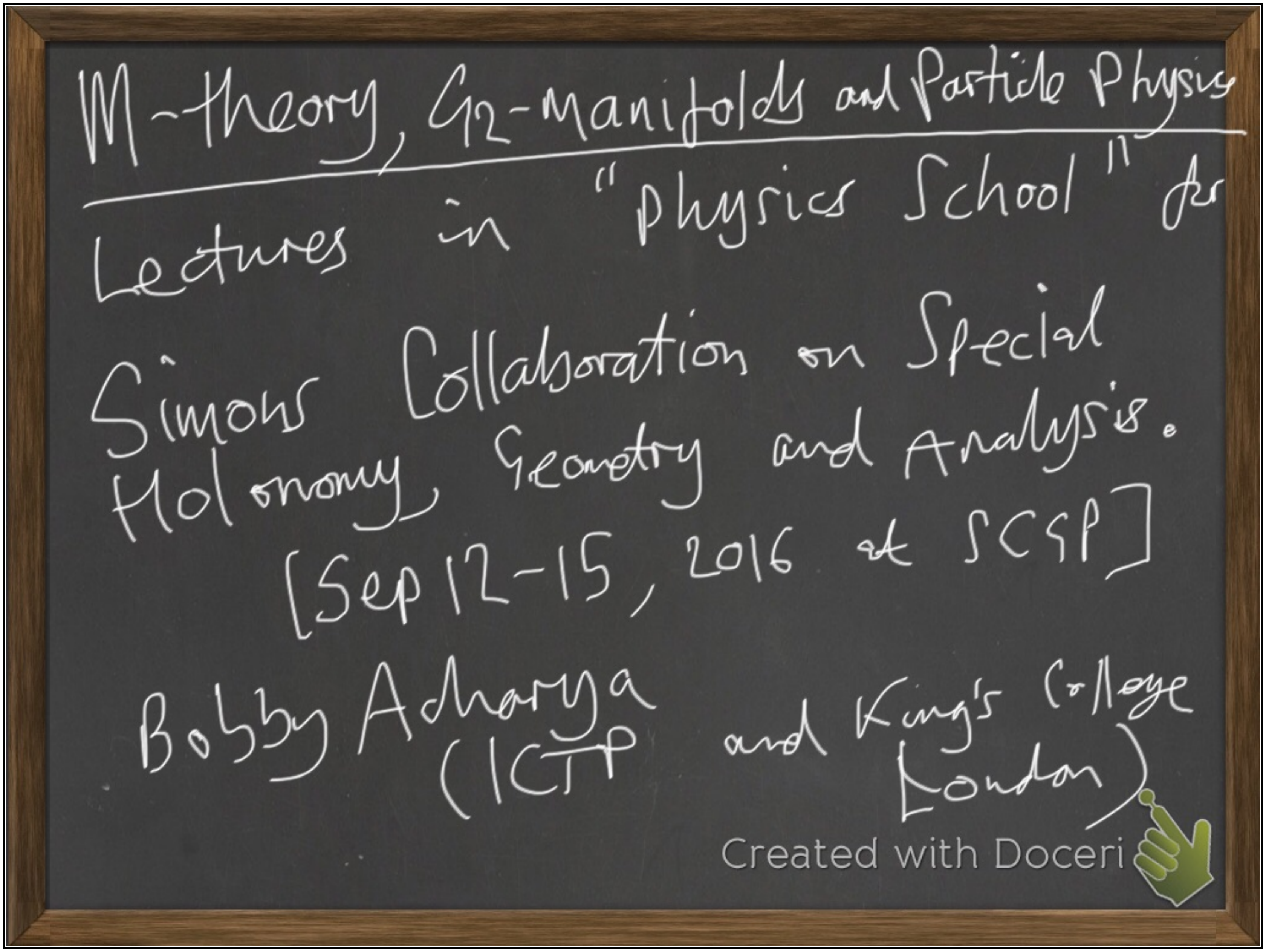

M-theory, G_2-manifolds and Particle Physics

Lectures in "Physics School" for
Simons Collaboration on Special
Holonomy, Geometry and Analysis.
[Sep 12-15, 2016 at SCGP]

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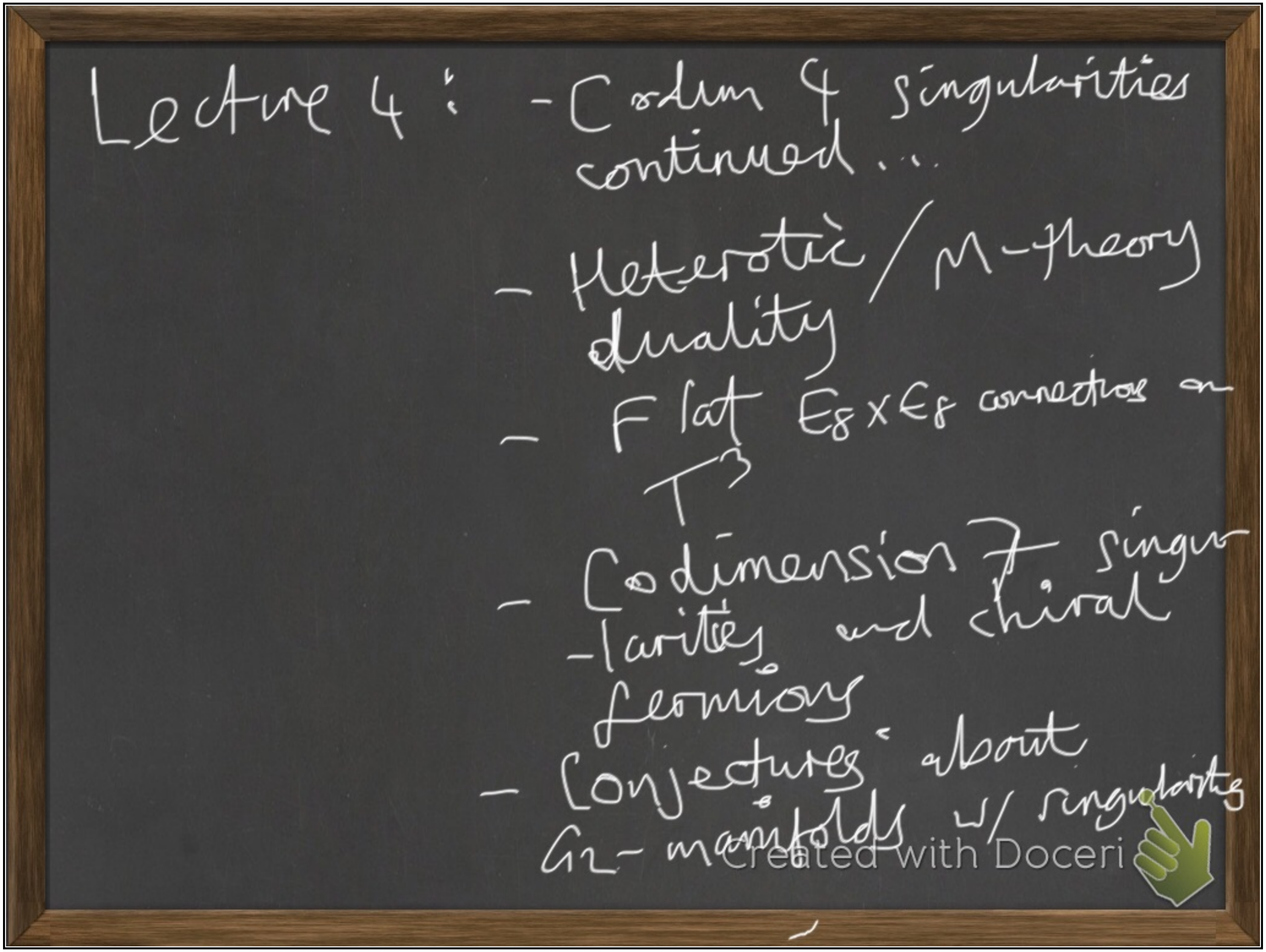
London)

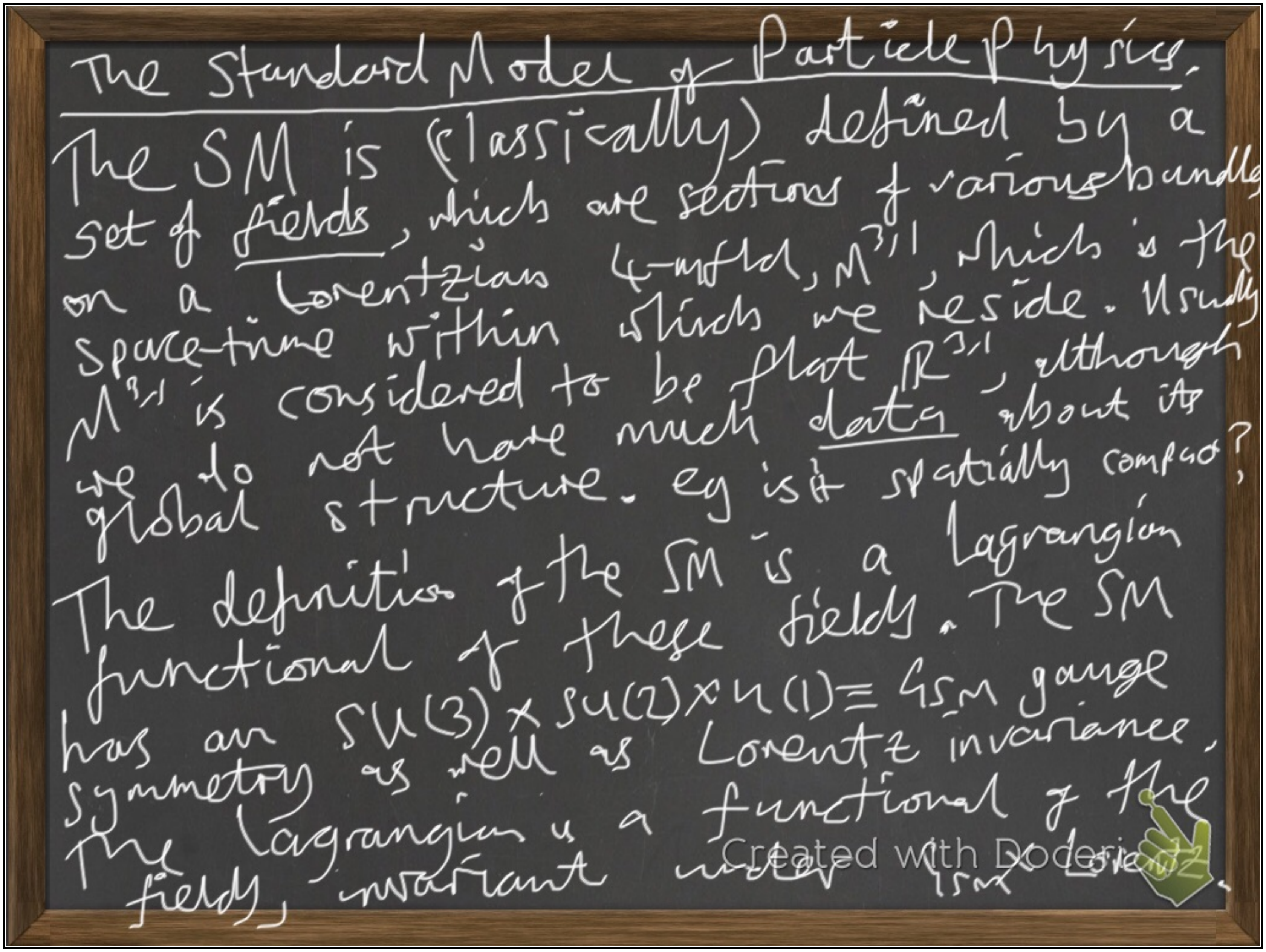
Outline

- Lecture 1 : Define the *fields* of the STANDARD MODEL of PARTICLE PHYSICS
- Lecture 2 : Define the SM and describe how it makes physical predictions
- Lecture 3 : Describe more general gauge theories, including supersymmetric ones, briefly in the same language. Describe how codim 4 singularities give rise to Non-Abelian gauge fields



Lecture 4: - Codim 4 singularities continued...

- Heterotic/M-theory duality
- Flat $E_8 \times E_8$ connections on T^3
- Codimension 7 singularities and chiral fermions
- Conjectures about G_2-manifolds w/ singularities



The Standard Model of Particle Physics.

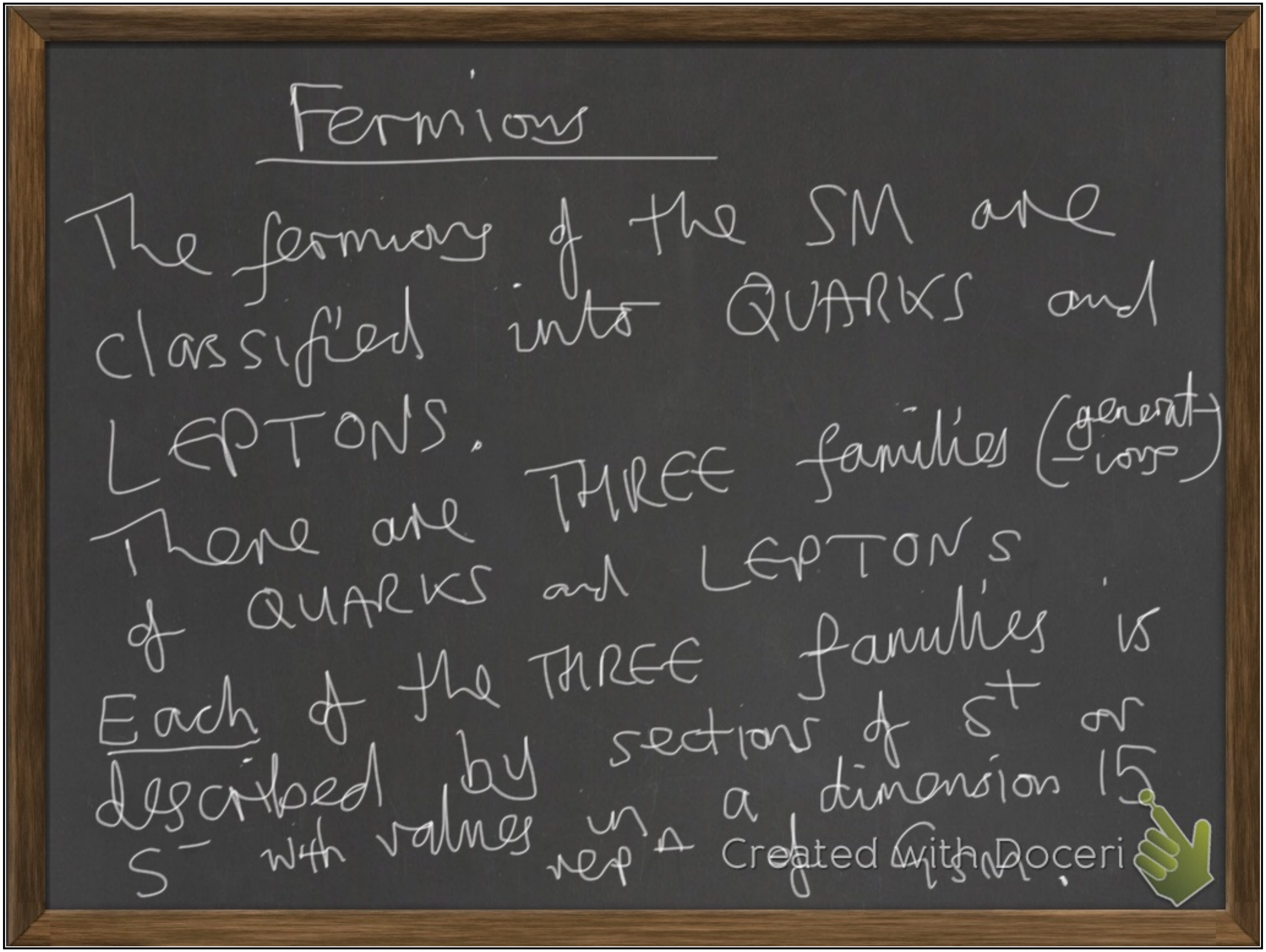
The SM is (classically) defined by a set of fields, which are sections of various bundles on a Lorentzian 4-mfld, $M^{3,1}$, which is the space-time within which we reside. Usually $M^{3,1}$ is considered to be flat $\mathbb{R}^{3,1}$, although we do not have much data about its global structure. eg is it spatially compact?

The definition of the SM is a Lagrangian functional of these fields. The SM has an $SU(3) \times SU(2) \times U(1) \equiv G_{SM}$ gauge symmetry as well as Lorentz invariance. The Lagrangian is a functional of the fields, invariant under $G_{SM} \times$ Lorentz.

Each particle in the SM is described by a field. One field for each particle.

A field is a section of some natural bundle $F \to M$ with values in a representation (R) of G_{SM} ie in an associated $R_{G_{SM}} \to M$ (associated to E)

F could be $S^+, S^-, \Omega^0, \Omega^1, \ldots$

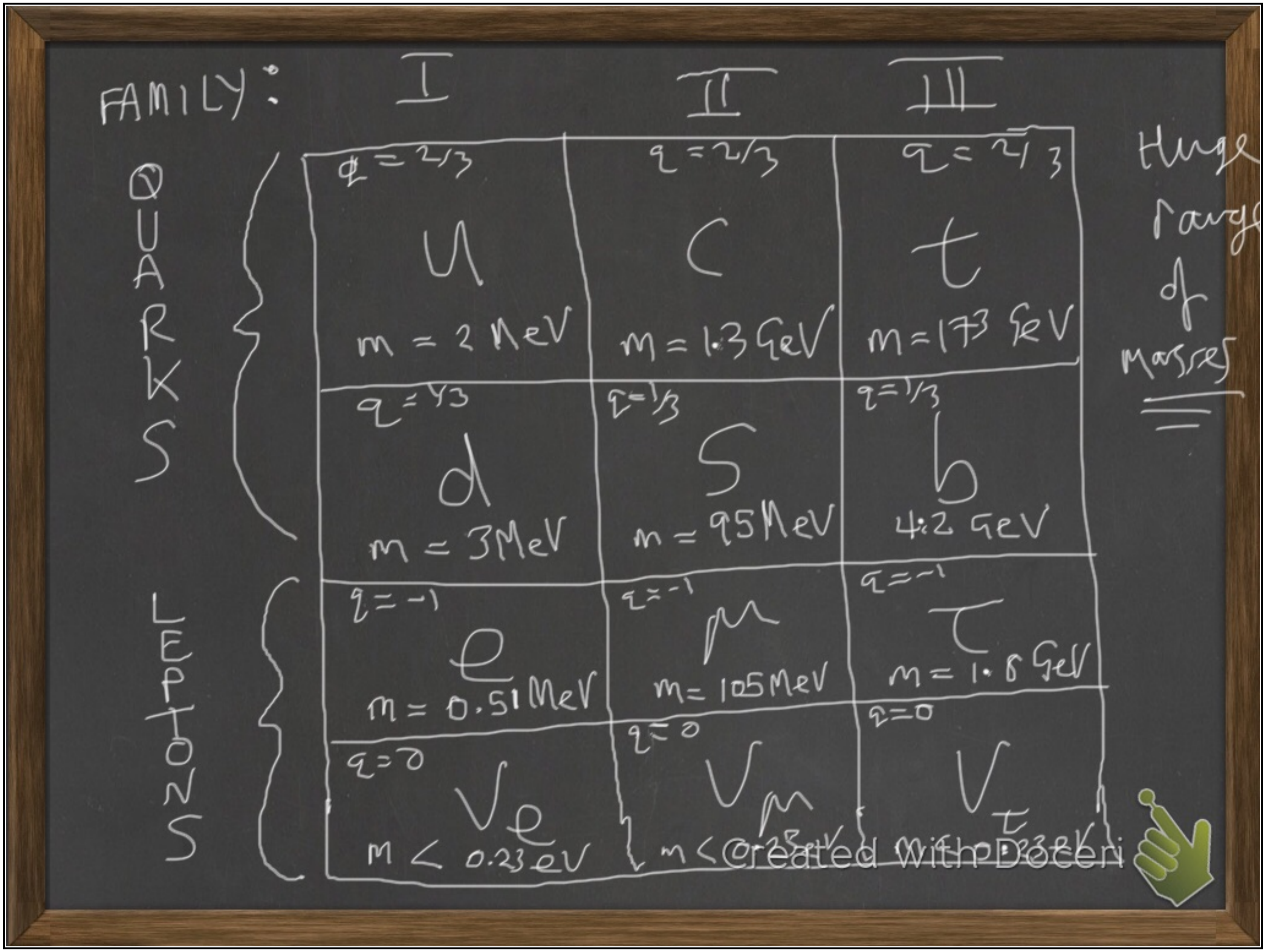


Fermions

The fermions of the SM are classified into QUARKS and LEPTONS.

