Problems with counting G_2-instantons and associatives, and generalized Seiberg–Witten equations

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Abstract. There is a naive hope that counting G_2-instantons (or associatives) might lead to invariants for G_2-manifolds. However, numerous geometric degenerations threaten the invariance of such counts (pointed out by Donaldson–Segal, Joyce, Nordström, Doan–W.). Initially unrelated to this, generalized Seiberg–Witten equations have been studied. A formal analysis suggests that the degeneration phenomena of the ADHM Seiberg–Witten equations and (some of) those encountered for G_2-instantons and associatives have the same origin. Based on this observation, one might hope to construct counting invariants for G_2-manifolds after all. The purpose of this talk is to explain this in more detail and to report on some progress regarding the analysis of generalized Seiberg–Witten equations (joint with Aleksander Doan and Boyu Zhang).

Y G_2-manifold

(1) dCS^Y closed 1-form on $\mathcal{A}/\mathcal{G}$

crit. pts: G_2-instantons

✓ ell. PDE ✓ top energy identity

(2) $d\,\mathcal{Y}$ closed 1-form on $\mathcal{S}$

crit. pts: associative submfds

✓ ✓

Q: Does counting crit pts give deformation invariant number?

KEY ISSUES: degenerations / non-compactness

(1) G_2-instantons can degenerate by:

(a) Bubbling of (Fueter sections of) ASD instantons along associatives (Donaldson–Segal)

This is a codim 1 phenomenon.

(b) Formation of non-Kähler singularities. Very poorly understood.

(2) Associatives can degenerate by:

(a) Developing singularities

(Joyce, Nordström)

(b) Multiple covering phenomena

In GW theory, such covers may not contribute.

UPSHOT: Counts of G_2-instantons or associatives cannot be invariant?

Algebraic data:

- $G \triangleleft H \to Sp(S)$ quaternionic rep.
- $K = H/G$ flavour group
- hyperkähler moment map (for $G \to Sp(S)$) $\mu: S \to (\operatorname{Im}\mathbb{H} \otimes \mathfrak{g})$

geometric data:

- M 3-mfd. with a $Spin^H(3)$-str. $\mathfrak{s}$
- connection B on flavour bundle

$$\mathfrak{F} := \mathfrak{s} \times_{Spin^H(3)} K$$

$\rightsquigarrow$ adjoint bundle

$$Ad(\mathfrak{s}) := \mathfrak{s} \times_{Spin^H(3)} Lie(G)$$

$\rightsquigarrow$ spinor bundle

$$\mathbb{S} := \mathfrak{s} \times_{Spin^H(3)} S$$

$\rightsquigarrow \mu: \mathbb{S} \to \Lambda^2 T^*M \otimes Ad(\mathfrak{s})$

The generalized Seiberg–Witten equation associated with this is the following PDE on a connection A on $\mathfrak{s}$ inducing B on $\mathfrak{F}$ and a "spinor" $\Phi \in \Gamma(\mathbb{S})$:

$$(gSW) \begin{cases} \not{D}_A \Phi = 0 \\ F_{Ad(A)} = \mu(\Phi) \end{cases}$$

(There is an 2D & 4D versions too; even non-linear.) cf. Haydys

Examples: (1) $G = H = U(1)$ & $S = \mathbb{H}$ $\rightsquigarrow$ classical (3D) Seiberg–Witten equation.

Counting solutions gives invariants $SW(M) \in \mathbb{Z}$ satisfying (*)

(a) $SW(M_1 \# M_2) = 0$

(b) $\sum_{j=1}^{3} \pm SW(M_j) = 0$ if M_j are obtained by Dehn filling M^0 along $\mu_j \in H_1(T^2)$ w/ $\mu_1 + \mu_2 + \mu_3 = 0$

(Meng–Taubes)

associatives

associatives with SW monopoles

(i)

$P_1 \# P_2$

$P_1 \amalg P_2$

(a) $\rightsquigarrow$

no contribs from $P_1 \# P_2$

SW monopoles on $P_1 \amalg P_2$

(ii)

P_1 P P_2 P_3

(b) $\rightsquigarrow$

mono. on P_1

mono. on P_2

mono. on P_3

(*) needs $b_1 \geq 1$ (≥ 2); in general, there is monopole homology w/ analogues of (a), (b)

(2) $G = H = U(1)$ & $S = \mathbb{H}^{\oplus n}$ $\rightsquigarrow$ Seiberg–Witten with n spinors

