Examples of Asymptotically Conical G_2-instantons

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Plan for the Second Half

1. Instanton equations for $M_{1,1}$
2. Abelian solutions
3. Solutions with gauge group SU(2)

K. Matthies, J. Nordströom and M. Turner. $\mathrm{SU}(2)^2\times$ U(1)-invariant G_2-instantons on the AC limit of the $\mathbb{C}_7$ family. arXiv e-prints, page arXiv:2202.05028, April 2022.

The principal orbits can be written as

$$\mathrm{SU}(2)^2/\mathbb{Z}_4 \cong \mathrm{SU}(2)^2 \times \mathrm{U}(1)/K_0$$

where the stabiliser of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-action on $M_{1,1}$ away from the singular orbit of the $\mathrm{SU}(2)^2$-action is the subgroup

$$K_0 \cong \mathbb{Z}_2 \times \mathrm{U}(1).$$

K_0 can be embedded in $T^2 \times \mathrm{U}(1) \subset \mathrm{SU}(2)^2 \times \mathrm{U}(1)$ by

$$(\xi, e^{i\theta}) \hookrightarrow (\xi e^{i\theta}, e^{i\theta}, \xi^{-1} e^{-2i\theta})$$

where $\xi \in \mathbb{Z}_2 \subset \mathrm{U}(1)$.

Wang's Theorem tells us that SU(2)-bundles up to isomorphism correspond to isotropy homomorphisms $K_0 \to \mathrm{SU}(2)$.

Away from the singular orbit, there are exactly two bundles P_1 and P_{-1} corresponding to

$$(\xi, e^{i\theta}) \mapsto \begin{pmatrix} \xi e^{i\theta} & 0 \\ 0 & \xi^{-1} e^{-i\theta} \end{pmatrix} \text{ and } (\xi, e^{i\theta}) \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

respectively.

Let A be an $\mathrm{SU}(2)^2 \times U(1)$-invariant G_2-instanton on $\mathbb{R}^+ \times (\mathrm{SU}(2)^2/\mathbb{Z}_4) \cong \mathbb{R}^+ \times (\mathrm{SU}(2)^2 \times U(1))/K_0$ with gauge group $\mathrm{SU}(2)$. Then A takes the form:

$$\begin{aligned} A = f(t)(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(t)(E_1 \otimes e_1' + E_2 \otimes e_2') \\ + g(t)(E_3 \otimes \tfrac{1}{2}(e_3 - e_3')) + h(t)(E_3 \otimes (e_3 + e_3')). \end{aligned}$$

$$\begin{aligned}
2\dot{a}^3\dot{b}^2\dot{f} &= f'(1-h+\tfrac{1}{2}g)\Phi_1 + f(g-1)\chi_1 + f'(\tfrac{1}{2}g+h)\Psi_1 \\
2\dot{a}^3\dot{b}^2\dot{f}' &= f(1-\tfrac{1}{2}g-h)\Phi_1 - f'(g+1)\chi_1 + f(h-\tfrac{1}{2}g)\Psi_1 \\
2\dot{a}^4\dot{b}\dot{g} &= (f^2-(f')^2-g)(\Phi_1-\Psi_1) \\
2\dot{a}^4\dot{b}\dot{h} &= (h-\tfrac{1}{2}(f')^2-\tfrac{1}{2}f^2)\Phi_1 - 2ff'\chi_1 \\
&\quad + (h-\tfrac{1}{2}f^2-\tfrac{1}{2}(f')^2)\Psi_1
\end{aligned}$$

$$\Phi_1 = 2a^2b + r_0^3(2a^2+b^2-r_0^6) \quad \Psi_1 = b(b^2-2a^2-r_0^6) - 2a^2r_0^3$$

$$\chi_1 = a(b^2+r_0^6) + 2abr_0^3$$

The singular orbit is

$$\mathrm{SU}(2)^2/K_{1,1} \cong (\mathrm{SU}(2)^2 \times \mathrm{U}(1))/K$$

where $K \cong \mathrm{U}(1)^2$ is the stabiliser of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-action on the singular orbit and is embedded in $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ by

$$(e^{i\psi}, e^{i\theta}) \hookrightarrow (e^{i\psi}e^{i\theta}, e^{i\theta}, e^{-i\psi}e^{-2i\theta}).$$