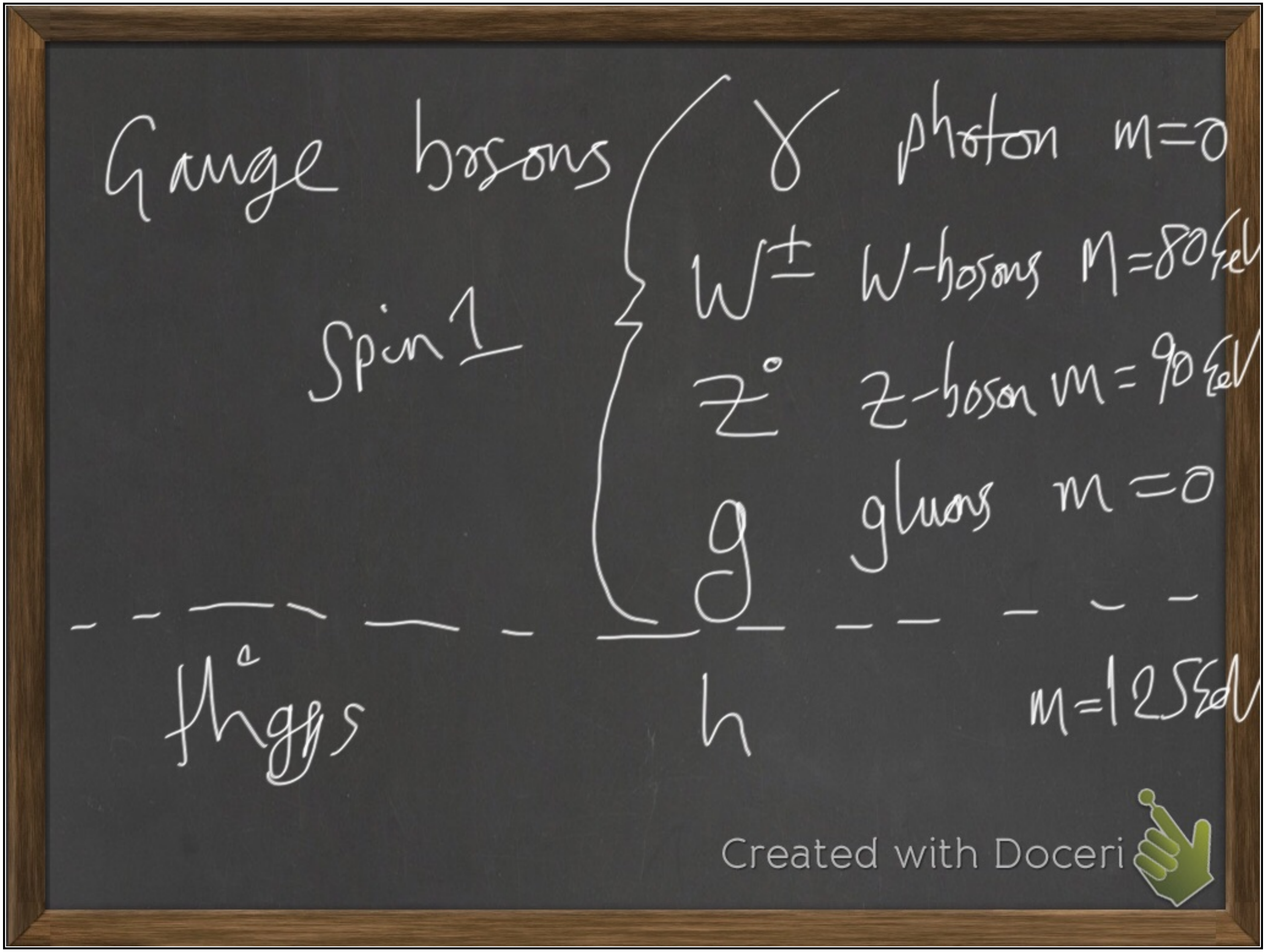
There are THREE families (generations) of QUARKS and LEPTONS

Each of the THREE families is described by sections of S^+ or S^- with values in a dimension 15 rep^n of G_{SM}.



	FAMILY:	I	II	III
QUARKS		q = 2/3 u m = 2 MeV	q = 2/3 c m = 1.3 GeV	q = 2/3 t m = 173 GeV
		q = 1/3 d m = 3 MeV	q = 1/3 s m = 95 MeV	q = 1/3 b 4.2 GeV
LEPTONS		q = −1 e m = 0.51 MeV	q = −1 μ m = 105 MeV	q = −1 τ m = 1.8 GeV
		q = 0 ν_e m < 0.23 eV	q = 0 ν_μ m < 0.23 eV	q = 0 ν_τ m < 0.23 eV

Huge range of masses



Gauge bosons

spin 1

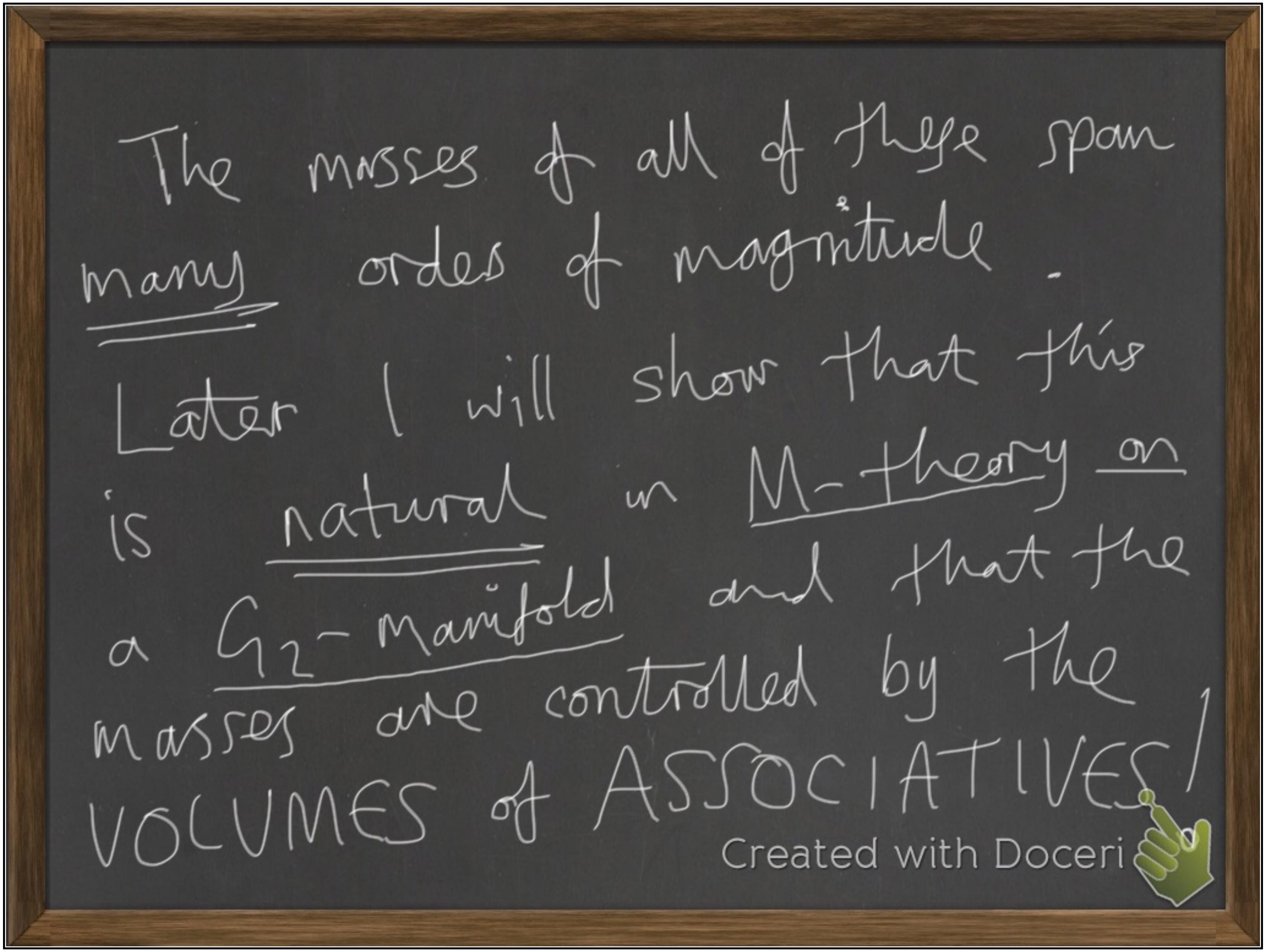
γ photon $m=0$

$W^{\pm}$ W-bosons $M=80$ GeV

Z^0 Z-boson $m=90$ GeV

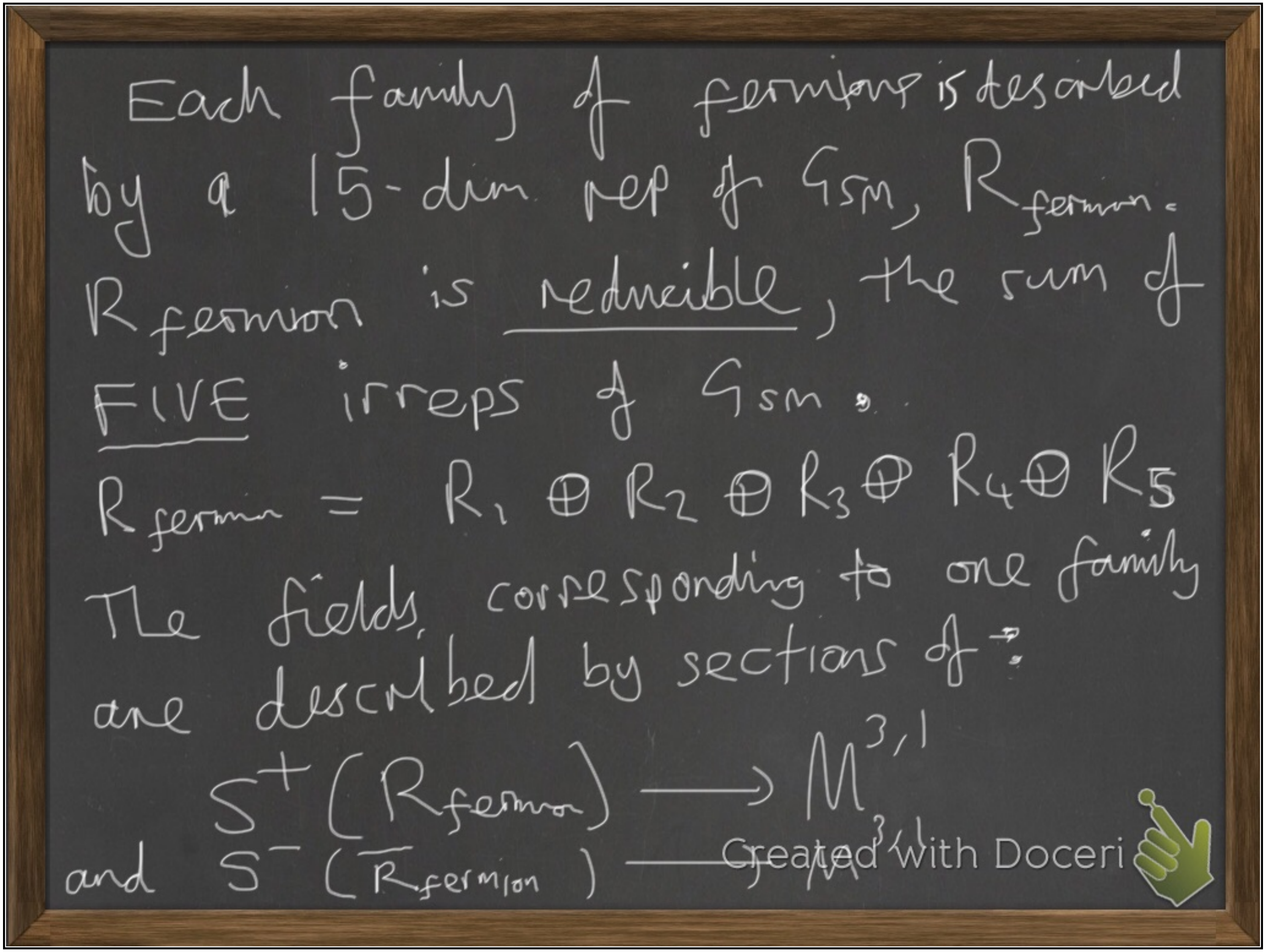
g gluons $m=0$

Higgs h $m=125$ GeV



The masses of all of these span many orders of magnitude.

Later I will show that this is natural in M-theory on a G_2-manifold and that the masses are controlled by the VOLUMES of ASSOCIATIVES!



Each family of fermions is described by a 15-dim rep of G_{SM}, R_{fermion}.

R_{fermion} is reducible, the sum of FIVE irreps of G_{sm}.

$$R_{\text{fermion}} = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5$$

The fields corresponding to one family are described by sections of:

$$S^+(R_{\text{fermion}}) \longrightarrow M^{3,1}$$

and $S^-(\overline{R}_{\text{fermion}}) \longrightarrow M^{3,1}$

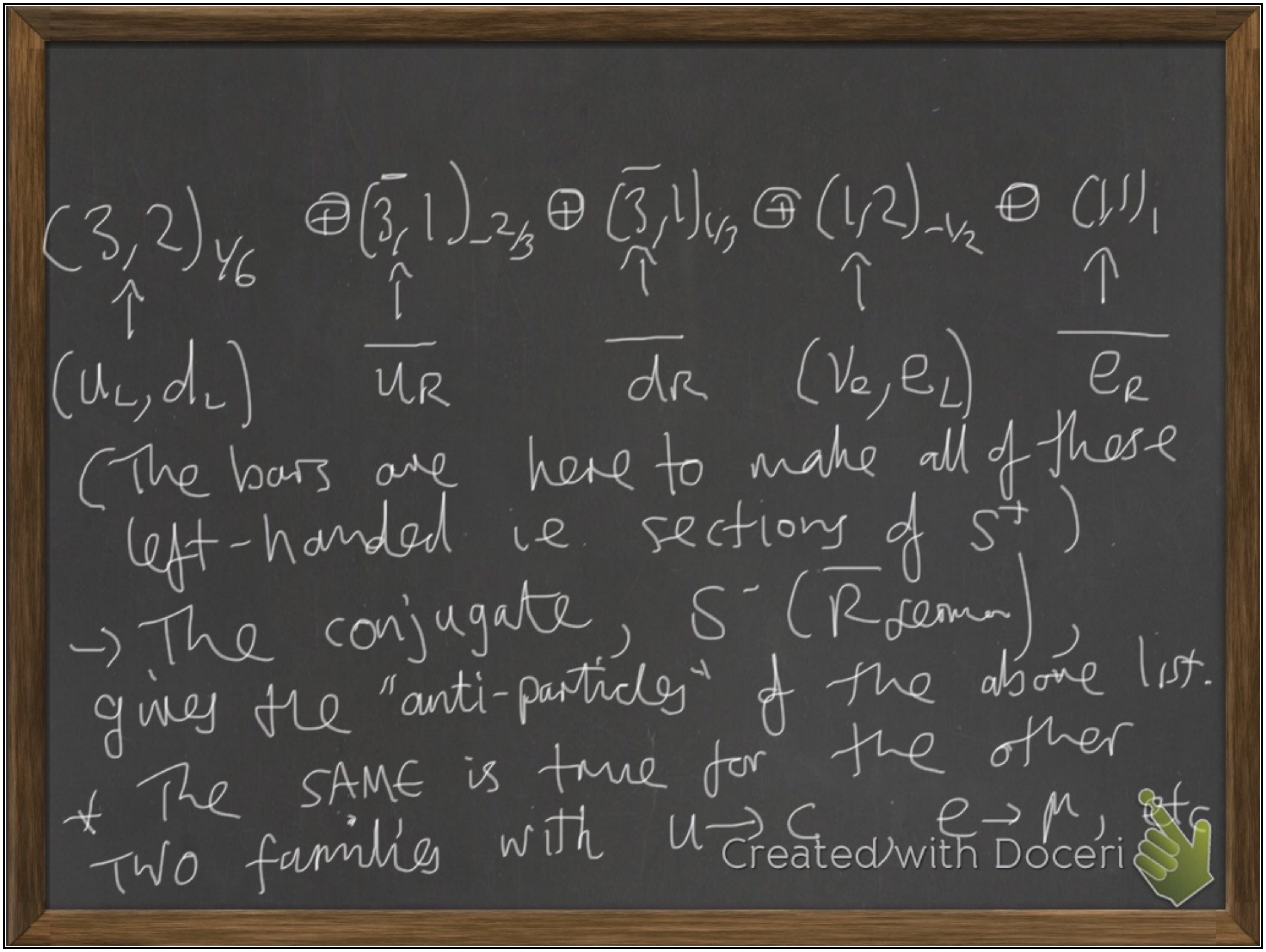
$$R_{\text{fermion}} = (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_{1}$$

The fermions of each family are described by sections of

$$S^{+}(R_{\text{fermion}}) \oplus S^{-}(\overline{R_{\text{fermion}}}) .$$

The particles are identified as

$$\underset{\underset{(u_L, d_L)}{\uparrow}}{(3,2)_{1/6}} \oplus \underset{\underset{\overline{u_R}}{\uparrow}}{(\bar{3},1)_{-2/3}} \oplus \underset{\underset{\overline{d_R}}{\uparrow}}{(\bar{3},1)_{1/3}} \oplus \underset{\underset{(\nu_e, e_L)}{\uparrow}}{(1,2)_{-1/2}} \oplus \underset{\underset{\overline{e_R}}{\uparrow}}{(1,1)_{1}}$$

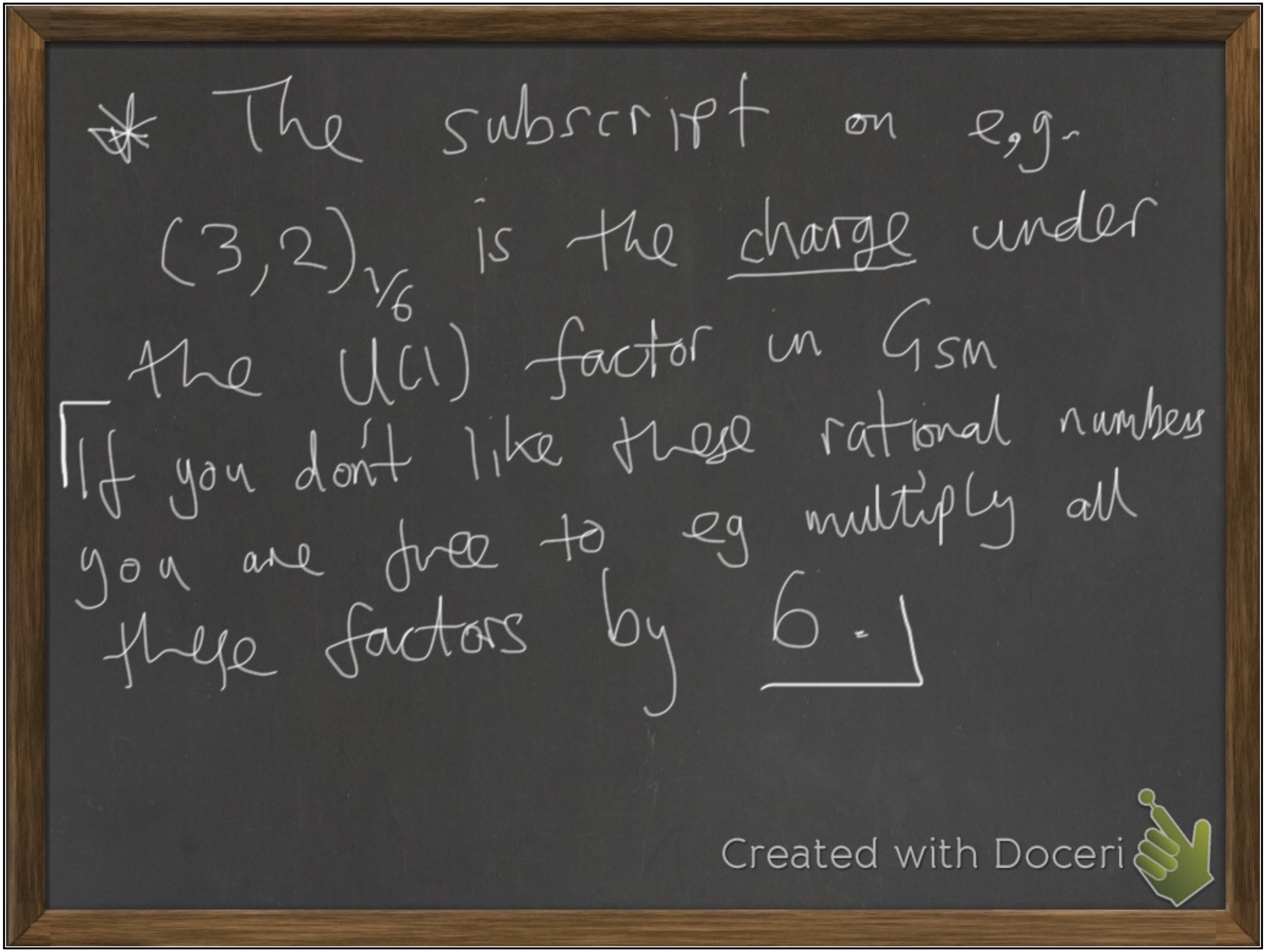


$$\begin{array}{ccccccccc}
(3,2)_{1/6} & \oplus & (\bar{3},1)_{-2/3} & \oplus & (\bar{3},1)_{1/3} & \oplus & (1,2)_{-1/2} & \oplus & (1,1)_1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
(u_L, d_L) & & \overline{u_R} & & \overline{d_R} & & (\nu_e, e_L) & & \overline{e_R}
\end{array}$$

(The bars are here to make all of these left-handed ie sections of S^+)

→ The conjugate, S^- ($\overline{\text{Riemann}}$), gives the "anti-particles" of the above list.

* The SAME is true for the other TWO families with $u \to c$, $e \to \mu$, etc



* The subscript on e.g. $(3,2)_{1/6}$ is the charge under the U(1) factor in G_{SM}

[If you don't like these rational numbers you are free to eg multiply all these factors by 6.]