(3) $G \triangleleft H = G \times Sp(1)$ & $\mathfrak{g} \otimes \mathbb{H}$

adjoint; right multiplication

$Spin^H(3) = SO(4) \times G$

$\rightsquigarrow$ on M, need rk 4 b'dle V w/ $\Lambda^+ V \cong T^*M$ & princ. G b'dle P

pick $V = \mathbb{R} \oplus T^*M$ and LC on $\Lambda^- V$

$\rightsquigarrow$ A conn on princ. G-bundle P

$\Phi = (\xi, a) \in \Omega^0(M, Ad(P)) \oplus \Omega^1(M, Ad(P))$

$$\begin{cases} d_A^* a = 0 \\ d_A a + * d_A \xi = 0 \\ F_A = \frac{1}{2}[a \wedge a] + *[\xi, a] \end{cases}$$

stable flat $G^{\mathbb{C}}$-conn. (w/ ξ)

(4) Haydys: $G = \mathcal{G}$ & $S = \mathcal{A}$

gauge group; conn on $\mathbb{R}^4$

$\rightsquigarrow$ G_2-instantons on $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$

Compactness for gSW?

✓ Weitzenböck formula $\rightsquigarrow$ a priori bound on $\mu(\Phi)$.

μ proper $\Rightarrow$ a priori bound on Φ $\Longleftrightarrow$ compactness

$\Updownarrow$

hyperkähler quotient

$S /\!/\!/ G = \mu^{-1}(0)/G = \{0\}$ fails for (2), (3), (4), ...

$\rightsquigarrow$ expect $|\Phi| \to \infty$

For $\Phi = \varepsilon^{-1} \Psi$ w/ $\|\Psi\| = 1$, (gSW) becomes $\varepsilon = \|\Phi\|^{-1}$

$$(g\widetilde{SW}) \begin{cases} \not{D}_A \Psi = 0 \\ \varepsilon^2 F_A = \mu(\Psi) \end{cases} \quad (\|\Psi\| = 1)$$

If $(A_j, \Psi_j, \varepsilon_j)$ is a seq. of sol. to $(g\widetilde{SW})$, w/ $\varepsilon_j \to 0$ then maybe / sometimes / in good cases:

$(A_j, \Psi_j) \xrightarrow[\text{gauge}]{\text{upto}} (A, \Psi)$ satisfying

$$(\infty) \begin{cases} \not{D}_A \Psi = 0 \\ \mu(\Psi) = 0. \end{cases}$$

This seems naive because $(g\widetilde{SW})$ becomes deg. ell. as $\varepsilon \to 0$. (∞) not elliptic.

But it is true for

- (3) w/ $G = SO(3), SU(2)$ (Taubes)
- (2) (Haydys–W.)
- "whenever $S /\!/\!/ G$ has precisely one singular stratum" (Boyu Zhang–W.)

with two caveats

- convergence only holds away from a codimension two subset $Z \subset M$
- convergence is not currently known to be in C^∞ (except for abelian G, cf. Doan–W.).

Morally, these results work because of the Haydys correspondence:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\pi"] \arrow[dr] & \mathfrak{X} = \bigcup \mathfrak{X}_{(T)} \arrow[d] \\ & M \end{tikzcd}$$

stabilizer type

b'dle of HK quotients

$X = S /\!/\!/ G = \bigcup X_{(T)}$

w/ $X_{(T)} = (S_{(T)} \cap \mu^{-1}(0))/G \cong S_T /\!/\!/ W_G(T)$

(A, Ψ) satisfying (∞) and $\Psi \in \Gamma(\mathbb{S}_{(T)})$ for fixed T

$\Updownarrow$ upto gauge

ell. problem: $s := \pi \circ \Psi$ satisfies Fueter eq'n (non-linear Dirac), $s \in \Gamma(\mathfrak{X}_{(T)})$ + extra data if $T \neq 1$;

and "Higgs mechanism" ($(\Delta_A + \varepsilon^{-2}$ non-neg. op. dep. on $\Phi) F_A =$ "lower order".)

Ex: • Let $r, k \in \mathbb{N}$.

$G = U(k) \triangleleft H = SU(r) \times Sp(1) \times U(k)$

&

$S_{r,k} = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{H} \otimes_{\mathbb{C}} \mathbb{C}^k) \oplus \mathbb{H}^* \otimes_{\mathbb{R}} \mathfrak{u}(k)$

def. inv. def. right mult. ad.