Appealing to Wang's Theorem again, on the singular orbit, there are countable bundles P_j for $j = 2\nu - 1$ odd, corresponding to

$$(e^{i\psi}, e^{i\theta}) \mapsto \begin{pmatrix} e^{i(\nu\psi+\theta)} & 0 \\ 0 & e^{-i(\nu\psi+\theta)} \end{pmatrix}.$$

Proposition

Let $Y \subset M_{1,1}$ contain the singular orbit $S^3 \times S^3/K_{1,1}$, and be endowed with an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant AC or ALC holonomy G_2-metric of Foscolo, Haskins and Nordström. There is a 2-parameter family of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instantons A_j, parameterised by $f_0, h_0 \in \mathbb{R}$, with gauge group $\mathrm{SU}(2)$ in a neighbourhood of the singular orbit in Y, extending smoothly over P_j, for $j = 2\nu - 1$ with $\nu \in \mathbb{Z}_+$. Furthermore for A_j, the defining functions f, f', g and h satisfy

$$
\begin{aligned}
f(t) &= f_0 t^{\nu-1} + O(t^{\nu+1}), \\
f'(t) &= \frac{\beta^3(1-2h_0) + 1 - \nu}{4\nu r_0 \beta^2} f_0 t^{\nu} + O(t^{\nu+2}), \\
g(t) &= 2\nu - 1 + O(t^2), \\
h(t) &= h_0 + O(t^2).
\end{aligned}
$$

Abelian Solutions

$$
\begin{aligned}
f(t) &= f'(t) = 0 \\
g(t) &= j \exp\left(\int_0^t \frac{(b+r_0^3)((b-r_0^3)^2-4a^2)}{2\dot{a}^4\dot{b}} d\tau\right), \\
h(t) &= h_0 \exp\left(\int_0^t \frac{(b+r_0^3)^2(b-r_0^3)}{2\dot{a}^4\dot{b}} d\tau\right).
\end{aligned}
$$

By taking $h_0 = 0$ on P_j, we get a particular solution

$$
A^{\mathrm{ab}} = \frac{1}{2} g E_3 \otimes (e_3 - e_3')
$$

Denote $z = (f, f', g, h)$. Rescaling time as $t(\tau) = \exp(\tau)$ means we can write

$$\frac{dz}{d\tau} = F(z) + G(z, \tau),$$

where

$$F(z) = \begin{pmatrix} 2f'(2 - 3h + \frac{1}{2}g) + 2f(g-1) \\ 2f(2 - 3h - \frac{1}{2}g) - 2f'(g+1) \\ 6(f^2 - (f')^2 - g) \\ 2(h - \frac{1}{2}(f')^2 - \frac{1}{2}f^2) - 4ff' \end{pmatrix}$$

and with the non-autonomous part satisfying

$$\lim_{\tau\to\infty} \exp(\tau) G(z, \tau) = 0$$
$$\lim_{\tau\to\infty} \exp(\tau) D_1 G(z, \tau) = 0.$$

We consider the truncated autonomous ODE

$$\dot{z} = F(z)$$

which has steady states

$$z_0 = (0,0,0,0), \quad z_+ = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, \tfrac{1}{3}\right), \quad z_- = \left(-\tfrac{1}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}\right).$$

- All steady states are hyperbolic saddles, which implies the existence of respective stable and unstable manifolds, tangential to the stable and unstable eigenspaces.
- For z_0, both the stable and unstable manifolds are two dimensional.
- For z_+ and z_-, the stable manifold is three dimensional and the unstable manifold is one dimensional.