Bosons

Higgs fields: $H \in \Omega^0(C(1,2)_{-1/2})$

Gauge bosons: $A \in \Omega^1(\underset{\substack{\uparrow \\ \text{adjoint rep of } G_{SM}}}{\mathfrak{g}_{SM}})$

$$\mathfrak{g}_{SM} = \underset{\text{gluons}}{(8,1)_0} + \underset{\text{W-bosons}}{(1,3)_0} + \underset{\substack{\uparrow \\ \text{Hypercharge gauge boson}}}{(1,1)_0}$$

A is a connection on $E \to M$

- There is a covariant derivative D_A

We have now introduced all the fields of the Standard Model.

Now we can define it! ☺

Essentially, it is defined by the set of all "operators" of dimension four which are invariant under

$$G = G_{SM} \times \underset{\substack{\uparrow \\ SO(3,1)}}{\text{Lorentz}}$$

- An operator is simply a product of any combination of fields plus derivatives acting on them.

- In natural units, H and A have mass dimension ONE. The spinors have dimension $3/2$. d, $\partial/\partial x_\mu$ have dimension one.

- This is because the Lagrangian has mass dimension four (it is the difference of kinetic and potential energy densities).

- $\mathcal{L}_{\text{DIRAC}} = i\overline{\psi}\not{\partial}\psi$, for a massless free fermion described by a section of $S^+ \oplus S^-$

- $\mathcal{L}_{\text{YANG-MILLS}} = -\frac{1}{4}\mathrm{tr}(F_A \wedge * F_A)$

$$\Rightarrow [\psi] = [M]^{3/2} \qquad [A] = [M]^1$$

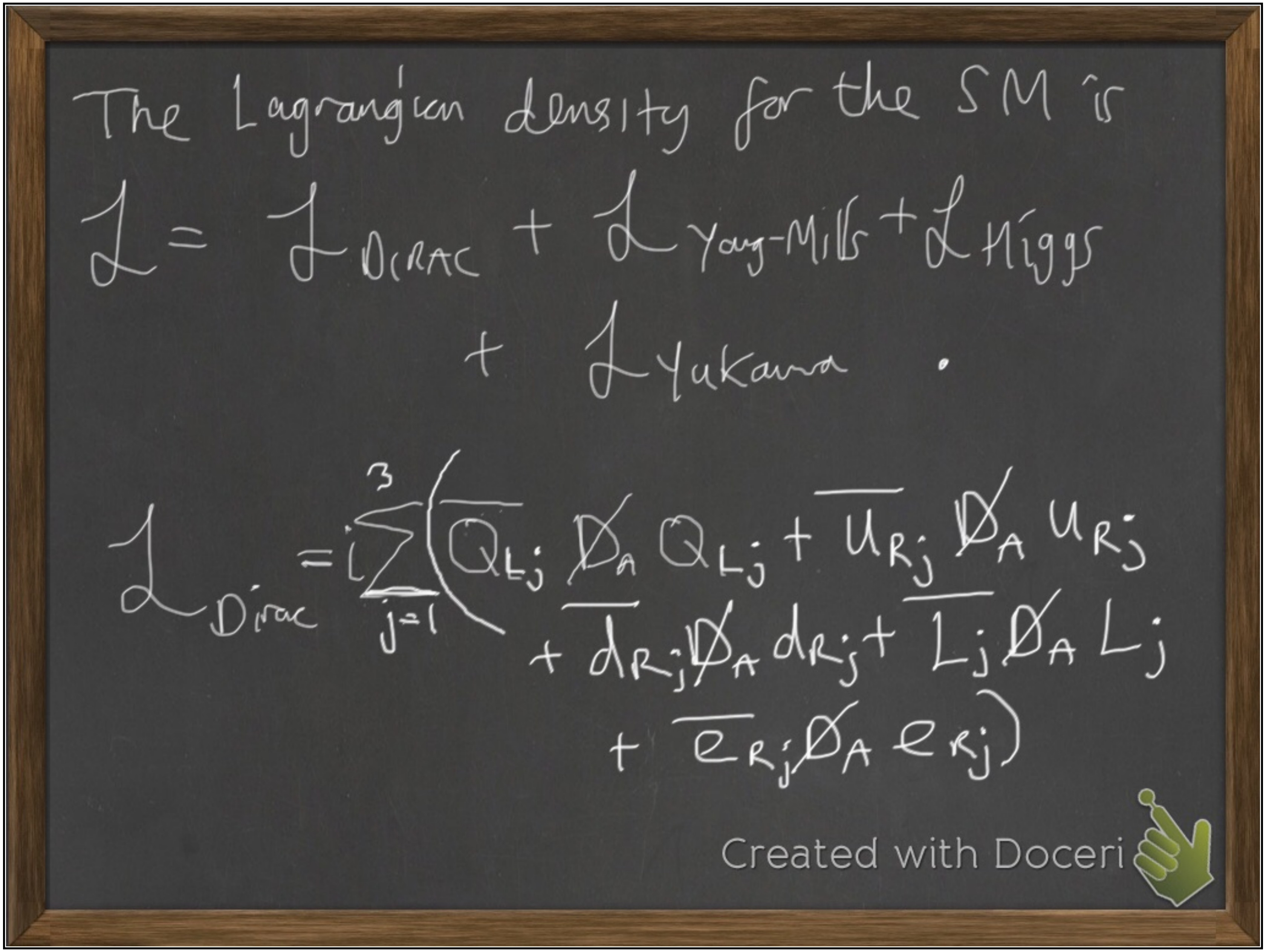
The particles are identified as

$$\begin{array}{ccccccccc} (3,2)_{1/6} & \oplus & (\bar{3},1)_{-2/3} & \oplus & (\bar{3},1)_{1/3} & \oplus & (1,2)_{-1/2} & \oplus & (1,1)_{1} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (u_L, d_L) & & \overline{u_R} & & \overline{d_R} & & (\nu_e, e_L) & & \overline{e_R} \\ \rotatebox{90}{$\equiv$} & & & & & & \rotatebox{90}{$\equiv$} & & \\ Q_L & & & & & & L & & \end{array}$$

Introduce an index $i = 1,2,3$ with

$$Q_{L1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad Q_{L2} = \begin{pmatrix} c_L \\ s_L \end{pmatrix} \text{ etc}$$

$$e_{R3} = \tau_R \quad \text{etc}$$



The Lagrangian density for the SM is

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} .$$

$$\mathcal{L}_{\text{Dirac}} = i\sum_{j=1}^{3}\Big(\overline{Q_{Lj}}\,\not{D}_A\, Q_{Lj} + \overline{u_{Rj}}\,\not{D}_A\, u_{Rj} + \overline{d_{Rj}}\,\not{D}_A\, d_{Rj} + \overline{L_j}\,\not{D}_A\, L_j + \overline{e_{Rj}}\,\not{D}_A\, e_{Rj}\Big)$$

Note: $D_A \equiv d + ig_1 \underset{\uparrow}{B} + ig_2 \underset{\uparrow}{W} + ig_3 \underset{\uparrow}{G}$

corresponding to $U(1) \times SU(2) \times SU(3)$

g_1, g_2, g_3 are dimensionless and define the unit of "charge" for the three forces corresponding to $U(1)$, $SU(2)$, and $SU(3)$.

$$\mathcal{L}_{\text{Yang Mills}} = \frac{-1}{4}\,\mathrm{tr}\left(F_A \wedge {*F_A}\right)$$

is a sum of three terms for $A = (B, W, G)$

$$\mathcal{L}_{Higgs} = D_A H \wedge \star D_A H - m^2 |H|^2 - \lambda |H|^4$$

The last two terms are minus the Higgs potential.

$$\mathcal{L}_{Yukawa} = \lambda^u_{ij} \overline{Q_{Li}} \tilde{H}^* u_{Rj} + \lambda^d_{ij} \overline{Q_{Li}} H d_{Rj} + \lambda^L_{ij} \overline{L_i} H e_{Rj}$$

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

$$\tilde{H}^* \equiv \begin{pmatrix} H_2^* \\ H_1^* \end{pmatrix}$$

Note: apart from Higgs, there are

$$\boxed{\text{NO MASS TERMS in } \mathcal{L}_{SM}}$$

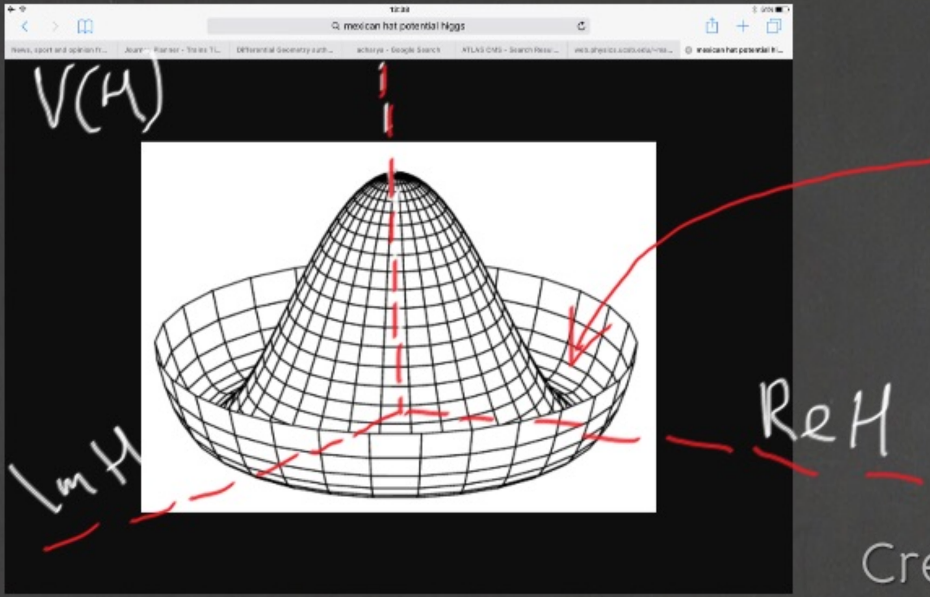
eg a fermion mass requiring ψ_L and ψ_R such that $R_L \otimes R_R$ containing a G_{SM} invariant, which it does NOT.

So, if G_{SM} is an exact symmetry, the fermions and gauge bosons would be massless.

The Higgs mechanism solves this problem

Higgs Mechanism

- $\mathcal{L}_{Higgs}$ contains a potential $V(H)$ for the Higgs field $V(H) = m^2|H|^2 + \lambda|H|^4$
- If the parameter $m^2 < 0$ and $\lambda > 0$


mexican hat potential higgs
V(H)
ReH
ImH

minima

///

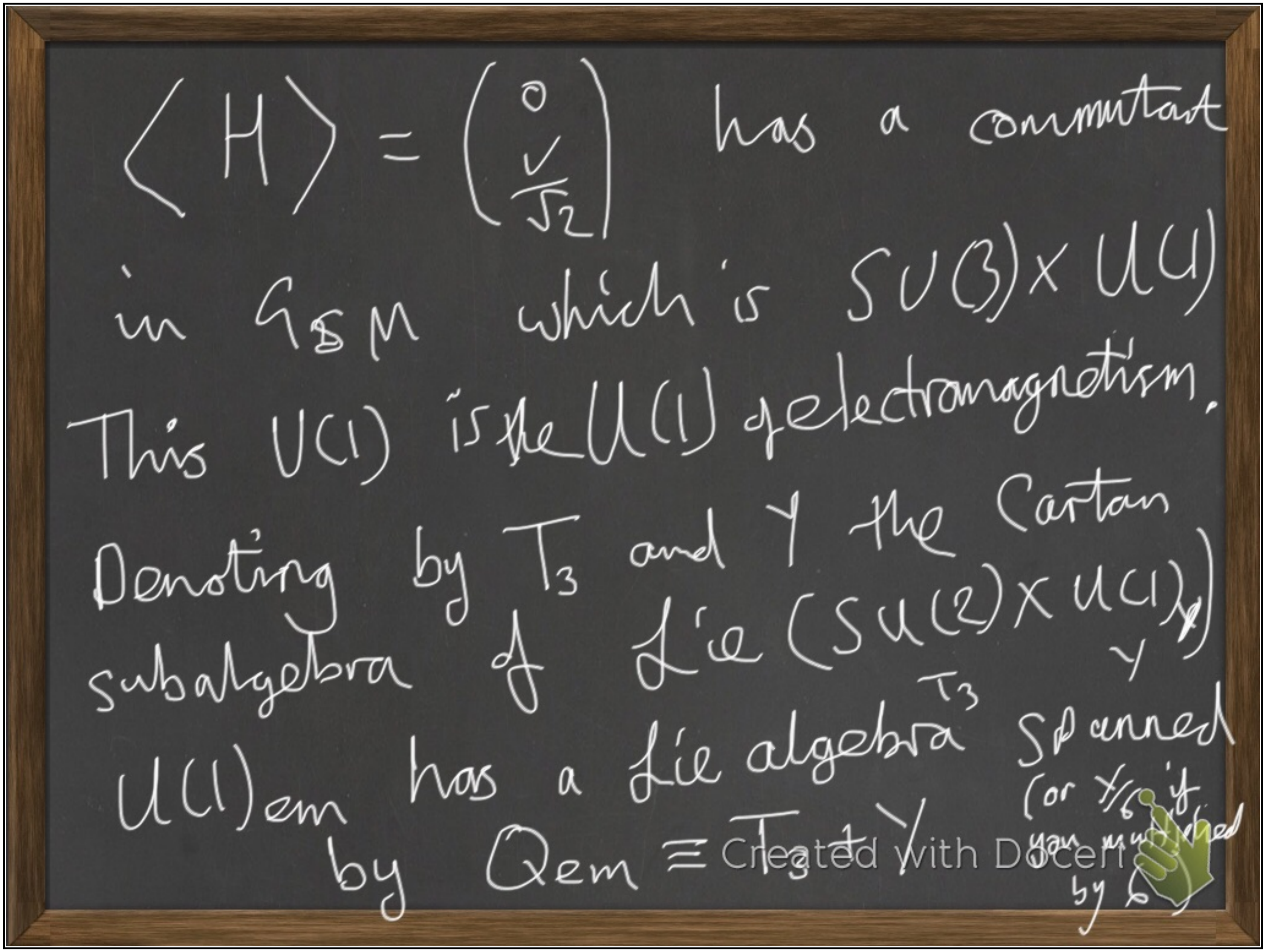
S^1 of vacua at non-zero H.

So, $\mathcal{L}_{SM}$ is G_{SM} invariant, but its minima (vacua) break the G_{SM} symmetry.

The choice of vacuum is spontaneous, without loss of generality we can write the direction as

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

where $v \sim \sqrt{\frac{-m^2}{2\lambda}}$

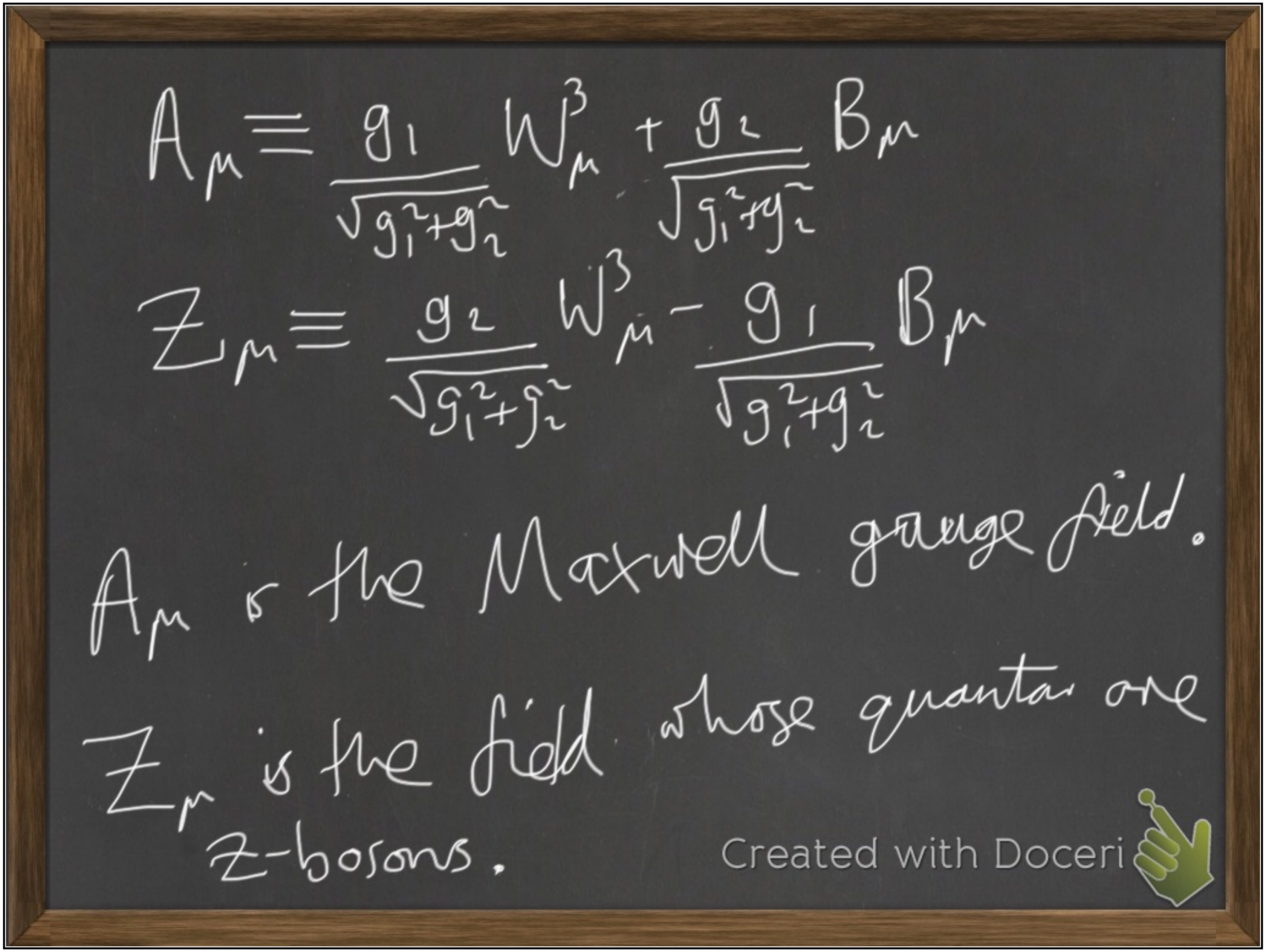


$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$ has a commutant in G_{SM} which is $SU(3) \times U(1)$

This $U(1)$ is the $U(1)$ of electromagnetism.

Denoting by T_3 and Y the Cartan subalgebra of $\mathcal{L}ie\,(\underset{T_3}{SU(2)} \times \underset{Y}{U(1)_Y})$

$U(1)_{em}$ has a Lie algebra spanned by $Q_{em} \equiv T_3 + Y$ (or $Y/6$ if you multiplied by 6)



$$A_\mu \equiv \frac{g_1}{\sqrt{g_1^2+g_2^2}} W^3_\mu + \frac{g_2}{\sqrt{g_1^2+g_2^2}} B_\mu$$

$$Z_\mu \equiv \frac{g_2}{\sqrt{g_1^2+g_2^2}} W^3_\mu - \frac{g_1}{\sqrt{g_1^2+g_2^2}} B_\mu$$

A_μ is the Maxwell gauge field.

Z_μ is the field whose quanta are Z-bosons.

The other two Lie(SU(2)) generators correspond to the W^+ and W^- bosons.

