbf{R}^{n-2}$ direction are tangential, but not ∂_r.
Let

$$\mathcal{E}^{k+2,\alpha} = \{f \in L^\infty : \Delta f \in \mathcal{D}^{k,\alpha}.$$

Then we have

$$A : \mathcal{E}^{k+2} \to C^{k+1,\alpha+1/2}.$$

Let $\mathcal{E}_0^{k+2} \subset \mathcal{E}^{k+2}$ be the kernel of A.

The Fréchet spaces $\mathcal{E}_0^\infty = \bigcap \mathcal{E}_0^k$ and $\mathcal{D}^\infty = \bigcap \mathcal{D}^k$ give a good setting for the nonlinear analysis.

If $u \in \mathcal{E}_0^{k+2}$ then one finds that

$$\Delta u, \sum u_{ij} u_i u_j \in \mathcal{D}_k$$

We also want estimates in some larger spaces.

If $r\rho \in C^{\,,\alpha}$ then A_ρ is defined and lies in $C^{\,,\alpha+1/2}$ (but B_ρ is not defined).

(The condition that $r\rho \in C^{,\alpha}$ allows ρ which are $O(r^{-1/2})$)

We define similar function spaces on the manifold M. Then we have set up our problem as

$$\mathcal{F} : \mathcal{U} \to \mathcal{D}^\infty,$$

with $\mathcal{U} \subset (\mathcal{E}_0^\infty \times C^\infty)$.

That is $\mathcal{F}(u, \psi)$ where $u \in \mathcal{E}_0^\infty, \psi \in C^\infty$.

We just have to check that this map, with these function spaces, satisfies the conditions to apply the Nash-Moser theory.

To invert the linearised operator we have to solve the equation

$$P\widetilde{v} + \mathcal{L}f = \rho,$$

for any $\rho \in \mathcal{D}^{\infty}$, where $\widetilde{v}$ is a vector field on M and the $O(r^{1/2})$ term $A(f)$ of f vanishes.
Following some analysis of the Laplace type operator $\mathcal{L}$, one finds that there is a solution of $\mathcal{L}g = \rho$ but with $A(g)$ not necessarily 0.

Recall that

$$P\widetilde{v} = \nabla_{\widetilde{v}}\mathcal{F} - \mathcal{L}(\nabla_{\widetilde{v}}u).$$

So for a solution we must have

$$\mathcal{L}(g + \nabla_{\widetilde{v}}u - f) = \nabla_{\widetilde{v}}\mathcal{F}$$

i.e.

$$g + \nabla_{\widetilde{v}}u - f = G(\nabla_{\widetilde{v}}\mathcal{F}),$$

where $G = \mathcal{L}^{-1}$.

The condition $A(f) = 0$ means that we need

$$A(\nabla_{\tilde{v}} u) - A(G\nabla_{\tilde{v}} \mathcal{F}) = -A(g).$$

Let v be the normal component of $\tilde{v}$ on Σ, so $v \in \Gamma(N_\Sigma)$.
THE CRUCIAL FACT is that

$$A(\nabla_{\tilde{v}} u) = \frac{3}{2}(v.B)$$

where . is the algebraic pairing

$$N_\Sigma \times N_\Sigma^{-3/2} \to N_\Sigma^{-1/2}.$$

This is essentially the formula

$$\frac{d}{dz} z^{3/2} = \frac{3}{2} z^{1/2}.$$

Since B is nowhere 0 we can write the equation as

$$\frac{3}{2}v - B^{-1}\left(A(G\nabla_{\widetilde{v}})\right) = B^{-1}A(g). \tag{1}$$

We choose an extension $\widetilde{v}$ of any normal vector field v in a standard fashion. Then this becomes an equation for v.

The derivative $\nabla_{\widetilde{v}}\mathcal{F}$ is not necessarily in $C^{\ ,\alpha}$ but it is in $r^{-1}C^{\ ,\alpha}$. Applying our estimates we find that

$$\|AG\nabla_{\widetilde{v}}\mathcal{F}\|_{C^{,\alpha+1/2}} \leq C\|\mathcal{F}\|_{\mathcal{D}^1}\|v\|_{C^{\ ,\alpha}}.$$

We can suppose that $\|\mathcal{F}\|_{\mathcal{D}^1}$ is as small as we please and this means that (1) has a unique solution v.