Within the lines $\ell_\pm = \{f = f' = \pm h\}$, there are hetroclinic orbits satisfying

$$z(\tau) \xrightarrow{\tau \to -\infty} z_0 \text{ and } z(\tau) \xrightarrow{\tau \to \infty} z_\pm .$$

These solutions lie in the intersections

$$W^u(z_0) \cap W^s(z_\pm)$$

and these intersections are transversal along $\ell_\pm$.

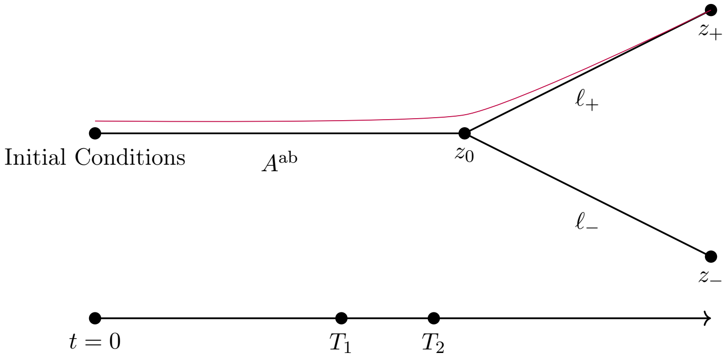
Solutions via a Dynamical Systems Approach

- The construction of the local stable manifolds relies on a contraction mapping argument.
- The limiting behaviour of the non-autonomous part of the system and it's derivative means that we can perturb solutions to the autonomous system to solutions of the full system for sufficiently large times.

Theorem

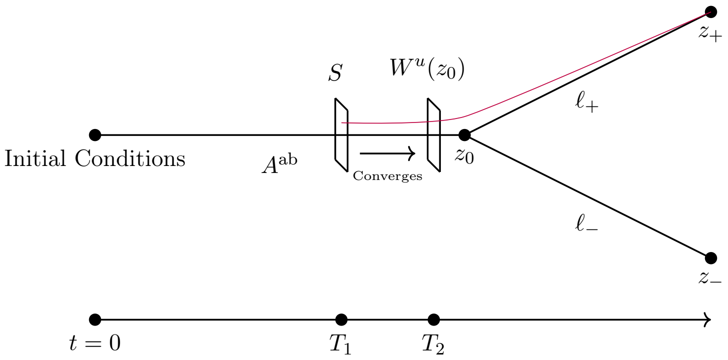
Let $M_{1,1}$ be the AC G_2-manifold of the $\mathbb{C}_7$ family and let A^{ab} be the unique abelian solution on each bundle P_j over $M_{1,1}$ for $j \in \mathbb{Z}$ odd.

- *There is a function $f_0 \mapsto h_0(f_0)$ such that the unique local solution near the singular orbit with initial condition $(f_0, h_0(f_0))$ extends to a global solution.*
- *It is a small perturbation of A^{ab} near the singular orbit.*
- *We obtain a 1-parameter family of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instantons with full gauge group $\mathrm{SU}(2)$ and bounded curvature.*
- *These instantons converge to $z_\pm$ near the lines $\ell_\pm$ along the conical end.*

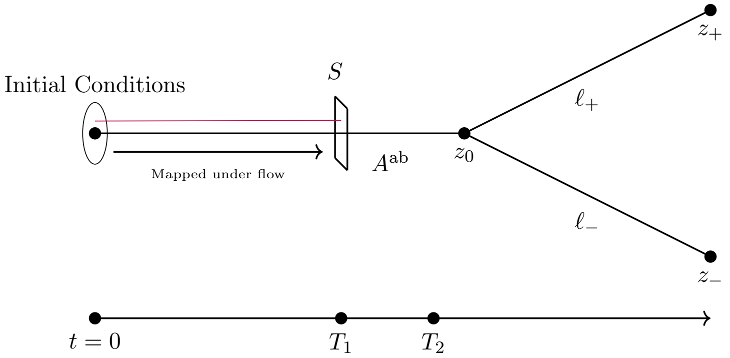

z_+
ℓ_+
Initial Conditions
A^{ab}
z_0
ℓ_-
z_-
$t = 0$
T_1
T_2

Lemma (The Inclination Lemma)