Expanding $\mathcal{L}_{Higgs} \sim D_A H \wedge * (D_A H)^{\dagger}$

gives $g_1^2 \, B H \wedge * (BH)^{\dagger} +$

$g_2^2 \, W H \wedge * (WH)^{\dagger} + \cdots$

expanding around $\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + h \end{pmatrix}$

gives $\frac{g_1^2}{2} B \wedge * B \, v^2 + \frac{g_2^2}{2} W \wedge * W \, v^2 + \cdots$

ie mass terms

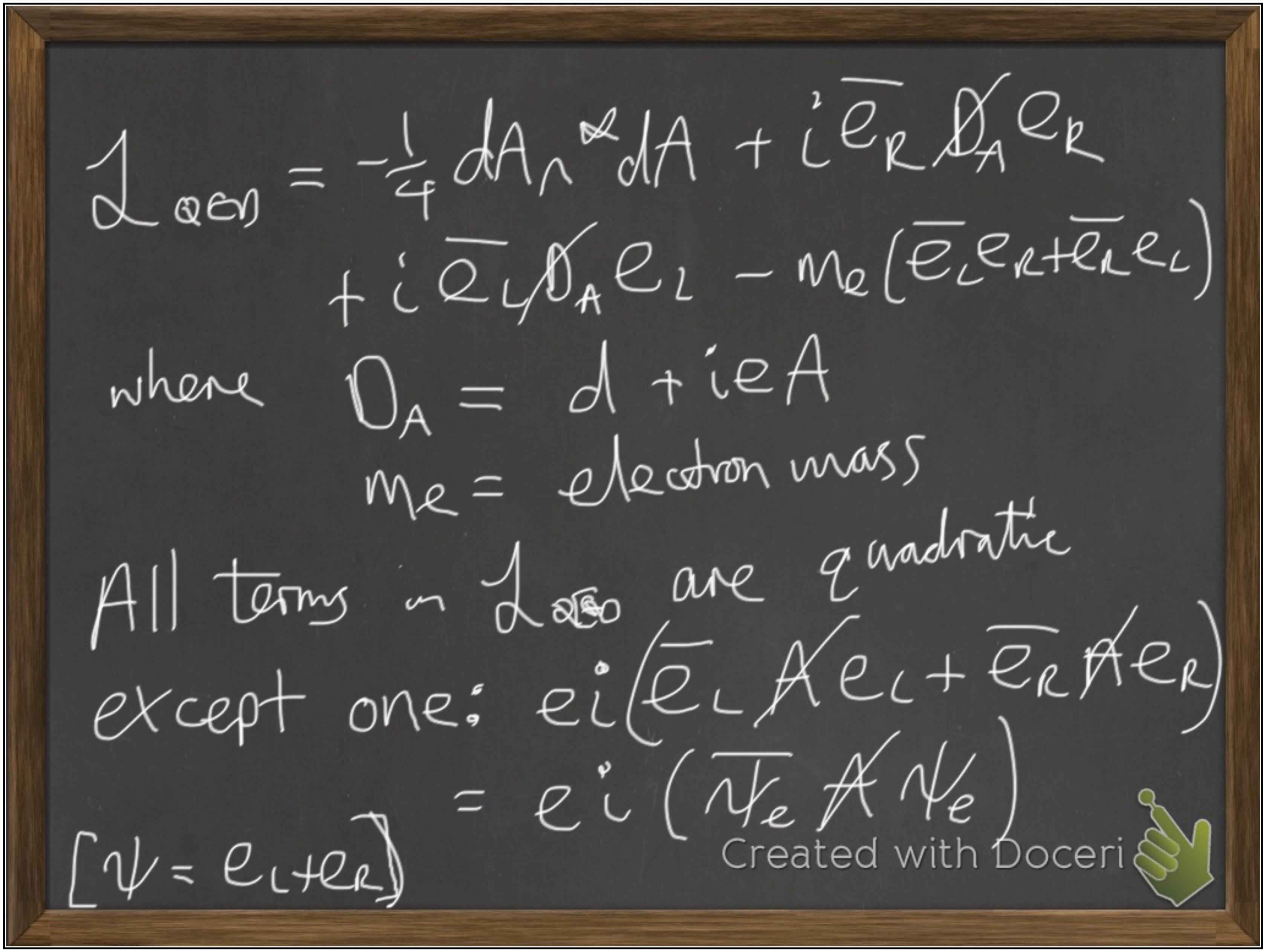
Interactions, vertices and Feynman Diagrams

Simple example QED:

Fields (A, e_L, e_R). A connection on U(1) bundle

$$\underset{S^+}{\uparrow} \qquad \underset{S^-}{\uparrow}$$

e_L, e_R both have charge -1 in units of electric charge e, where $\alpha = e^2/4\pi$ is the fine structure constant (at low energies



$$\mathcal{L}_{QED} = -\frac{1}{4} dA \wedge * dA + i \bar{e}_R \not{D}_A e_R + i \bar{e}_L \not{D}_A e_L - m_e (\bar{e}_L e_R + \bar{e}_R e_L)$$

where $D_A = d + ieA$

$m_e =$ electron mass

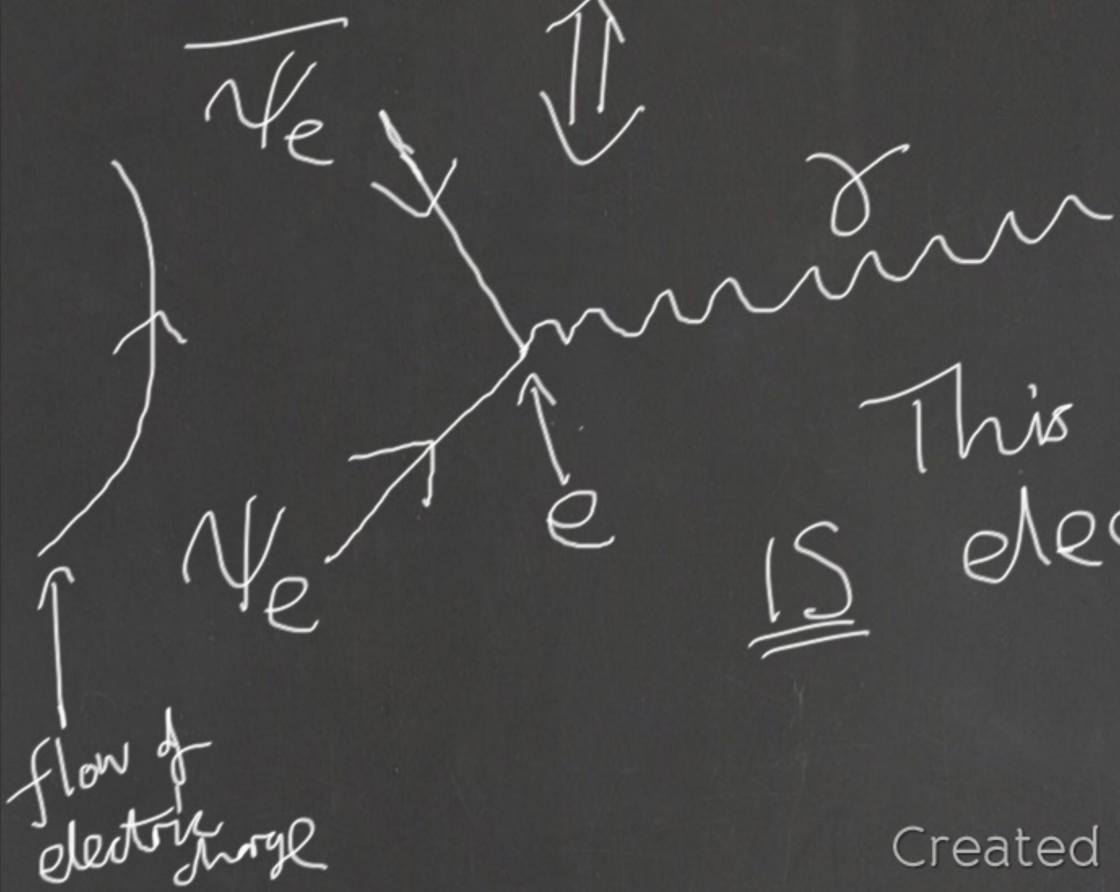
All terms in $\mathcal{L}_{QED}$ are quadratic except one: $ei(\bar{e}_L \not{A} e_L + \bar{e}_R \not{A} e_R)$

$$= ei(\bar{\psi}_e \not{A} \psi_e)$$

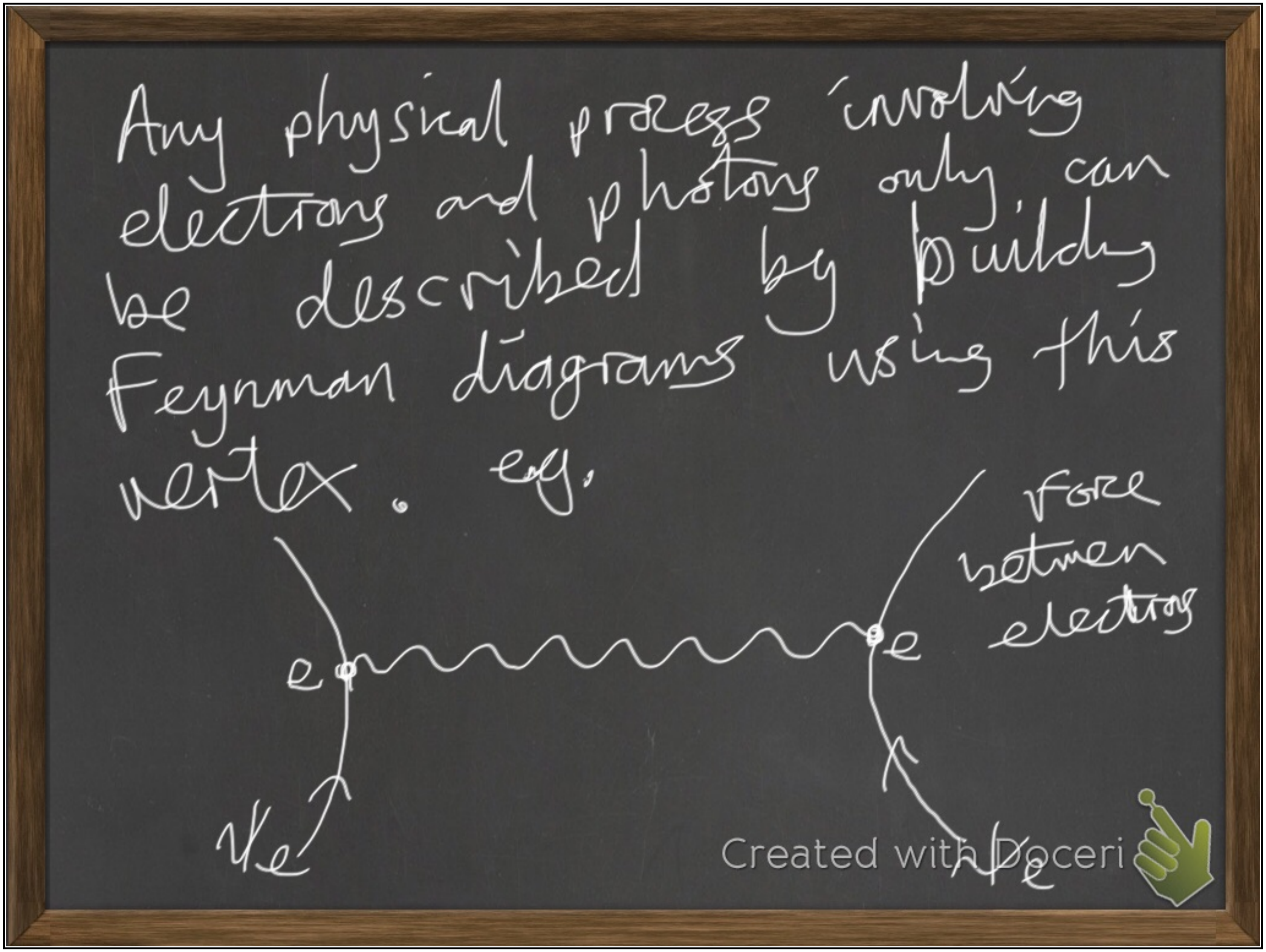
$[\psi = e_L + e_R]$

$$\mathcal{L}_{int} = ei\bar{\Psi}_e \not{A} \Psi_e$$

γ = photon

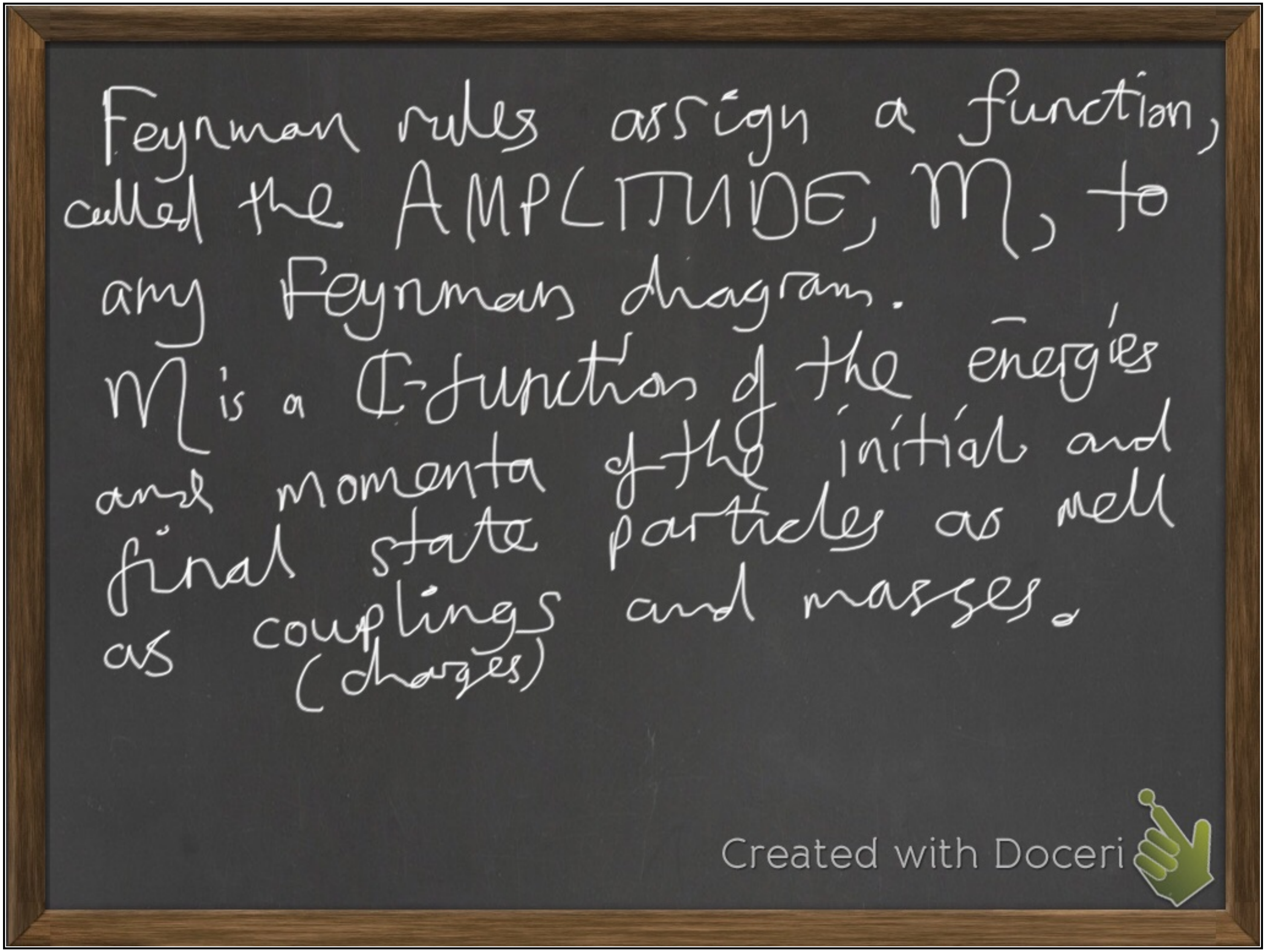

$\Updownarrow$
$\bar{\Psi}_e$
γ
e
Ψ_e
flow of electric charge

This vertex IS electromagnetism!



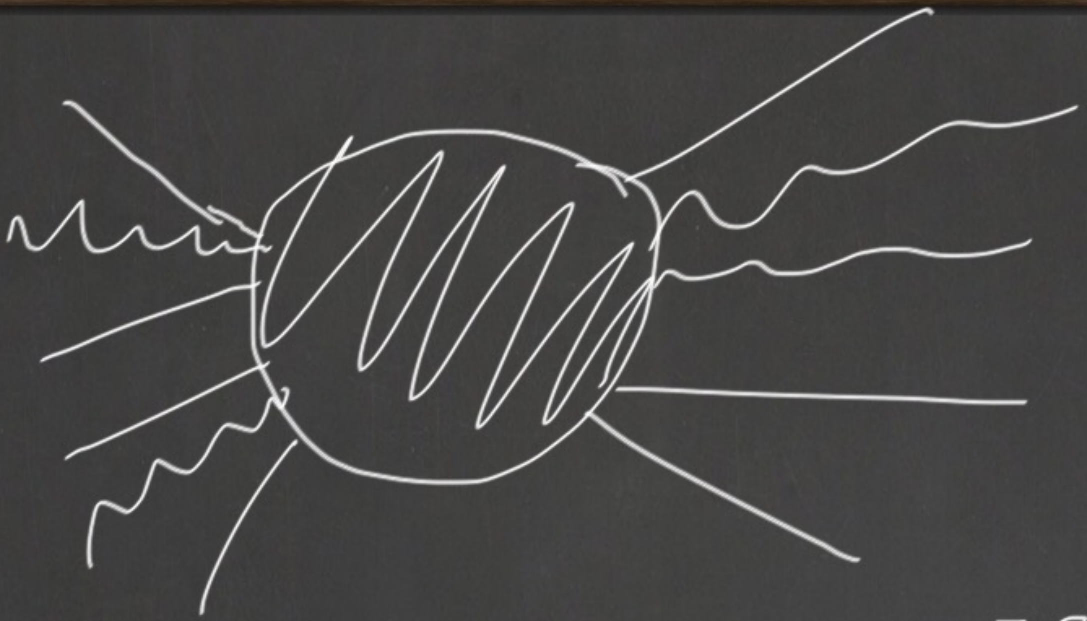
Any physical process involving electrons and photons only can be described by building Feynman diagrams using this vertex. eg.

Force
between
electrons
e
e
ν_e
ν_e



Feynman rules assign a function, called the AMPLITUDE, $\mathcal{M}$, to any Feynman diagram.

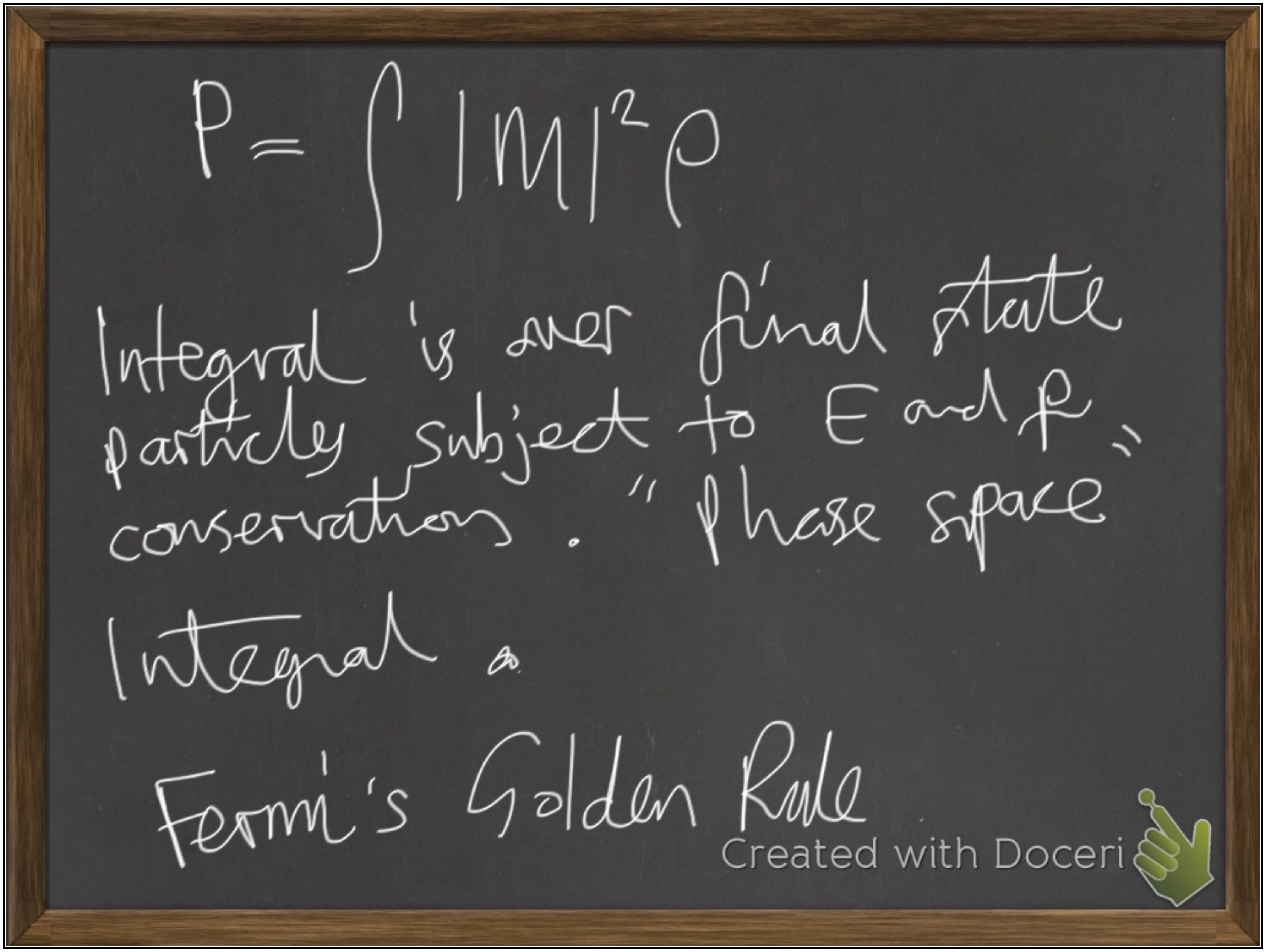
$\mathcal{M}$ is a $\mathbb{C}$-function of the energies and momenta of the initial and final state particles as well as couplings (charges) and masses.



probability for any process is

$$P = \int |M|^2 \rho(E,P)$$

where $\rho(E,p)$ is the density of states.



$$P = \int |M|^2 \rho$$

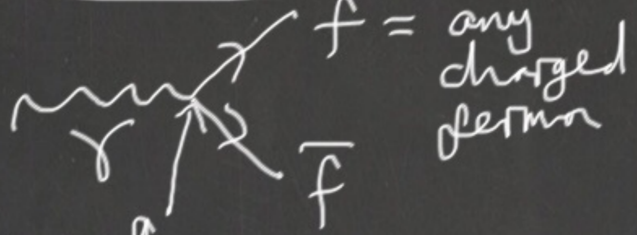
Integral is over final state particles subject to E and p conservation. "Phase space"

Integral a

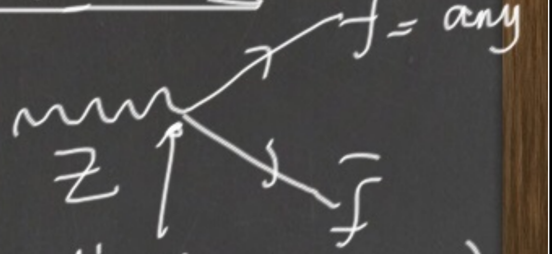
Fermi's Golden Rule

Vertices in Standard Model!

