Towards enumerative theories for structures on 4-manifolds

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Talk based on a paper:

Fredholm topology and enumerative geometry: reflections on some words of Michael Atiyah

To appear in Proceedings of the Gokova conference on geometry and topology, special issue in memory of Atiyah.

The paper and talk are to some degree speculative.

The words in question:

“What we have here is a nonlinear way to use the index theorem”.

“You need to be careful compactifying moduli spaces: people spend their lives doing that”.

MOTIVATION 1

In our collaboration we have some ambitious long term goals e.g describe moduli spaces of $G_2, Spin(7)$ structures, ...

There are many basic things we do not understand. Thus it is interesting to study related questions which are in some respects well-understood.

MOTIVATION 2

We like “enumerative” or “nonlinear Fredholm” theories such as holomorphic curves in symplectic manifolds, gauge theories on 4-manifolds. . . .

We would like to find more examples.

In this talk we will discuss such ideas in connection with structures on 4-manifolds.

- complex structures of general type (possible there is a rigorous theory using known technology);
- self-dual structures (no prospect of a full theory in near future, but some interesting features).

BACKGROUND 1

The situation we have in mind (ignoring all technicalities) is that we have a nonlinear elliptic PDE which can be written as $S = 0$ where S is a section of a infinite-dimensional vector bundle $\mathcal{E}$ over an infinite-dimensional manifold $\mathcal{B}$. Ellipticity means that the linearised equation is defined by a linear Fredholm operator, which has an index μ.

We also want to have some kind of mechanism which gives at least a hope that the set of solutions, the zero set $Z = S^{-1}(0)$, is compact or can be compactified in a reasonable way.

Example 1 Holomorphic curves in a symplectic manifold with compatible almost complex structure.
The linearised operator is (roughly) the $\overline{\partial}$-operator on sections of the normal bundle. Partial compactness is given by control of area.

Example 2 Yang-Mills Instantons on a 4-manifold.
The linearised theory is expressed in terms of an elliptic complex

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A^+} \Omega^2_+$$

corresponding to

$$\text{gauge transformation} \qquad \text{connections} \qquad F^+(A).$$

Partial compactness is given by control of $\|F\|_{L^2}$.

Example 2a Holomorphic vector bundles over complex surfaces.
This is very close to Example 2. The linearised theory is expressed in terms of a complex

$$\Omega^0 \xrightarrow{\overline{\partial}_A} \Omega^{0,1} \xrightarrow{\overline{\partial}_A} \Omega^{0,2}$$

For technical reasons we restrict to "stable" bundles. These have Hermitian-Yang-Mills connections which gives (partial) compactness.

In such situations we hope to define an “Euler class” of the infinite dimensional bundle $\mathcal{E}$ which (roughly) should be an element

$$\zeta \in H_\mu(\mathcal{B}).$$

If we know cohomology classes $\psi \in H^\mu(\mathcal{B})$ we get **numbers** $\langle \psi, \zeta \rangle$.

DIFFERENTIAL GEOMETRY APPROACH.

If necessary, perturb S to a generic section S'. Then the zero set Z' is a manifold of dimension μ and ζ is its fundamental class. If we want to use integral (or rational) cohomology we have to check *orientations*.

ALGEBRAIC GEOMETRY APPROACH (for relevant theories).

Endow Z with extra structure which allows the definition of a "virtual fundamental class" $\zeta \in H_\mu(Z) \to H_\mu(\mathcal{B})$.

A general condition under which this can be done is when there is a "perfect obstruction theory".

In example 2a the (real) index μ is minus twice the Euler characteristic $h^0 - h^1 + h^2$ which is given by the Riemann-Roch formula. For our purposes we can assume that $h^0 = 0$ so the complex dimension of the virtual fundamental class is $h^1 - h^2$. There is a "Kuranishi model" for a neighbourhood in Z as the zero set of a holomorphic map $f : H^1 \to H^2$ and the extra structure is extracted from that. For example if:

$$H^1 = H^2 = \mathbf{C} \quad , \quad f(t) = t^2.$$

Then $\mu = 0$ and the point is counted with multiplicity 2.

Another kind of example is when H^2 has constant dimension ν over Z and all Kuranishi models have f identically zero (the "unobstucted case"). Then Z is a manifold of complex dimension $\mu + \nu$ and the $H^{2\prime}s$ define a rank ν vector bundle $\underline{H} \to Z$. The homology class ζ is the Poincaré dual of the Euler class of $\underline{H}$.

The situations where these ideas apply are special and "low-dimensional".

In higher dimensions we typically get overdetermined equations (such as for complex submanifolds of dimension > 1) and deformation theories with higher cohomology groups. We do not usually have a well-defined "expected dimension".