Section III, Context 2.
Consists of five parts (a)-(e).
(a) Small deformation are just the beginning of what would ultimately like to understand. In particular *convergence theory* for sequences and regularity. B. Zhang showed that, for the harmonic 1-form set-up, starting with an *a priori* very general definition of the singular set that set is in fact rectifiable.

These questions involving codimension-2 singular sets have are analogous to *free boundary problems*. The regularity theory for free boundary problems is difficult.

(b) *Gauge theory* gives another way in which these questions interact with special holonomy.

Beginning around 2012, Taubes found that multivalued 1-forms, spinors etc. with branch sets arise from limits of solutions of various coupled equations for connection + additional field, over 3-manifolds and 4-manifolds.

The programme of Haydys and Walpuski is to use the count of solutions of generalised Seiberg-Witten equations over calibrated codimension 4-submanifolds to (at least partially) get around the compactness problems in putative enumerative theories for Yang-Mills instantons on G_2 and $\mathrm{Spin}(7)$ manifolds.

It seems likely that there are “stable” solutions to the G_2 and $\mathrm{Spin}(7)$ instanton equations with codimension 6 singularities (see (e) below) and that these should be related to the codimension 2 singularities on calibrated submanifolds via “Taubes/Walpuski gluing”.
In the Hermitian Yang-Mills case, over Calabi-Yau 3-folds, there are some results of Doan and of Yang Li in this direction.

(c)
Another aspect is that these branched solutions may model sequences of calibrated submanifolds collapsing to a multiple cover, as is familiar for complex curves.
There is work of Doan and Walpuski in this direction, in the case of associative submanifolds, and work in progress by Siqi He, for Special Lagrangians.

(d)
There are parabolic versions of both the problems that we discussed. In the second case this is given by mean curvature flow away from the singular set, and one expects that this flow is related to Bryant's G_2-Laplacian flow and the 6-dimensional reduction of that.

The Nash-Moser theory can also be applied to parabolic equations and probably can be used to establish short time existence.

Even the simplest situation on a Riemannian surface with a moving set of points Σ and the ordinary heat equation, seems interesting.

(e)
The work of Yuanqi Wang on singular G_2-instantons (arxiv 2011.15042, and his presentation at the September 2020 collaboration meeting) suggests a general framework for studying “free” singular sets.

Suppose that we have some natural equation with a model singular solution having a point singularity in dimension p *i.e.* over $\mathbf{R}^p \setminus \{0\}$. Suppose that this has a conical structure, so is defined by data over S^{p-1} (the “tangent cone”).

Let $\mathcal{L}$ be the linearised operator, so there are separated solutions of $\mathcal{L}f = 0$ of the form $f = r^\lambda g$ where g is an eigenfunction with eigenvalue λ of a corresponding operator over S^{p-1}.

For large enough λ these will correspond to genuine deformations to solutions of the non-linear problem (near 0), with the same point singularity. But for some λ they will not.

On the other hand, to invert $\mathcal{L}$, i.e. to solve $\mathcal{L}f = \rho$ near 0 for general ρ, we need to use functions that grow according to a suitable collection of these eigenvalues.

Let H be the space spanned by eigenfunctions on S^{p-1} which:

- *Are* required to invert $\mathcal{L}$;
- *Do not* correspond to deformations with the same point singularity.

Translations of $\mathbf{R}^p$ give obvious deformations of the model solution, not fixing the singular point. So in suitable cases we will get a linear map

$$W : \mathbf{R}^p \to H.$$

Suppose that W is surjective. Then if we have a solution of our equation on a manifold of dimension $p+q$ with a codimension-p singular set Σ and a singularity modelled, transverse to Σ, on that above it should be possible to analyse the deformations using techniques like those we have discussed in these lectures.

In Wang's case the relevant tangent cones come from holomorphic bundles E over $\mathbf{CP}^2$. He shows that the space H is $H^1(\mathrm{End}E(-1))$ and the map W is defined by product with the "Atiyah class" of E.

Another interesting case to consider is that of harmonic maps. The spectral discussion is then related to results on conformal transformations and stability of harmonic maps of spheres (and thence to work of Simons for other variational problems).
In codimension 4, the Hopf map $S^3 \to S^2$ is a possible "tangent cone". It seems likely that the condition above is satisfied. Singular harmonic maps with this tangent cone would be differential geometric generalisations of rational functions in algebraic geometry.