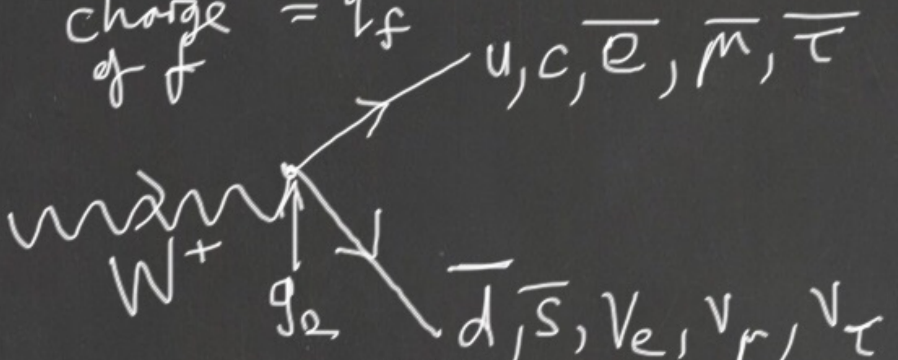
gauge


γ
f = any charged fermion
$\bar{f}$

charge of f = q_f

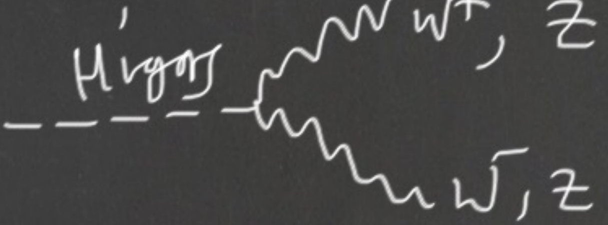


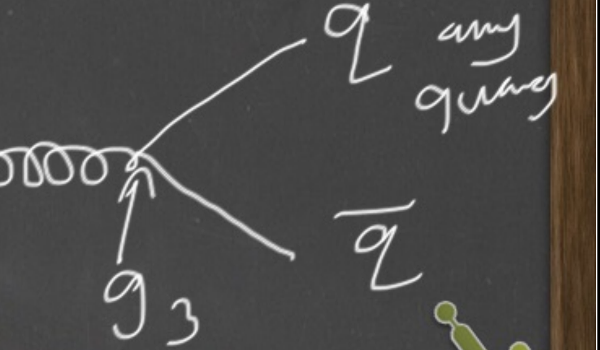
combination of (g_1, g_2)


W^+
g_2
$u, c, \bar{e}, \bar{\mu}, \bar{\tau}$
$\bar{d}, \bar{s}, \nu_e, \nu_\mu, \nu_\tau$

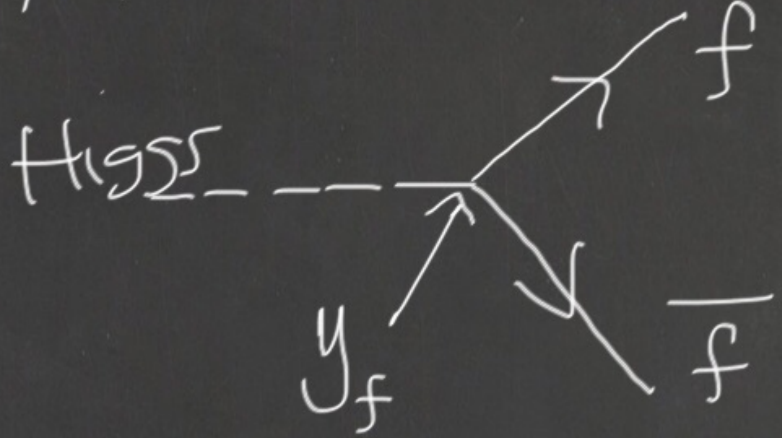
(ignoring CKM mixing)

Higgs



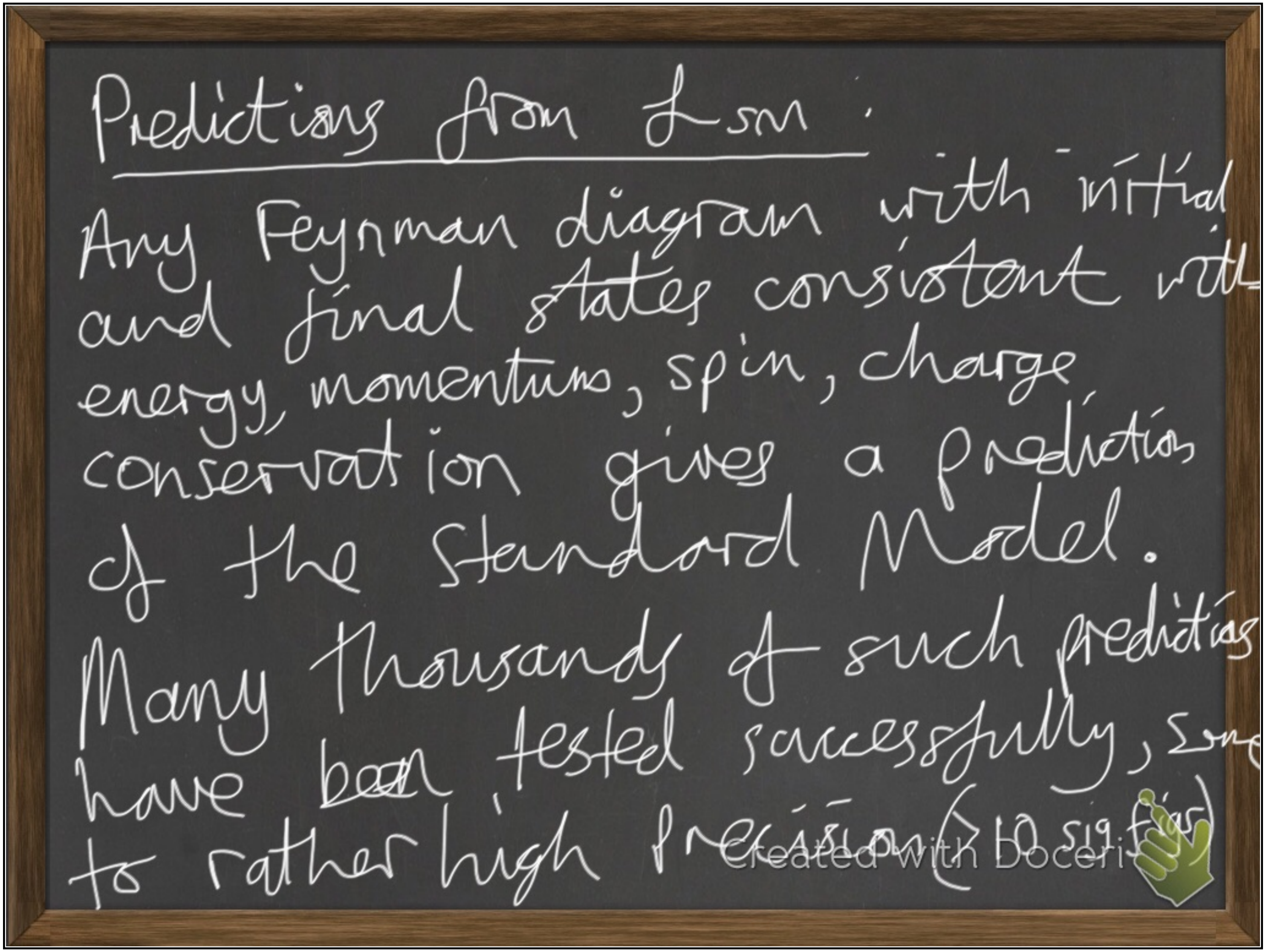

gluon
g_3
q any quark
$\bar{q}$

Yukawa Interactions



$$y_f = \frac{m_f}{\sqrt{2}\, v}$$

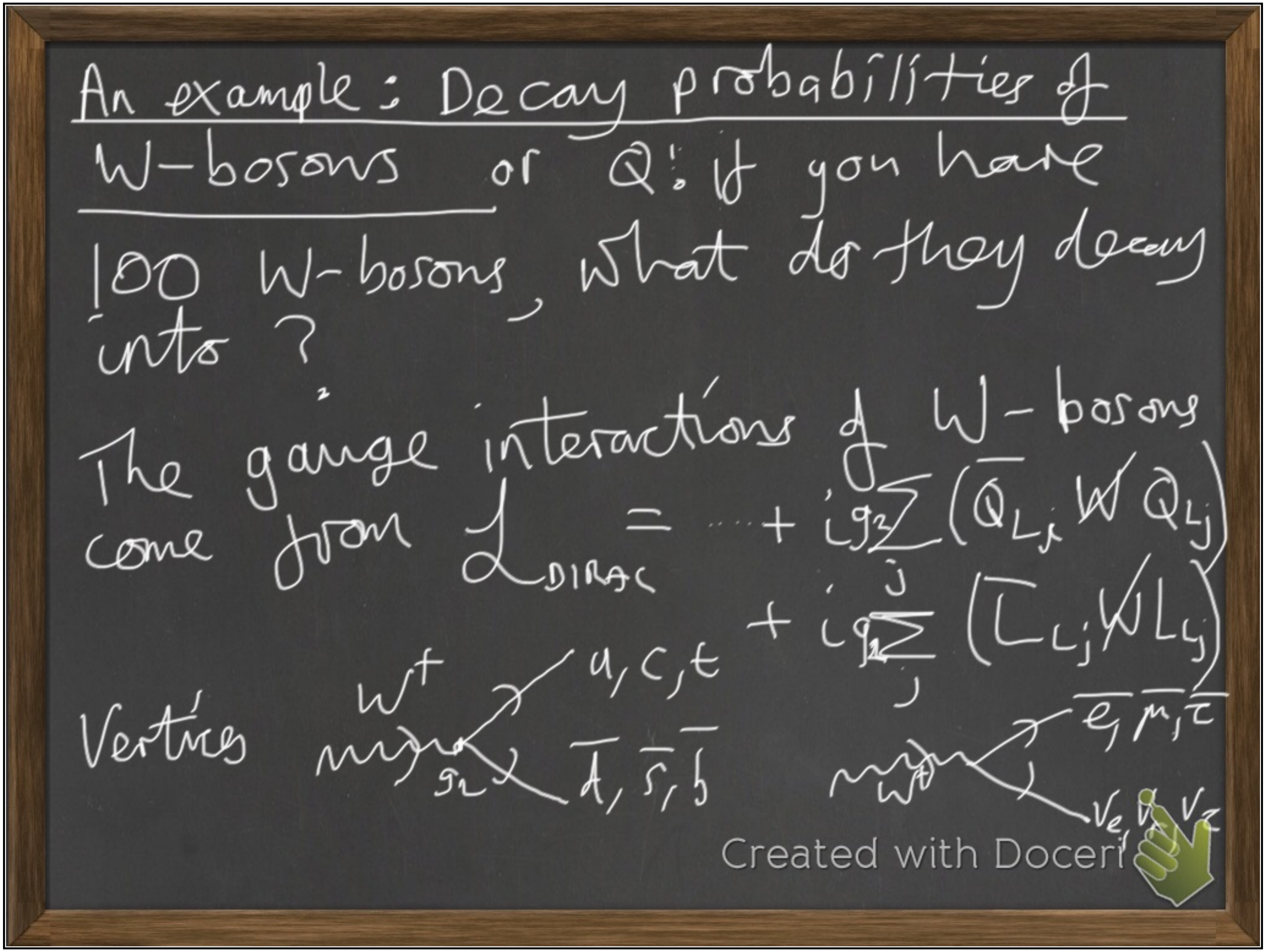
There are also quartic vertices but I won't discuss these.



Predictions from $\mathcal{L}_{SM}$:

Any Feynman diagram with initial and final states consistent with energy, momentum, spin, charge, conservation gives a prediction of the Standard Model.

Many thousands of such predictions have been tested successfully, some to rather high precision ($>$ 10 sig figs)



An example: Decay probabilities of W-bosons or Q: if you have 100 W-bosons, what do they decay into?

The gauge interactions of W-bosons come from

$$\mathcal{L}_{DIRAC} = \cdots + i g_2 \sum_j \left(\bar{Q}_{L_j} \not{W} Q_{L_j}\right) + i g_2 \sum_j \left(\bar{L}_{L_j} \not{W} L_{L_j}\right)$$

Vertices

W^+
u, c, t
$\bar{d}, \bar{s}, \bar{b}$
g_2
$\bar{e}, \bar{\mu}, \bar{\tau}$
W^+
ν_e, ν_μ, ν_τ
Created with Doceri

The vertices are actually Feynman diagrams for the processes:

$W^+ \to u\bar{d}$
$W^+ \to c\bar{s}$

called hadronic as the quarks "hadronise" into jets of hadrons

$W^+ \to \bar{e}\,\nu_e$
$W^+ \to \bar{\mu}\,\nu_\mu$
$W^+ \to \bar{\tau}\,\nu_\tau$

called leptonic channels

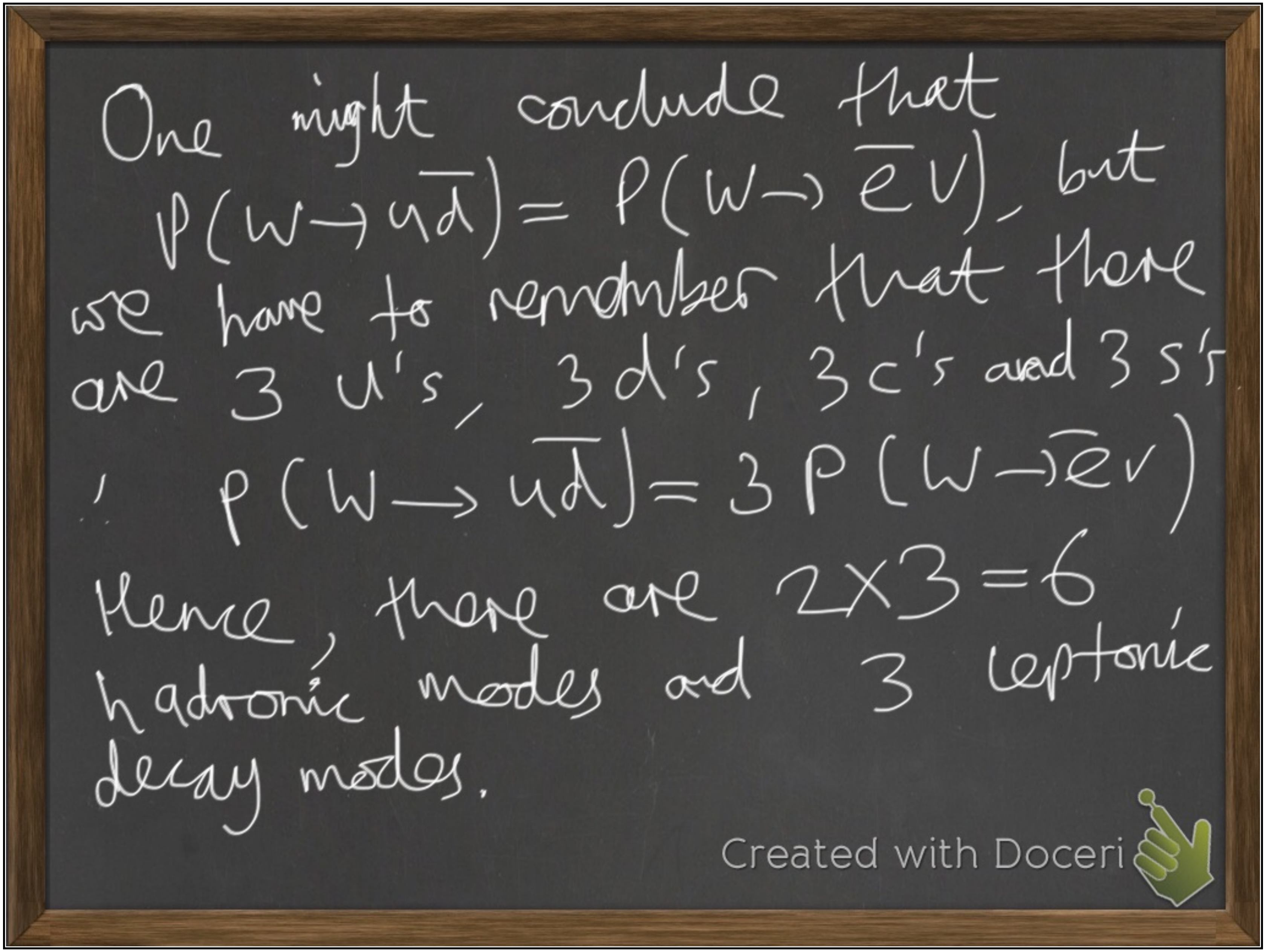
(But not $W^+ \to t\bar{b}$ as $M_t >> M_W$)

The vertex factors for all of these are the same $(= g_2)$ and so the SU(2) interactions are universal.

Moreover $M_W \gg M_b, M_c, M_s, M_u, M_d, m_e, \ldots$

So the fermions can be taken as massless to a good approximation

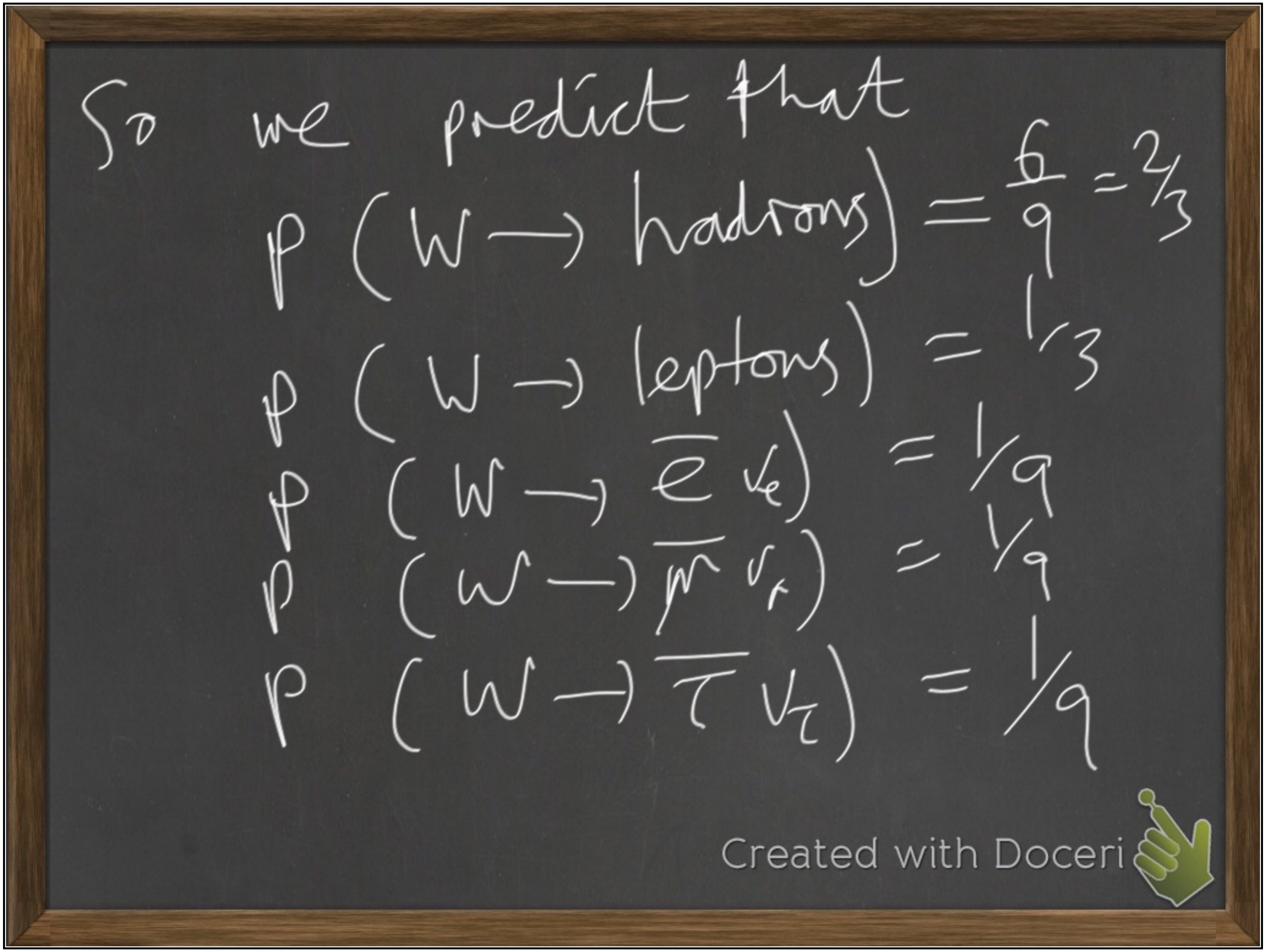
So, expect $P(W \to \bar{e}\nu) = P(W \to \bar{\mu}\nu)$
$= P(W \to \bar{\tau}\nu)$



One might conclude that
$P(W \to u\bar{d}) = P(W \to \bar{e}\nu)$, but
we have to remember that there
are 3 u's, 3 d's, 3 c's and 3 s's.

∴ $P(W \to u\bar{d}) = 3P(W \to \bar{e}\nu)$

Hence, there are 2X3=6
hadronic modes and 3 leptonic
decay modes.