BACKGROUND 2

Suppose that we have some kind of moduli space M of structures on a compact oriented n-manifold X, modulo diffeomorphism. Suppose that these structures have no symmetries.

We can form a universal family $U \to M$ which is fibre bundle with fibre X. The tangent bundle along the fibres is a bundle $T_v \to U$. Let c be an r-dimensional characteristic class for the relevant structure on TX. Then we have $c(T_v) \in H^r(U)$. There is an integration-over-the-fibre map

$$\mathrm{Int} : H^*(U) \to H^{*-n}(M)$$

so we get

$$I(c) = \mathrm{Int}(c(T_v)) \in H^{r-n}(M).$$

Then we can take cup products of these classes in $H^*(M)$.

If the structures have only finite symmetry groups this works provided we use cohomology with rational coefficents.

The classical case is when $n = 2$ and $M = M_g$ is the moduli space of complex structures on curves of genus $g \geq 2$. Then we take $c = c_1^p \in H^{2p}$ and we get *Miller-Mumford-Morita classes* in $H^{2p-2}(M_g)$.

There is a very extensive literature about these, and many famous results.

In higher dimensions the classes are called generalised Miller-Mumford-Morita (MMM) classes or tautological classes.

If we are considering structures on a manifold X satisfying some equation modulo diffeomorphisms the linearised theory will look like

$$\mathrm{Vect}(X) \xrightarrow{L} \mathcal{V}_1 \xrightarrow{D} \mathcal{V}_2$$

where L is the Lie derivative, $\mathcal{V}_1$ is the space of infinitesimal deformations of the structure in question, $\mathcal{V}_2$ is the space where the equation lives and D is the linearised equation. We want to find cases where this makes a 2-step elliptic complex:

$$0 \to \mathrm{Vect}(X) \xrightarrow{L} \mathcal{V}_1 \xrightarrow{D} \mathcal{V}_2 \to 0 \qquad (1)$$

To sum up so far, we want to look for structures on manifolds with some partial compactness and a deformation theory with a 2-step complex (1).

Then we may try to define a class ζ which can be paired with the MMM classes to give numbers.

In dimension 4 we have

- Complex structures. The equation is $N(J) = 0$ where J is an almost complex structure and $N(J)$ is the Nijenhuis tensor. The linearised theory (1) is the $\overline{\partial}$ complex with values in the tangent bundle. The relevant (complex) index is
$$\frac{5}{6}c_2 - \frac{7}{6}c_1^2.$$
- Self-dual conformal structures. Recall that the Weyl curvature of an oriented Riemannian 4-manifold splits into $W^+ \oplus W^-$ and is conformally invariant. $W^\pm$ is a section of the rank 5 bundle $s^2_0(\Lambda^2_\pm)$. A conformal structure is self-dual if $W_- = 0$. We have an elliptic complex (1) with $\mathcal{V}_1$ the sections of $s^2_0 T^*$ and $\mathcal{V}_2$ the sections of $s^2_0(\Lambda^2_-)$. (Check: $4 - 9 + 5 = 0$.) The index is
$$\frac{1}{2}\left(29\sigma(X) - 15e(X)\right).$$

Remarks on self-dual structures

- We have MMM classes in dimensions 0 modulo 4.
- There is some mechanism in the direction of compactness, through a bound on the L^2 norm of W.
- Compactness is not well understood in general. One probably wants to fix a metric in the conformal class, such as the Yamabe metric of constant scalar curvature. Then there are results of Tian and Viaclovsky, assuming a Sobolev constant bound.

- One example known of a moduli space is when X is the $K3$ manifold with orientation reversed. Then the Yau metrics are self-dual and the moduli space is

$$M_{K3} = \mathrm{Gr}_{-}(3, \mathbf{R}^{19,3})/\Gamma,$$

where Γ is a known discrete subgroup of the orthogonal group $O(19, 3)$.
- The dimension of M_{K3} is $3 \times 19 = 57$ whereas the virtual dimension is 52. The deformation complex (1) has 5-dimensional H^2, represented by the parallel sections of $s_0^2(\Lambda_-^2)$.

- The obstruction bundle $\underline{H} \to M_{K3}$ is orientable but the moduli space M_{K3} is not (from our knowledge of Γ). So **we do not have an orientable theory**.
- It is an interesting problem in index theory to understand the orientation of self-dual moduli spaces in general.

- We can consider moduli spaces of pairs (g, A) where g is a self-dual conformal structure and A is an anti-self-dual Yang-Mills instanton for g. In the K3 example, and perhaps in general, this theory is orientable.
- In the same vein we may couple the equations for (g, A) to

$$F_g^+(A) = 0 \quad , \quad W^-(g) = \epsilon F^-(A) * F^-(A). \tag{2}$$

Here we are assuming that we have somehow fixed a metric in the conformal class. The notation * denotes the combination of the Killing form on the Lie algebra and the quadratic map $\Lambda^2_- \mapsto s^2_0(\Lambda^2_-)$.