Let B^s, B^u be balls contained in the local stable and unstable manifolds, respectively, of a hyperbolic fixed point 0; set $V = B^s \times B^u$. Consider a point q in the local stable manifold, and a disc D^u of the same dimension as the local unstable manifold which is transversal to the local stable manifold at the point q. Let D^u_t be the connected component of $V \cap \Phi(t, D^u)$ to which $\Phi(t, q)$ belongs. Given $\epsilon > 0$, there exists $T \in \mathbb{R}$ such that if $t > T$, then D^u_t is ϵ C^1-close to B^u.



As S evolves with the flow, its intersection with each of the stable manifolds $W^s(z_\pm)$ is a 1-dimensional subset; this subset tends to a segment of the corresponding line $\ell_\pm$ as $\tau \to \infty$.


Initial Conditions
S
ℓ_+
z_+
z_0
A^{ab}
Mapped under flow
ℓ_-
z_-
$t = 0$
T_1
T_2

Linearise the system along the solution A^{ab}:

$$\begin{pmatrix} \widetilde{\chi}_1(g-1) & \widetilde{\Phi}_1 + \frac{1}{2}g(\widetilde{\Phi}_1 + \widetilde{\Psi}_1) & 0 & 0 \\ \widetilde{\Phi}_1 - \frac{1}{2}g(\widetilde{\Phi}_1 + \widetilde{\Psi}_1) & -\widetilde{\chi}_1(g+1) & 0 & 0 \\ 0 & 0 & \widehat{\Psi}_1 - \widehat{\Phi}_1 & 0 \\ 0 & 0 & 0 & \widehat{\Phi}_1 + \widehat{\Psi}_1 \end{pmatrix}.$$

Given $Z_0 = (f_0, h_0) \in \mathbb{R}^2$, let $Z(t, Z_0)$ be a solution on P_j with initial conditions determined by Z_0 at $t = 0$.

Lemma

Let $N(t, w) := D_2 Z(t, 0)w$ for $w \in \mathbb{R}^2$. For large times T, the image of the linear map $w \mapsto N(T, w)$ is transverse to the tangent space of $W^s(z_0)$.

We show that $N(t, w)$ is orthogonal to both of the stable eigenvectors for the fixed point z_0, namely

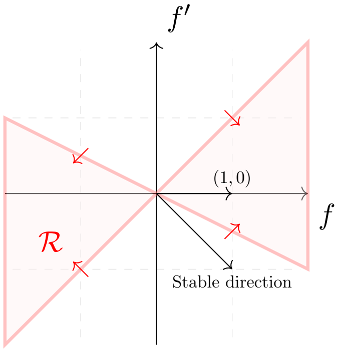
$$(0, 0, 1, 0) \qquad \text{and} \qquad (1, -1, 0, 0).$$

Firstly, since g is independent of h, we see that $N(t, (0, 1))$ is proportional to the vector $(0, 0, 0, 1)$ for all t, and hence is orthogonal to $W^s(z_0)$.

$N(\epsilon, (1,0))$ lies in the region

$$\mathcal{R} = \{f \geq f', f \geq -2f'\} \cup \{f \leq f', f \leq -2f'\}$$

for some small $\epsilon > 0$.



The fixed point z_+ defines a connection on the nearly Kähler $(S^3 \times S^3)/\mathbb{Z}_4$, namely

$$A_\infty = \frac{1}{3} \sum_{i=1}^{3} E_i \otimes (e_i + e_i').$$

The solutions on $M_{1,1}$ are asymptotic to the pull back of this nearly Kähler instanton on the cone.

There are a number of reasons why this method cannot be easily generalised to $M_{m,n}$.

- The explicit abelian solution A^{ab} has no analogue when m and n are distinct.
- The functions g and h don't decouple in the general case.

Further study: can this method be more substantially adapted to find solutions for any $M_{m,n}$?