So we predict that

$$p\,(W \to \text{hadrons}) = \frac{6}{9} = {}^{2}\!/_{3}$$

$$p\,(W \to \text{leptons}) = {}^{1}\!/_{3}$$

$$p\,(W \to \bar{e}\,\nu_e) = {}^{1}\!/_{9}$$

$$p\,(W \to \bar{\mu}\,\nu_\mu) = {}^{1}\!/_{9}$$

$$p\,(W \to \bar{\tau}\,\nu_\tau) = {}^{1}\!/_{9}$$

Lets compare this to real data:

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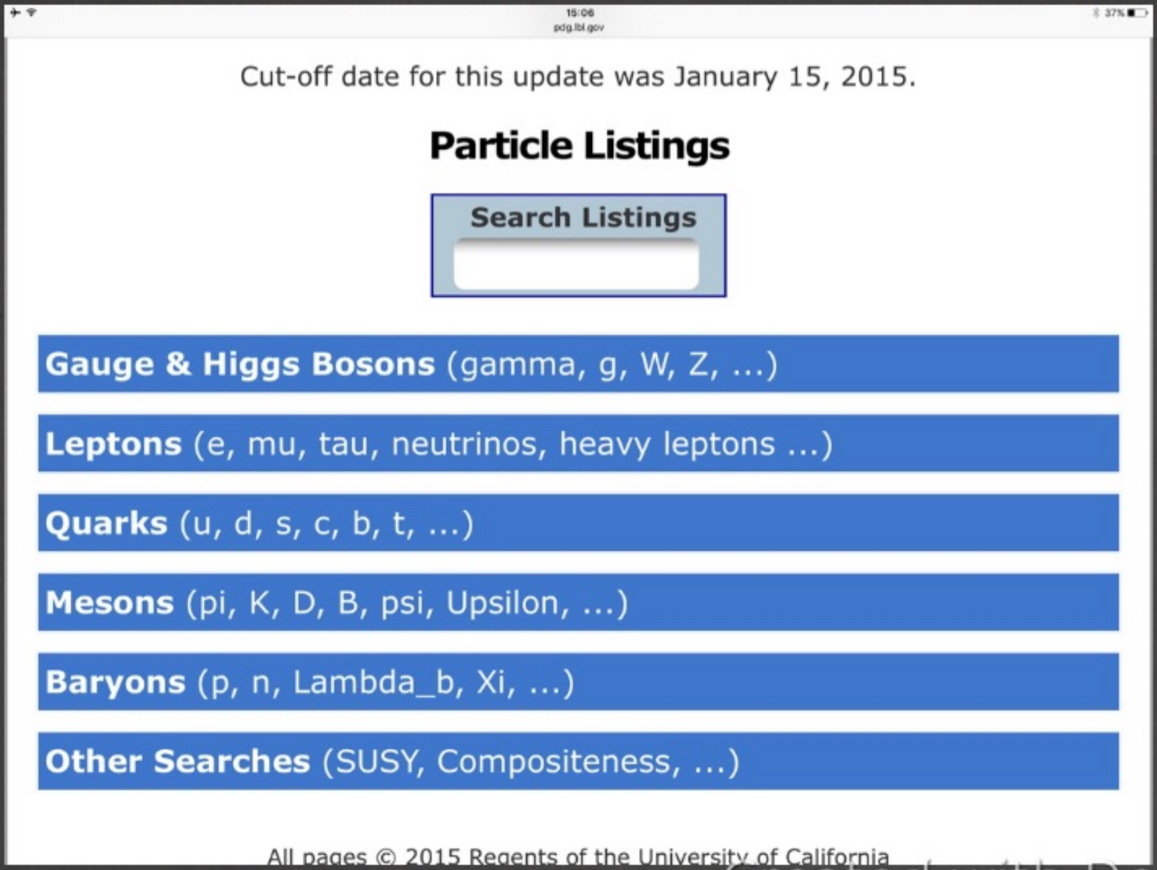

Cut-off date for this update was January 15, 2015.

Particle Listings

Search Listings

Gauge & Higgs Bosons (gamma, g, W, Z, ...)

Leptons (e, mu, tau, neutrinos, heavy leptons ...)

Quarks (u, d, s, c, b, t, ...)

Mesons (pi, K, D, B, psi, Upsilon, ...)

Baryons (p, n, Lambda_b, Xi, ...)

Other Searches (SUSY, Compositeness, ...)

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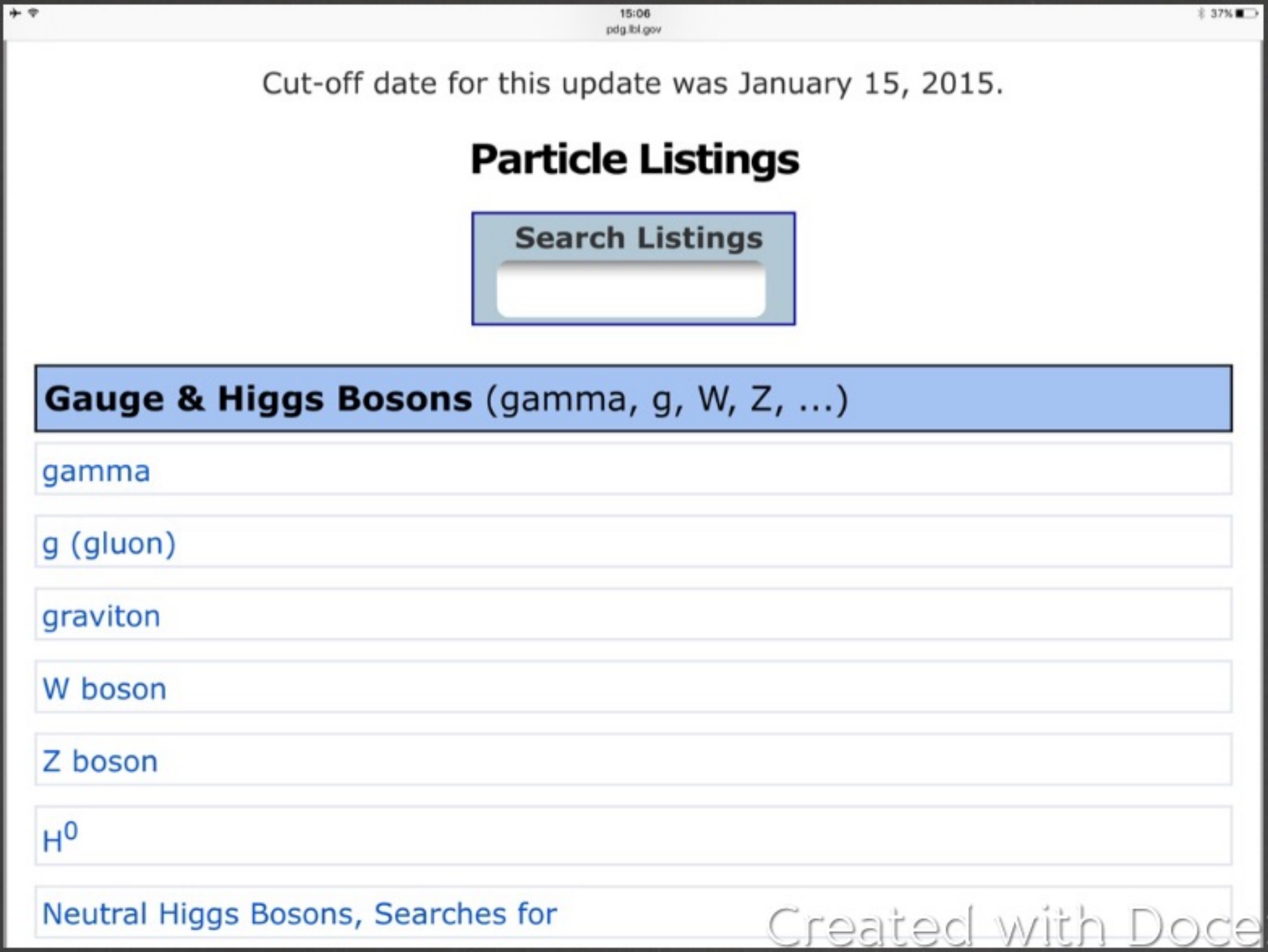
Open the section on bosons: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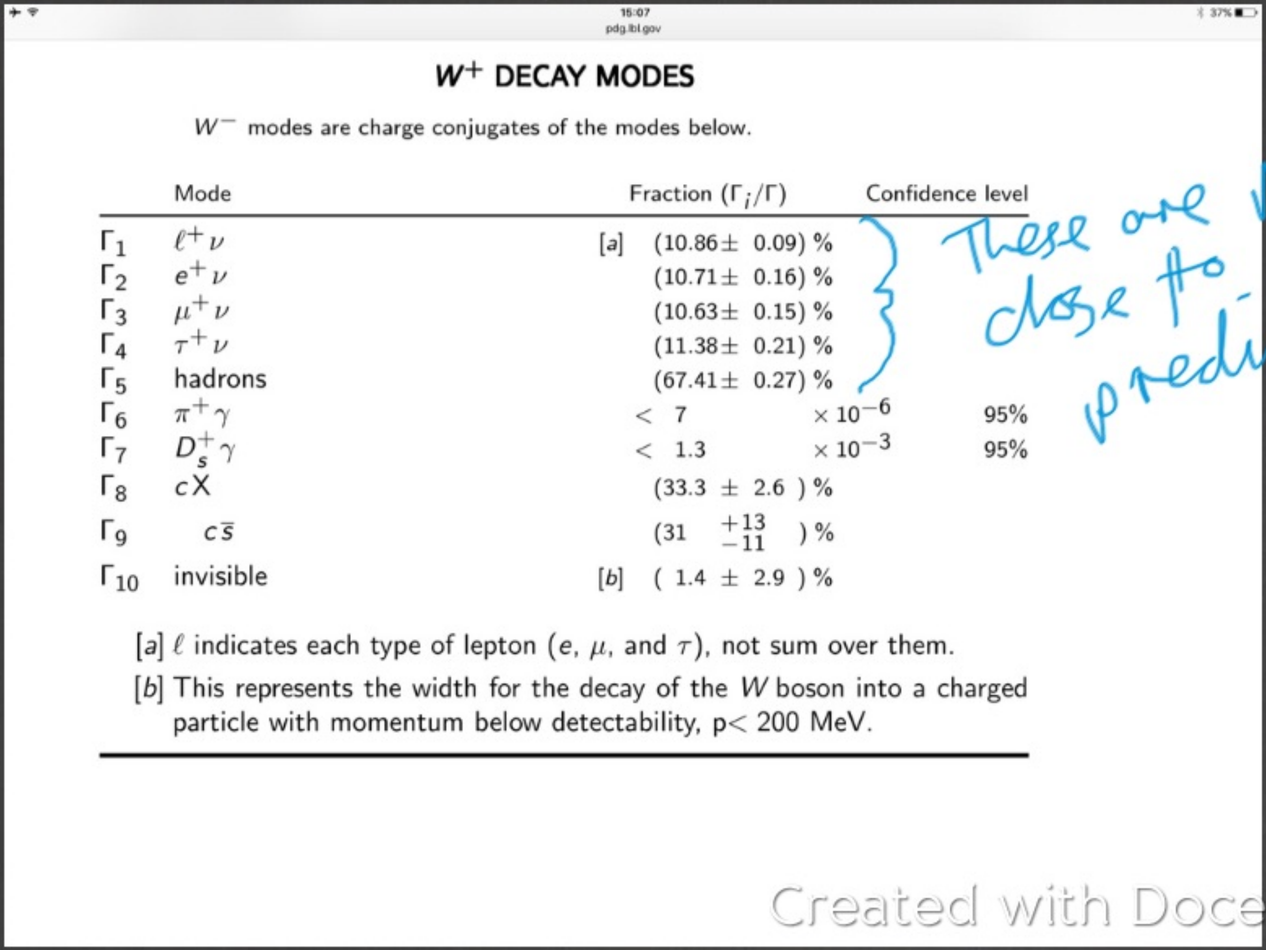
Particle Listings

Search Listings

Gauge & Higgs Bosons (gamma, g, W, Z, ...)

- gamma
- g (gluon)
- graviton
- W boson
- Z boson
- H^0
- Neutral Higgs Bosons, Searches for

Scroll down to W-decay modes:



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W^+ DECAY MODES

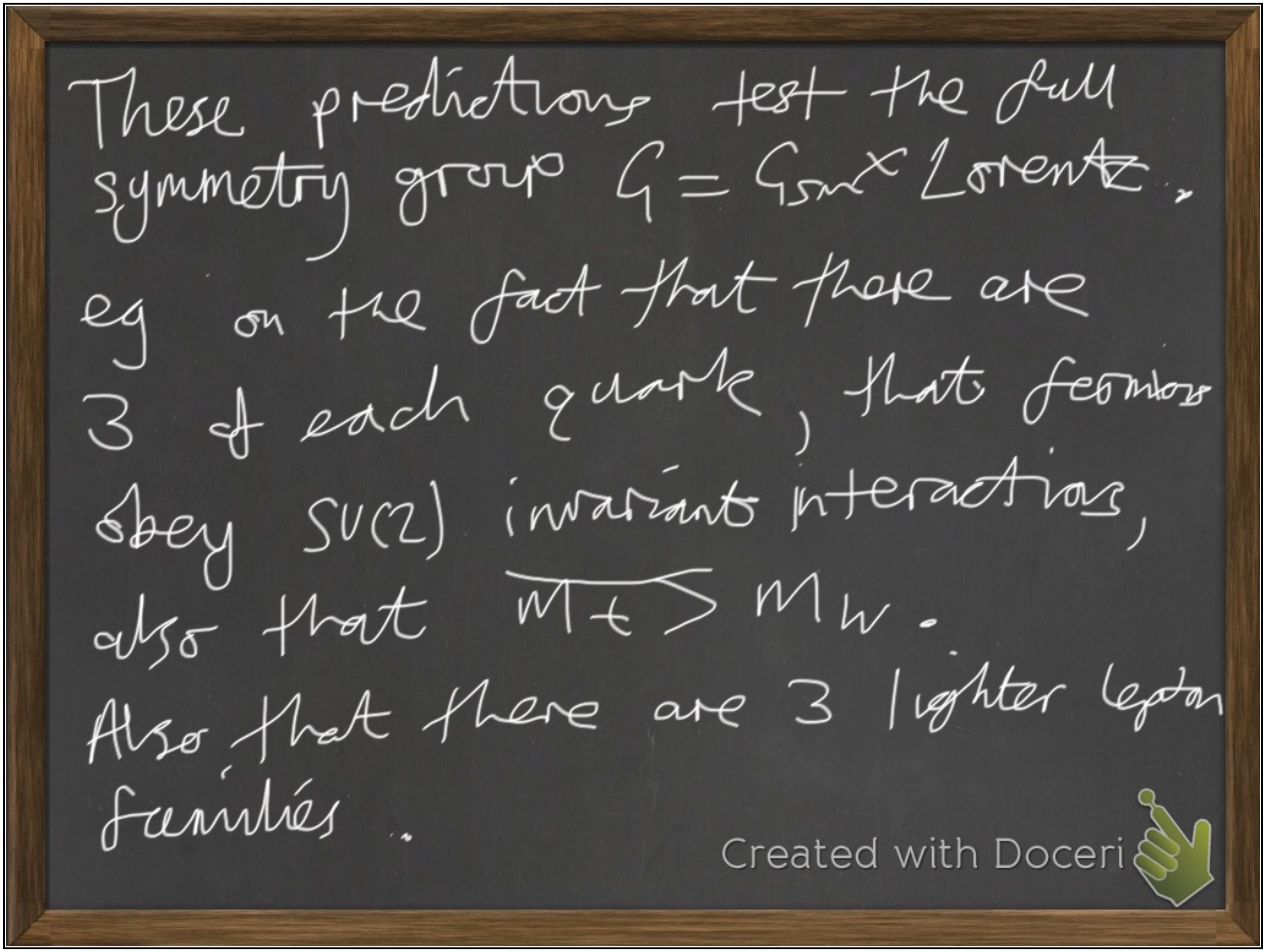
W^- modes are charge conjugates of the modes below.

	Mode		Fraction (Γ_i/Γ)	Confidence level
Γ_1	$\ell^+\nu$	[a]	(10.86 ± 0.09) %	
Γ_2	$e^+\nu$		(10.71 ± 0.16) %	
Γ_3	$\mu^+\nu$		(10.63 ± 0.15) %	
Γ_4	$\tau^+\nu$		(11.38 ± 0.21) %	
Γ_5	hadrons		(67.41 ± 0.27) %	
Γ_6	$\pi^+\gamma$		$< 7 \times 10^{-6}$	95%
Γ_7	$D_s^+\gamma$		$< 1.3 \times 10^{-3}$	95%
Γ_8	cX		(33.3 ± 2.6) %	
Γ_9	$c\bar{s}$		$(31 {}^{+13}_{-11})$ %	
Γ_{10}	invisible	[b]	(1.4 ± 2.9) %	

[a] ℓ indicates each type of lepton (e, μ, and τ), not sum over them.

[b] This represents the width for the decay of the W boson into a charged particle with momentum below detectability, p< 200 MeV.

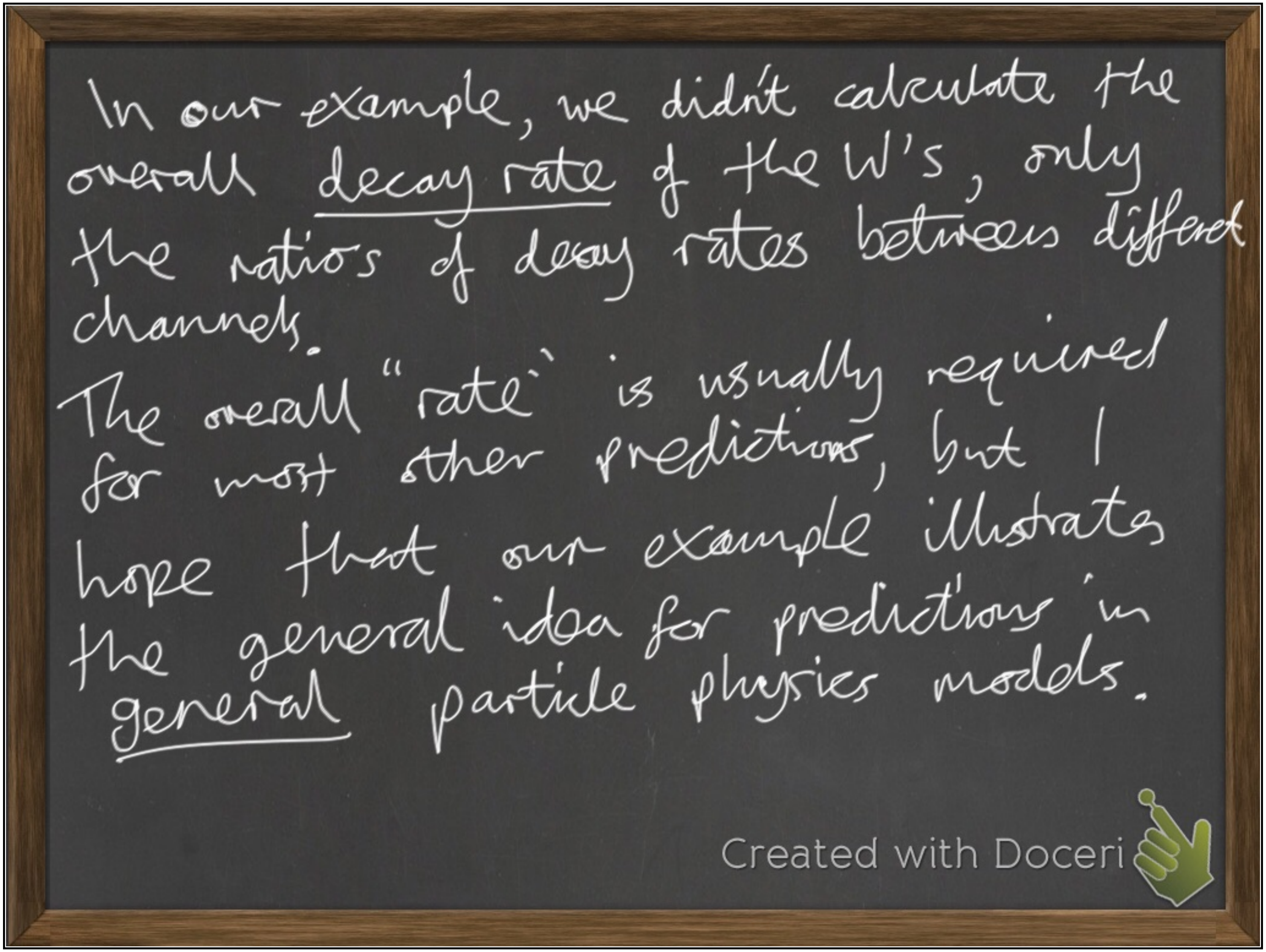
These are pretty close to our predictions!



These predictions test the full symmetry group $G = G_{SM} \times \text{Lorentz}$.