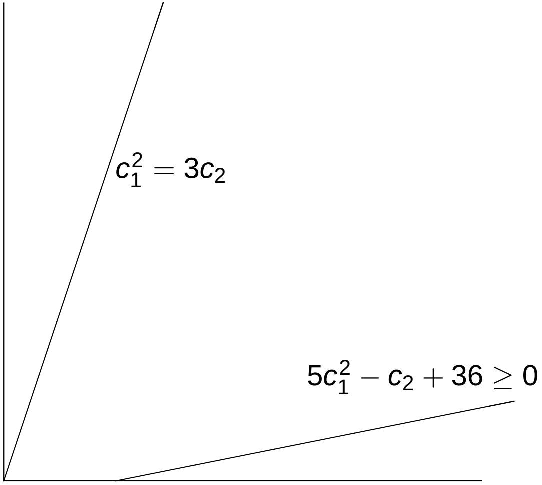
- It seems likely that for generic ϵ the moduli spaces for the coupled equation (2) will be a manifold of the expected dimension $13\sigma(X) - 9e(X) + 8k$. For the K3 case this is $8k - 8$.
- The coupled equation (2) has some interesting features. The Weitzenbock formula on Λ^2_- gives

$$\int_X |\nabla_A F^-|^2 + R|F(A)|^2 + \epsilon|W^-|^2 \leq \int_X |F^-|^3.$$

MINIMAL COMPLEX SURFACES OF GENERAL TYPE.

Roughly, these are surfaces with Kähler-Einstein metrics $\mathrm{Ricci} = -g$.
There is an algebro-geometric *KSBA compactification* $\overline{M}_{a,b}$ of such surfaces with $c_2 = a, c_1^2 = b$.

There is a huge literature (“geography”) about the values of (a, b) which occur.


$c_1^2 = 3c_2$
$5c_1^2 - c_2 + 36 \geq 0$

The KSBA compactification is analogous to the Deligne-Mumford compactification $\overline{M}_g$.

Recall that points in $\overline{M}_g$ are built out of curves with marked points, glued in normal crossings. A component of genus 0 must have at least 3 marked points and a component of genus 1 at least 1 marked point.

The KSBA compactification is similar, built out of components which are surfaces with marked divisors. There is a numerical condition on each (surface,divisor) component. In codimension 1 the components are glued in normal crossings. In codimension 2 (i.e. a finite number of points) there is a condition on the kind of singularities allowed.

The points in $\overline{M}_{a,b}$ can be identified with certain singular/infinite diameter Kähler-Einstein metrics (Berman-Guenancia), extending the hyperbolic geometry description of $\overline{M}_g$.

There is a lot of scope for work on the asymptotics of these metrics.

The proposal is to define for each relevant (a, b) a virtual fundamental class $\zeta_{a,b}$ which can be paired with the MMM classes.

The latter make up a polynomial ring on infinitely many generators

$$\sigma_{pq} = I(c_1^p c_2^q).$$

This is a very large ring, so we would get a very large collection of pairings.

The class $-I(c_1^3)$ is particularly interesting because it is first Chern class of an ample line bundle on the moduli space.

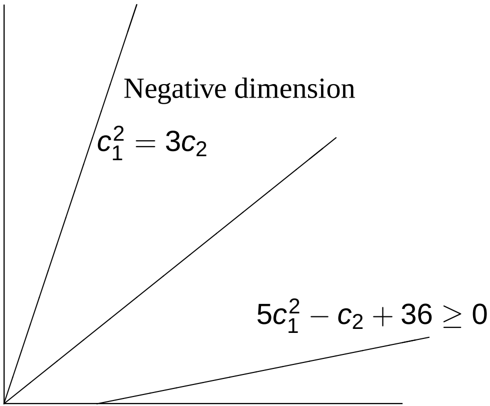
Within algebraic geometry we can ask if there is a perfect obstruction theory on $\overline{M}_{a,b}$. There is work of Yunfeng Jiang in this direction.

We can also imagine attacking these questions differential geometrically.
Whatever the outcome, it may shed light on related questions such as differential geometric approaches to DT theory.

Possible criticisms

- Shaky foundations,
- Impossible to calculate,
- What is the point?

Many surfaces of interest lie in moduli spaces of negative virtual dimension, where our discussion is nugatory.


Negative dimension
$c_1^2 = 3c_2$
$5c_1^2 - c_2 + 36 \geq 0$

Related to that, many other familiar surfaces lie in moduli spaces of the “wrong” dimension.