ADHM: $S_{r,k} /\!/\!/ G = \overline{M}_{k,r} \overset{r \geq 2}{=}$ Uhlenbeck compactification of framed moduli space of $SU(r)$ ASD instantons of charge k on $\mathbb{R}^4$

$\| r=1$

$Sym^k \mathbb{H} = \mathbb{H}^k / S_k$

$\rightsquigarrow$ ADHM SW equation

- $r = k = 1$ $\rightsquigarrow$ classical SW
- Specialize to $r = 1, k = 2$
  - M spin (for simplicity) 3-mfd
  - $V \to M$ or. Euclidean vector bundle of rank 4 with conn. B on $\Lambda^- V$.
  - $\Lambda^+ V \cong TM$
  - $\mathcal{H} \to M$ Hermitian vector bundle of rank 2

A conn. on $\mathcal{H}$, $\Phi = (\Psi, \xi)$ w/ $\Psi \in \Gamma(\mathbb{S} \otimes \mathcal{H})$, $\xi \in \Gamma(V \otimes Ad_0(\mathcal{H}))$

$$(SW_{1,2}) \begin{cases} \not{D}_A \Psi = 0 \\ \not{D}_A \xi = 0 \\ F_A = \mu(\Psi) + \mu(\xi) \end{cases}$$

Boyu Zhang–W.: If (A_j, Ψ_j, ξ_j) is a seq. of solutions of $(SW_{1,2})$ with $\|(\Psi_j, \xi_j)\| \to \infty$, then (after passing to subseq. and upto gauge):

(1) There are $Z \subset M$ rectifiable codim 2, A conn on $\mathcal{H}|_{M \setminus Z}$, $\Psi \in \Gamma(\mathbb{S} \otimes \mathcal{H}|_{M \setminus Z})$, $\xi \in \Gamma(V \otimes Ad_0(\mathcal{H})|_{M \setminus Z})$

(a) $\begin{cases} \not{D}_A \xi = 0 \\ \mu(\xi) = 0 \\ \|\xi\|_{L^2} = 1 \end{cases}$ (b) $|\xi|$ extends cont. to M and $Z = |\xi|^{-1}(0)$

Haydys corr. $\rightsquigarrow$ ξ mod gauge defines a 2-valued harmonic section h of V

$\widetilde{M} \setminus \widetilde{Z} := \operatorname{graph}(h)$

$\pi \downarrow 2:1$

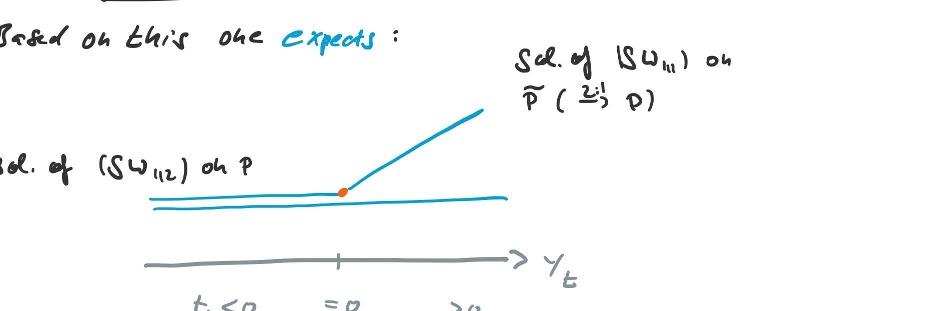
$M \setminus Z$

(c) $\mathcal{H}|_{M \setminus Z} = \pi_* \mathcal{L}$ for a line bundle on $\widetilde{M} \setminus \widetilde{Z}$, $A = \pi_* \tilde{A}$, $\Psi = \pi_* \tilde{\Psi}$, and $(\tilde{A}, \tilde{\Psi})$ sat. classical SW

$$\begin{cases} \not{D}_{\tilde{A}} \tilde{\Psi} = 0 \\ F_{\tilde{A}} = \mu(\tilde{\Psi}) \end{cases}$$

(2) $A_j \to A$, $\Psi_j \to \Psi$ (no rescaling!), and $\dfrac{\xi_j}{\|(\Psi_j, \xi_j)\|_{L^2}} \to \xi$ away from Z.

Based on this one expects:



@ • : • (Ψ, ξ) has blown-up $\rightsquigarrow$ 2-val. harmonic sect. h of NP appears, as well as SW monopole on $\tilde{P} = \operatorname{graph}(h)$

"$(SW_{1,2})$ sol. on P dies at $t = 0$, but lives on as $(SW_{1,1})$ sol. on $\tilde{P}$ for $t > 0$"