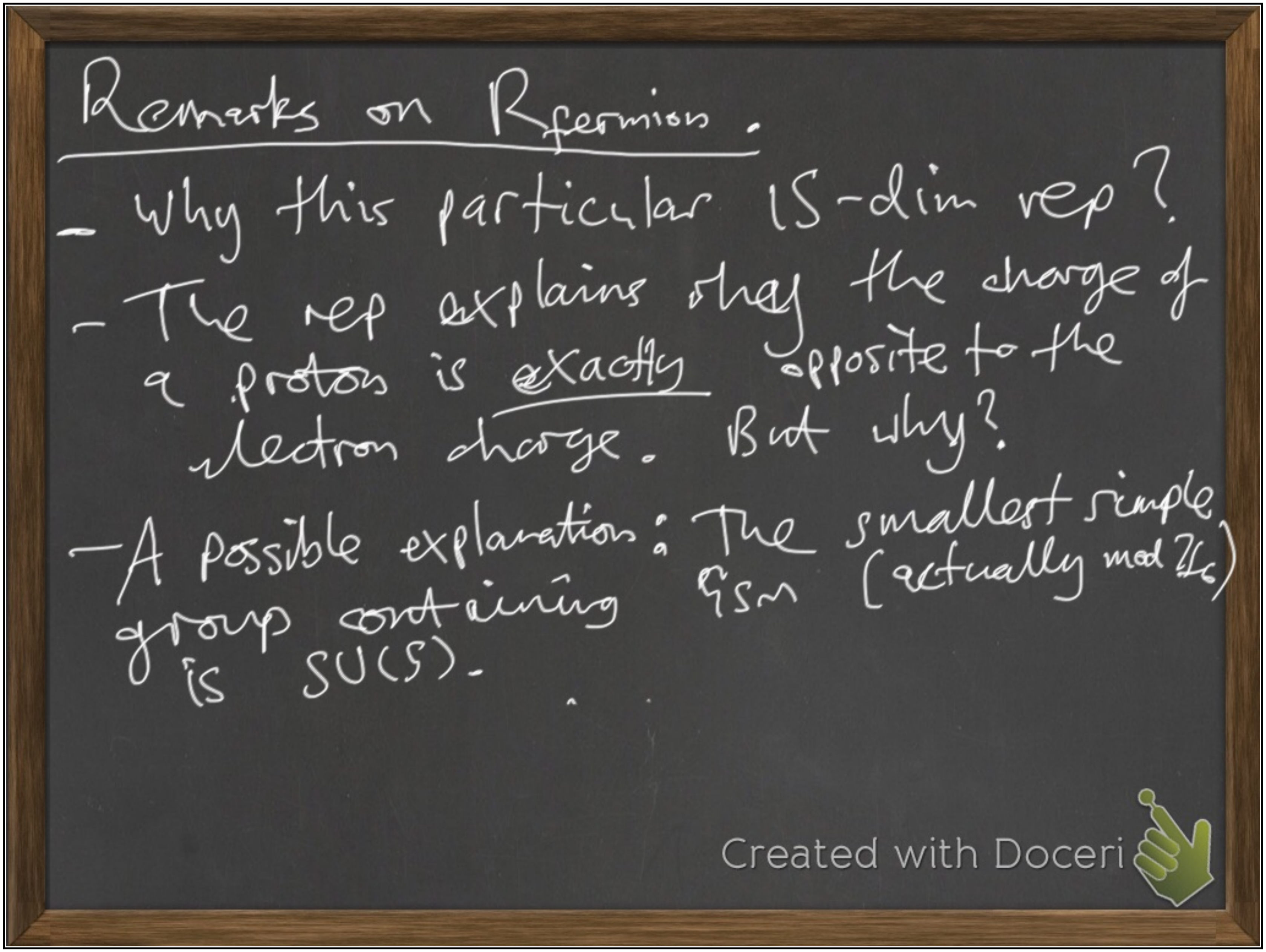
eg on the fact that there are 3 of each quark, that fermions obey SU(2) invariant interactions, also that $m_t > m_W$.

Also that there are 3 lighter lepton families.



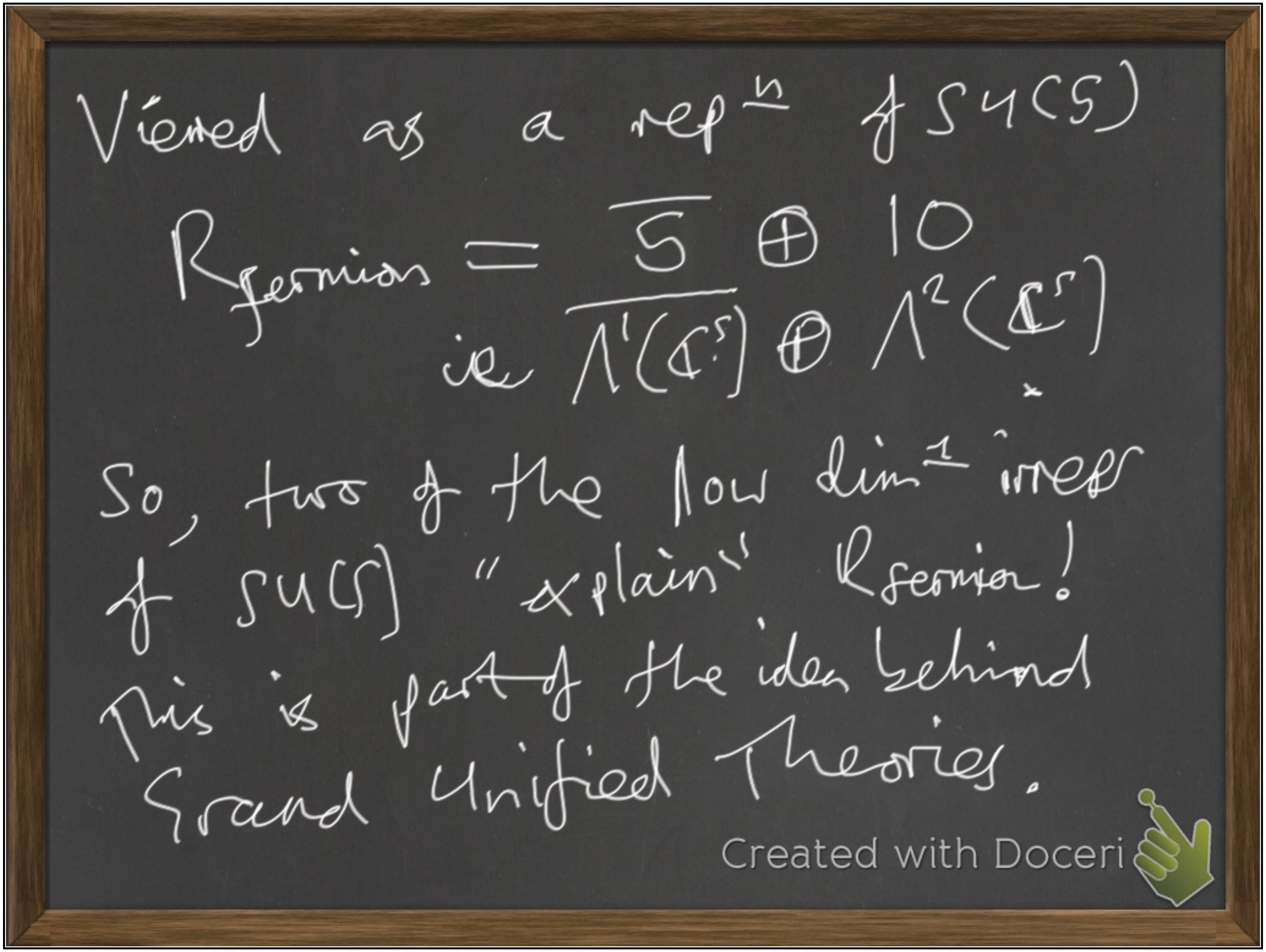
In our example, we didn't calculate the overall decay rate of the W's, only the ratios of decay rates between different channels.

The overall "rate" is usually required for most other predictions, but I hope that our example illustrates the general idea for predictions in general particle physics models.



Remarks on R_{fermion}.

- Why this particular 15-dim rep?
- The rep explains why the charge of a proton is exactly opposite to the electron charge. But why?
- A possible explanation: The smallest simple group containing G_{SM} (actually mod $\mathbb{Z}_6$) is $SU(5)$.



Viewed as a rep$^{\underline{n}}$ of $SU(5)$

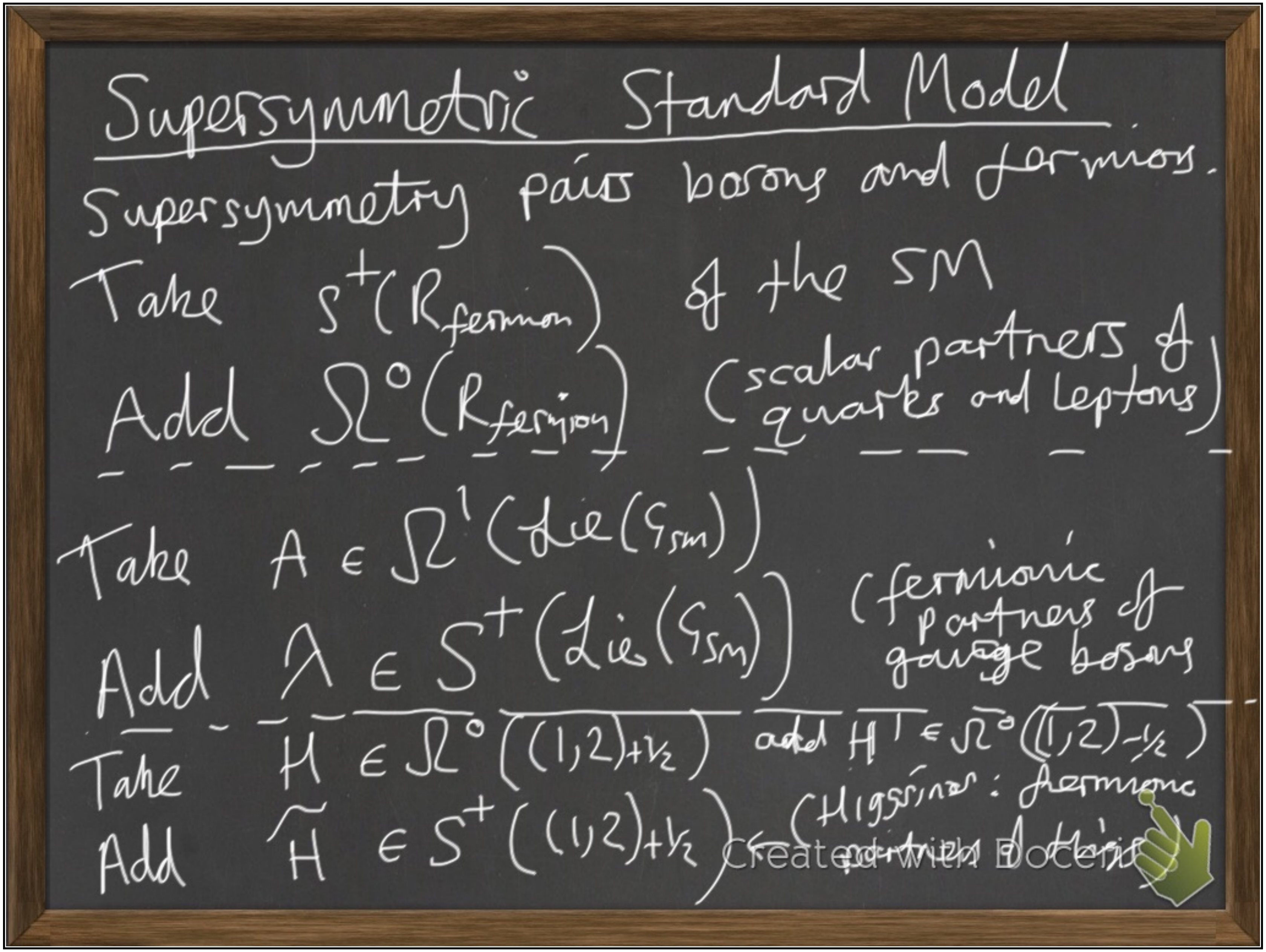
$$R_{\text{fermion}} = \overline{5} \oplus 10$$

$$\text{ie } \overline{\Lambda^1(\mathbb{C}^5)} \oplus \Lambda^2(\mathbb{C}^5)$$

So, two of the low dim$^{\underline{l}}$ irreps of $SU(5)$ "explain" R_{fermion}!
This is part of the idea behind Grand Unified Theories.

Other gauge theories can be obtained by replacing G_{SM} by another compact group and choosing $S^+(R)$, $\Omega^0(R')$, etc …

[There are restrictions called anomalies, but we will not get into that]



Supersymmetric Standard Model

Supersymmetry pairs bosons and fermions.

Take $S^+(R_{\text{fermion}})$ of the SM

Add $\Omega^0(R_{\text{fermion}})$ (scalar partners of quarks and leptons)

Take $A \in \Omega^1(\mathrm{Lie}(G_{SM}))$

Add $\lambda \in S^+(\mathrm{Lie}(G_{SM}))$ (fermionic partners of gauge bosons)

Take $H \in \Omega^0((1,2)_{+1/2})$ add $H^\dagger \in \Omega^0(\overline{(1,2)}_{-1/2})$

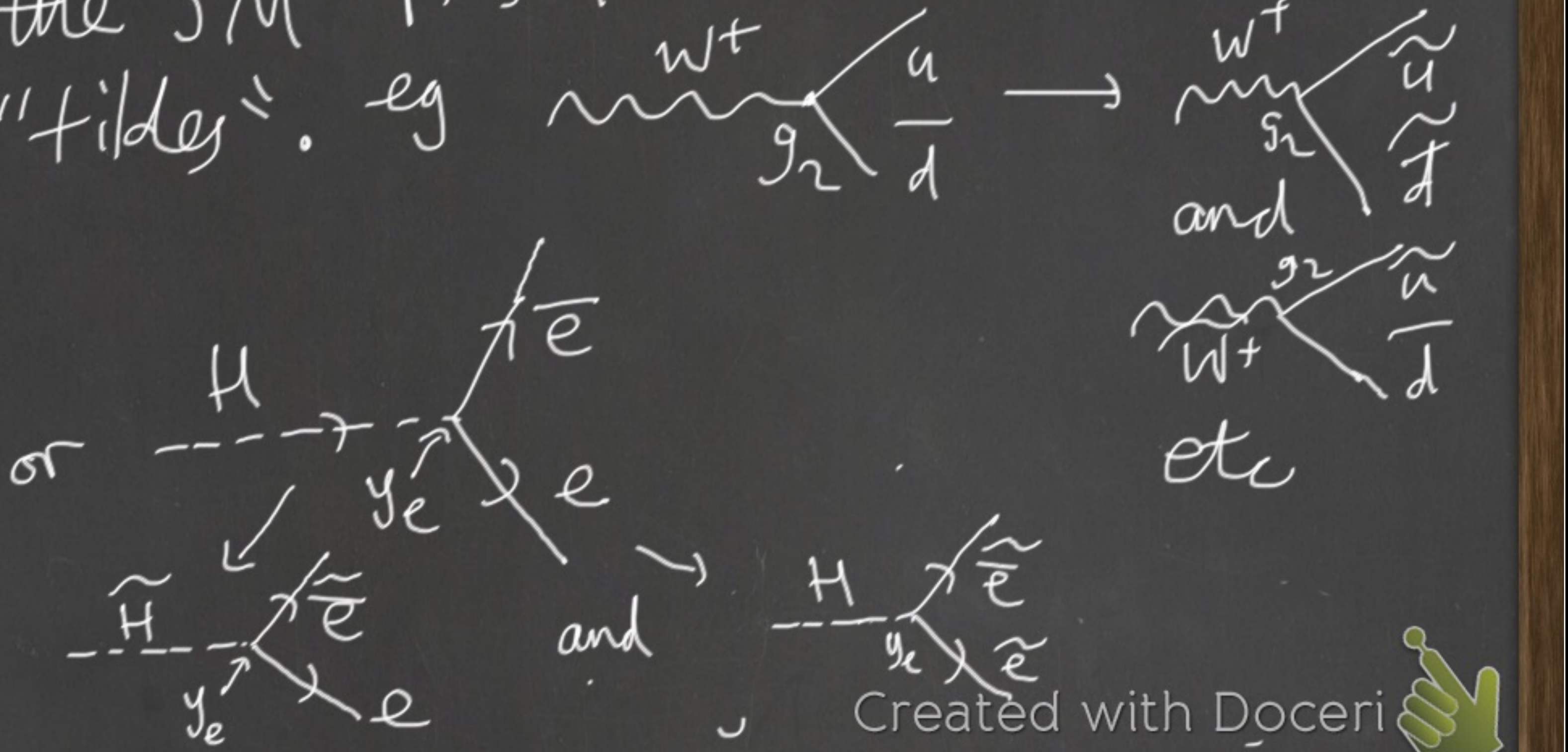
Add $\tilde{H} \in S^+((1,2)_{+1/2}) \leftarrow$ (Higgsinos: fermionic partners of these)

The superpartners of $Q_L, \overline{u_R}, \overline{d_R}, L, \overline{e_R}$ are denoted $\widetilde{Q_L}, \widetilde{\overline{u_R}}, \widetilde{\overline{d_R}}, \widetilde{L}, \widetilde{\overline{e_R}}$ and are sections of $\Omega^0(R_{\text{fermion}})$.

Redenote $H \in \Omega^0(R=(1,2)_{-1/2})$ as H_d and add $H_u \in \Omega^0(R=(1,2)_{1/2})$

The superpartners of H_u and H_d are described by fields $\widetilde{H_u} \in S^+((1,2)_{1/2})$ and $\widetilde{H_d} \in S^+((1,2)_{-1/2})$

Many vertices of the Minimal Supersymmetric Standard Model consist of those of the SM plus the same vertices with Two "tildes". eg


W^+
g_2
u
$\bar{d}$
$\longrightarrow$
W^+
g_2
$\tilde{u}$
$\tilde{\bar{d}}$
and
g_2
W^+
$\tilde{u}$
$\bar{d}$
etc
or
H
y_e
$\bar{e}$
e
$\tilde{H}$
y_e
$\tilde{\bar{e}}$
e
and
H
y_e
$\tilde{\bar{e}}$
$\tilde{e}$

There are also some additional quadratic terms which give masses

eg

$$M^2_{AB}\,\overline{\tilde{\Psi}_A}\,\tilde{\Psi}_B + \mu H_u H_d + B\mu H_u H_d$$

$$+\, i M_1 \overline{\tilde{B}}\, B + i M_2 \overline{\tilde{W}}\, W + i M_3 \overline{\tilde{g}}\, g$$

and some cubic ones

eg

$$A_{ij} H_u \tilde{Q}_{L_i} \tilde{u}_{R_j} + \dots$$

Particle Physics from G_2-manifolds

Consider M-theory on $X_7 \times \mathbb{R}^{3,1}$ with metric $g_{10,1} = g(X) + \eta(\mathbb{R}^{3,1})$ where $\mathrm{Hol}(g(X)) = G_2$ and η is flat.

If X is smooth (and large), we can use 11d supergravity to describe the low energy physics.

11d Sugra fields

$$\Big(\underset{\substack{\uparrow \\ \Omega^3}}{C_3}\,,\ \underset{\substack{\uparrow \\ \text{metric}}}{g_{11}}\,,\ \underset{\substack{\uparrow \\ S(TM) \\ \text{gravitino}}}{\psi_{3/2}} \Big)$$

1. We fix a backround solution of the E-L equations of the form

$$(C_3, g_{11}, \psi_{3/2}) = (0, g^0(x) + \eta(\mathbb{R}^{3,1}), 0)$$

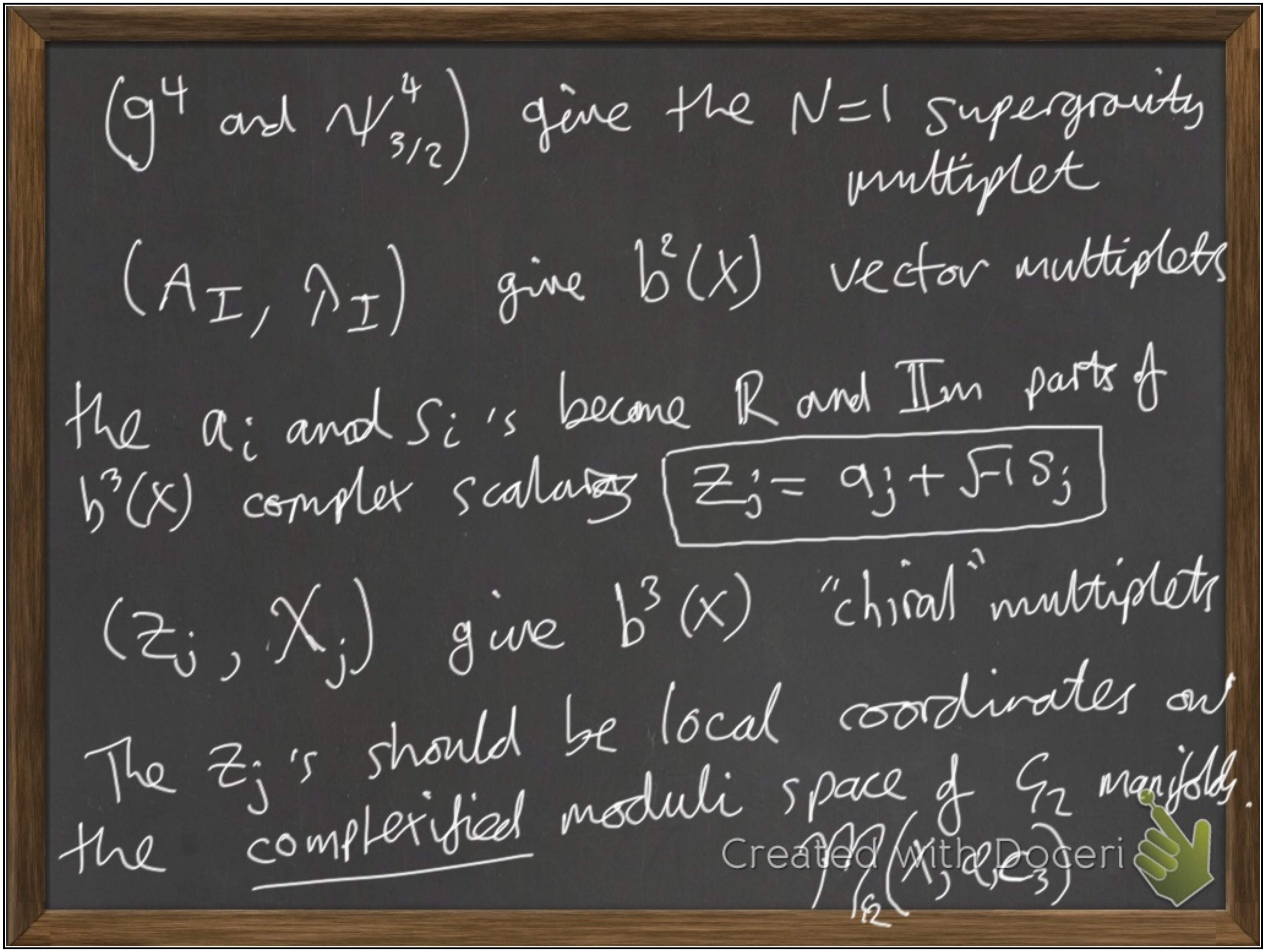
2. We perturb
3. We extract the ~~zero~~ modes
4. We integrate over $\int_X \sqrt{g^0}\,\mathcal{L}^{11d}$
5. We get left with $\int d^4x\,\mathcal{L}^{4d}$

The zero modes:

$$\text{– From } C_3 \left\{ \begin{array}{lll} b^2(X) & U(1)\text{ gauge fields} & A_I \in \Omega^1(M^{3,1}) \\ b^3(X) & S^1\text{-valued scalars} & \underset{\substack{\uparrow \\ \text{axions}}}{a_i} \in \Omega^0(M^{3,1}) \end{array} \right.$$

$$\text{– From } g_{11} \left\{ \begin{array}{lll} 1 & 4\text{d metric} & g^{3+1}_{\mu\nu} \in S^2(M^{3,1}) \\ b^3(X) & \text{scalars} & \underset{\substack{\uparrow \\ G_2\text{-moduli}}}{s_i} \in \Omega^0(M^{3,1}) \end{array} \right.$$

$$\text{– from } \Psi_{3/2} \left\{ \begin{array}{lll} 1 & 4\text{d gravitino} & \psi^4_{3/2} \in S^+(TM) \\ b^2(X) & \text{spinors} & \lambda_I \in S^+(M) \\ b^3(X) & \text{spinors} & \chi_i \in S^+(M) \end{array} \right.$$



(g^4 and $\psi^4_{3/2}$) give the $N=1$ supergravity multiplet

(A_I, λ_I) give $b^2(X)$ vector multiplets

the a_i and s_i's become $\mathbb{R}$ and $\mathbb{I}$m parts of $b^3(X)$ complex scalars $\boxed{z_j = a_j + \sqrt{-1}\, s_j}$

(z_j, χ_j) give $b^3(X)$ "chiral" multiplets

The z_j's should be local coordinates on the complexified moduli space of G_2 manifolds.