For example if $S \subset \mathbf{CP}^3$ is a surface of degree $d \geq 6$ then

$$H^2(TS) \cong \mathrm{coker}\, s^{d-4}(\mathbf{C}^4) \to s^{d-5}(\mathbf{C}^4) \otimes \mathbf{C}^4,$$

so there is an excess dimension of $(d-2)(d-3)(d-5)/2$.

There are also examples where the moduli space has many components of *different* dimensions.

AN EXAMPLE.

Take homogeneous coordinates (x_1, y_1, x_2, y_2) on $\mathbf{P}^3$. Let G be the finite group generated by $x_i \mapsto \lambda_i x_i, y_i \mapsto \lambda_i^{-1} y_i$ with $\lambda^6 = 1$ and by interchanging (x_1, y_1) and (x_2, y_2).

We consider G-invariant surfaces of degree 6. Leaving out some “unstable’ surfaces we get a 2-parameter family S_{AB} with equation

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + AH_+^3 + BH_+H_-^2 = 0,$$

where $H_\pm = x_1 y_1 \pm x_2 y_2$.

But the obstruction space is isomorphic to the G-invariant part of $\Lambda^2 \mathbf{C}^4$ which is 1-dimensional, spanned by $dx_1 dy_1 + dx_2 dy_2$. So the virtual complex dimension is 1 and we want to compute the pairings with the two dimensional cohomology classes $I(c_1^3), I(c_1 c_2)$.

The naive, or GIT, compactification $\overline{M}_{GIT}$ of the moduli space is $\mathbf{CP}^2$, adding a line at infinity.
(In fact there is a symmetry $S_{AB} = S_{-A,-B}$ which we will ignore here.)
The points at infinity correspond to certain algebraic cycles, which are unions of quadrics $Q_\lambda = \{x_1 y_1 + \lambda x_2 y_2 = 0\}$ possibly with multiplicity.
These quadrics all meet in a configuration $\Diamond$ of 4 lines.

The KSBA compactification $\overline{M}_{KSBA}$ is different.

The easiest part to understand is the family

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + AH_+^3 = 0$$

with $A \to \infty$. We should take the limit to be a 3-fold branched cover of the quadric Q_{+1}, rather than the quadric with multiplicity 3. The branch curve is the intersection of Q_{+1} with $S_{0,0}$.

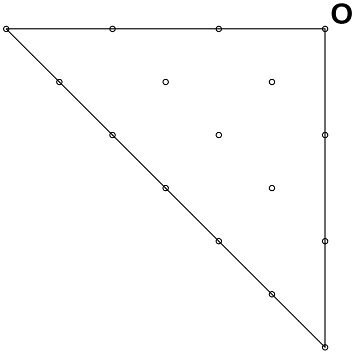
There is a similar, slightly more complicated, discussion for the KSBA limit of the family

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + BH_+H_-^2 = 0$$

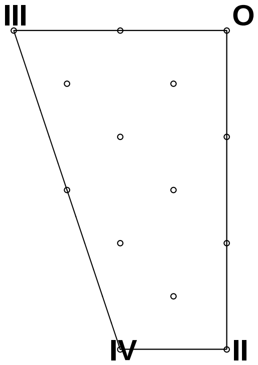
as $B \to \infty$.

Let $C = Q_{-1} \cap S_{0,0}$ and form the double cover of the quadric Q_{-1} branched over $C \cup \Diamond$.
This has 24 ordinary double points. Blowing these up and then collapsing the proper transform of $\Diamond$ gives the KSBA limit.

The compactifications $\overline{M}_{GIT}, \overline{M}_{KSBA}$ have toric descriptions. The first is standard.



For $\overline{M}_{KSBA}$ we get



The most interesting point is the vertex IV which corresponds to a surface S_{IV} with 4 components.

Let $\mathrm{Cone}(\Diamond)$ be the projective cone over $\Diamond$. The surface S_{IV} is a triple branched cover of $\mathrm{Cone}(\Diamond)$. There is a well-known degeneration of the quadric to $\mathrm{Cone}(\Diamond)$. Going backwards, S_{IV} deforms to a family of triple covers of the quadric which correspond to the edge from IV to III.

The surface S_{IV} can also be written as a (birational) double cover of $\mathrm{Cone}(\Diamond)$ and this gives another deformation to a family of double covers, corresponding to the edge from IV to II.

Now we can go ahead and calculate everything.

We have $H_2(\overline{M}_{KSBA}, \mathbf{Q}) = \mathbf{Q}^2$ with generators D_{II}, D_{III} corresponding to the edges OII and OIII of the toric quadrilateral. One finds that

$$\zeta.D_{III} = -1/6 \quad , \zeta.D_{II} = -1/4$$

and that

$$\langle I(c_1^3), \zeta\rangle = 12 \quad , \quad \langle I(c_1 c_2), \zeta\rangle = -11/192.$$