$\mathcal{M}_{G_2}(X, \varphi, C_3)$

$$Z_j \sim \int_{\Sigma_j} C_3 + i\,\Omega_3$$

$\uparrow$ G_2 form

are periods of a complexified G_2 3-form over a basis $\{\Sigma_j\}$ for $H_3(X)$.

The $\mathcal{L}^4$ that we get from this has a metric on the moduli space, which is a Kähler metric with potential:

$$\boxed{K(z_j,\bar{z}_j) = -3\ln\int_X \Omega\wedge *\Omega}$$

- We would very much like to know the properties of this Kähler metric!
- Can we compute it approximately for the TCS G_2-manifolds?

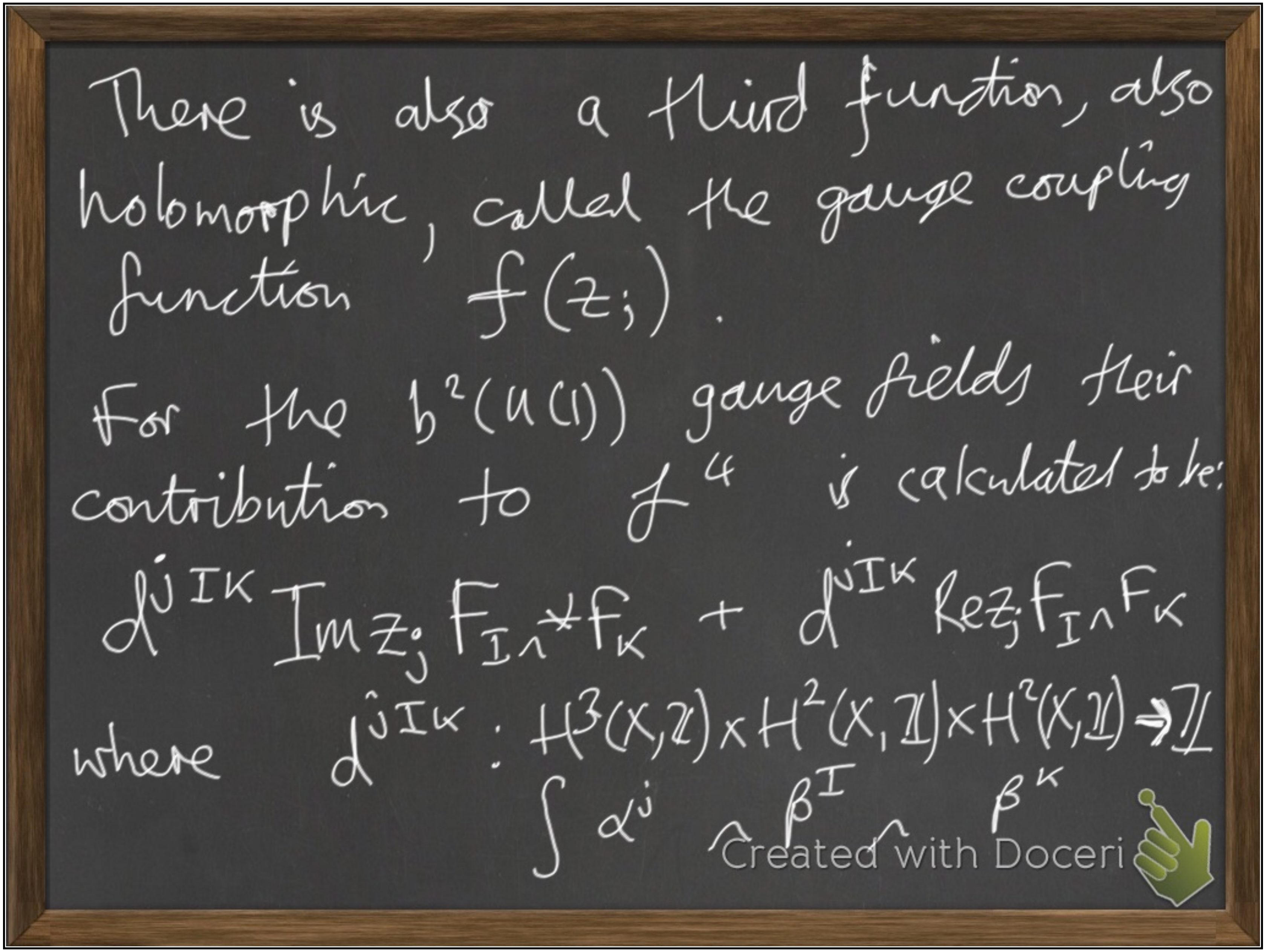
- The components of the moduli space metric are homogeneous of degree minus two, so it looks like the metric has "−ve curvature" in some sense that would be good to make precise.

In general, $N=1$ $d=4$ supergravity theories also depend on a superpotential $W(z_i)$, $\partial_j W = 0$, locally holomorphic. Witten/Bagger: W is a section of a line bundle $L \to \mathcal{M}_{G_2}(X, \ldots)$.

Because the $a_i = \mathrm{Re}\, z_i$ is periodic,

$W = \text{zero}$, up to instantons effects.

- Instantons are associative submanifolds (suitably rigid)



There is also a third function, also holomorphic, called the gauge coupling function $f(z_j)$.

For the $b^2(U(1))$ gauge fields their contribution to $\mathcal{L}^4$ is calculated to be:

$$d^{jIK} \operatorname{Im} z_j F_I \wedge * F_K + d^{jIK} \operatorname{Re} z_j F_I \wedge F_K$$

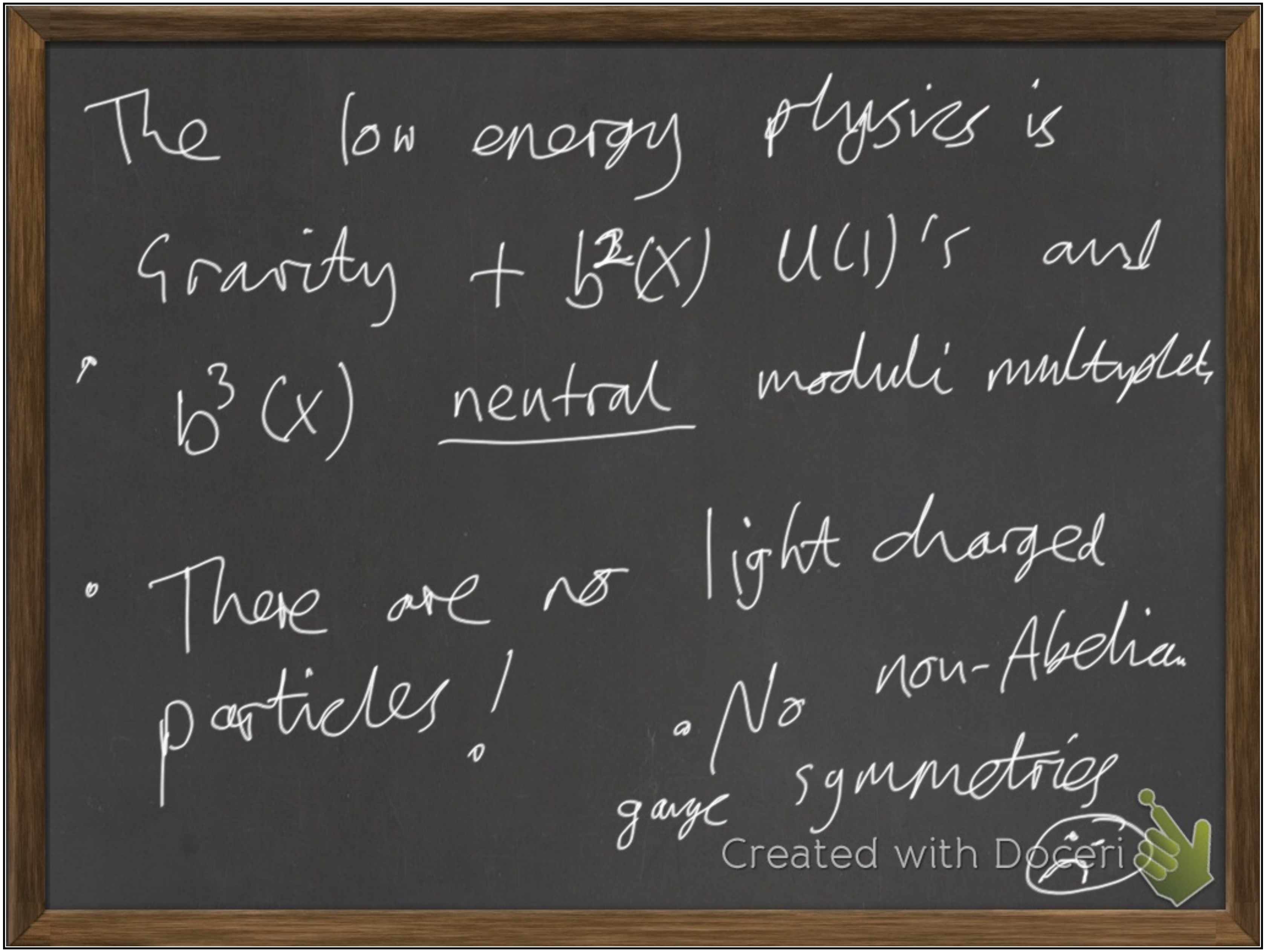
where $d^{jIK} : H^3(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$

$$\int \alpha^j \wedge \beta^I \wedge \beta^K$$

So $f(Z) = \int_X (C_3 + i\phi) \wedge \beta^I \wedge \beta^K$

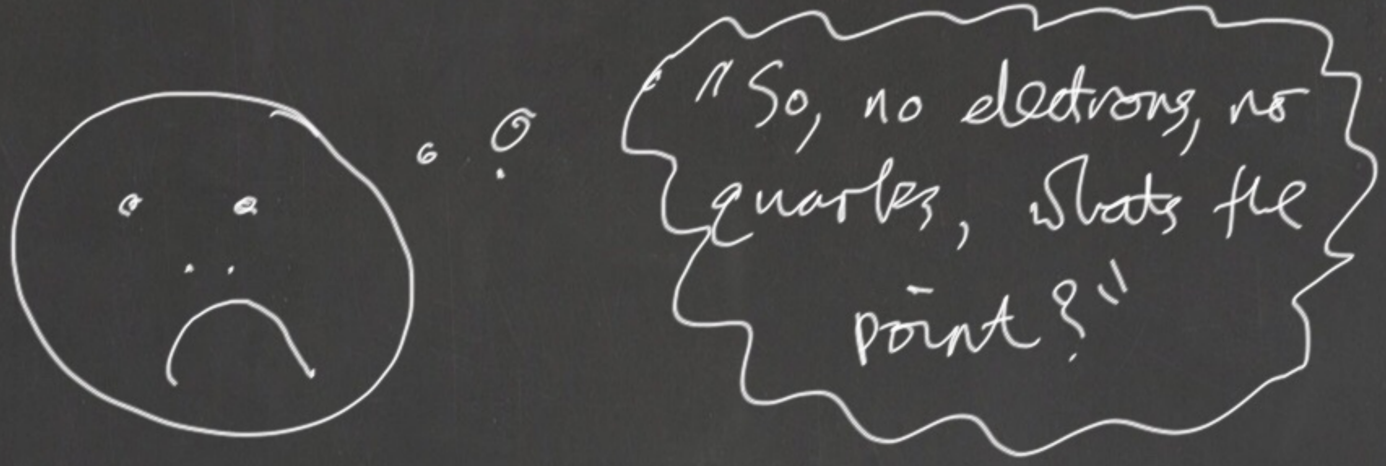
$\{\beta^I\}$ basis for $H^2(X, \mathbb{Z})$

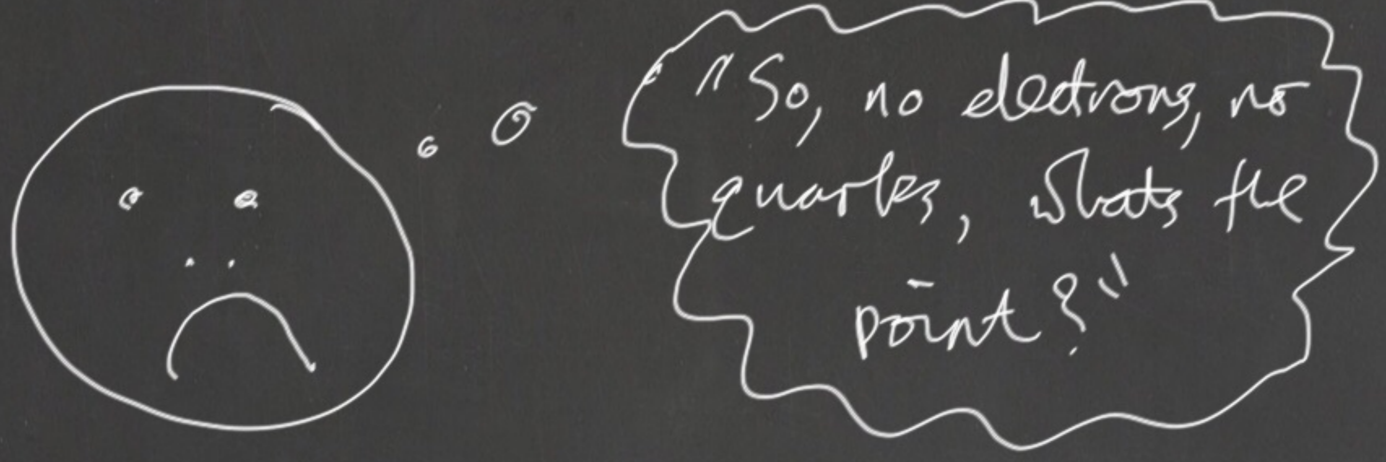
Can ~~wr~~ K



The low energy physics is Gravity + $b^2(X)$ U(1)'s and

- $b^3(X)$ neutral moduli multiplets
- There are no light charged particles!
- No non-Abelian gauge symmetries


"So, no electrons, no quarks, whats the point?"


"So, no electrons, no quarks, whats the point?"

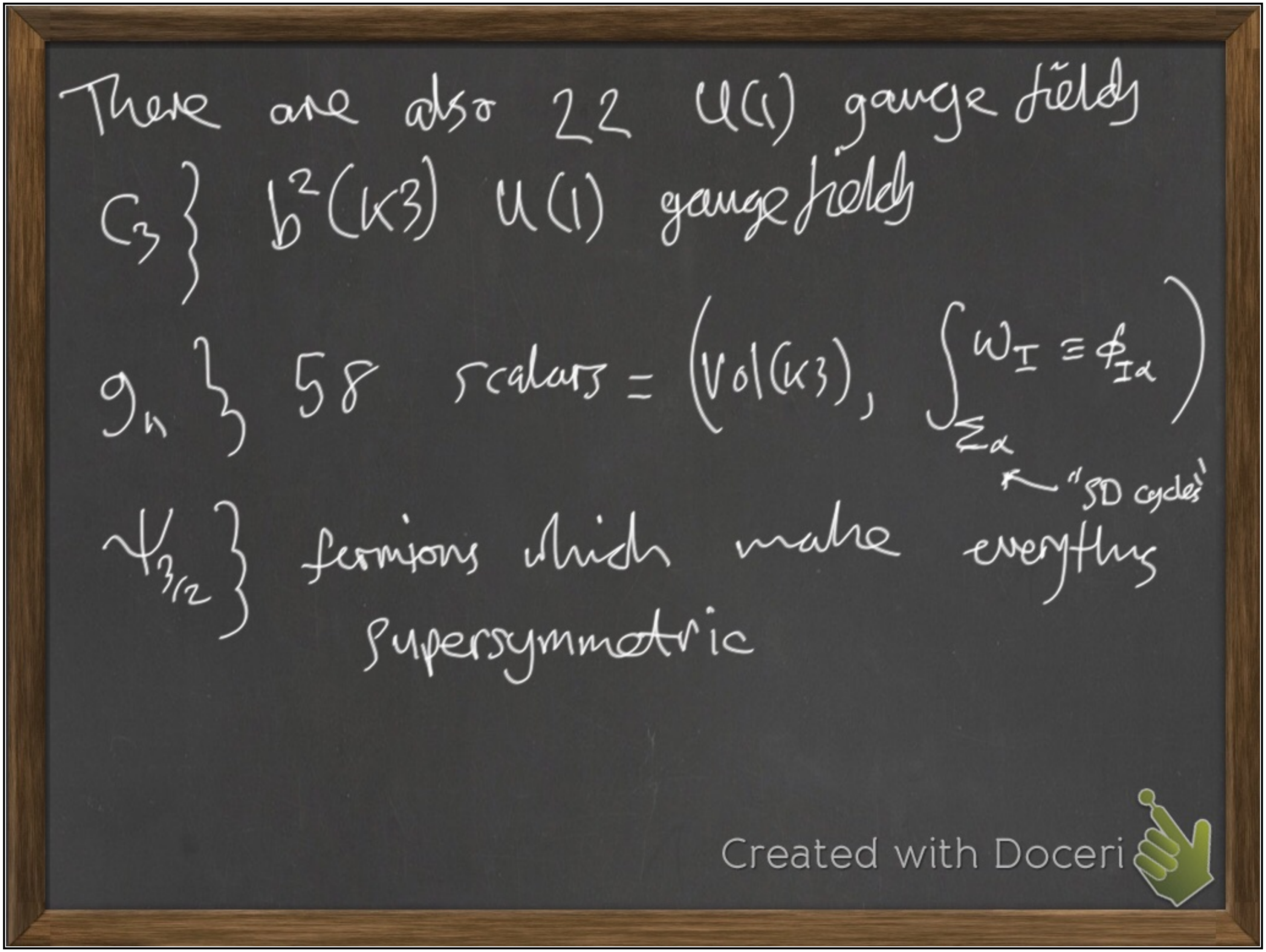
We will try and show that particles like quarks and leptons and non-Abelian gauge bosons emerge from special kinds of singularities of G_2-manifolds.

Yang-Mills Fields from codim 4 singularities

Consider a special case when $X = K3 \times \mathbb{R}^3$ w/ product metric

Then $M^{10,1} = X \times \mathbb{R}^{3,1} = K3 \times \mathbb{R}^{6,1}$, leading to a 6+1 d Lagrangian.

In this case the moduli space = space of Einstein metrics on $K3 = \mathbb{R}^+ \times \dfrac{SO(3,19)}{SO(3) \times SO(19)}$



There are also 22 U(1) gauge fields

$C_3 \}$ $b^2(K3)$ U(1) gauge fields

$g_n \}$ 58 scalars $= \left(\mathrm{Vol}(K3),\ \int_{\Sigma_\alpha} \omega_I \equiv \phi_{I\alpha} \right)$

$\nwarrow$ "SD cycles"

$\psi_{3/2} \}$ fermions which make everything supersymmetric

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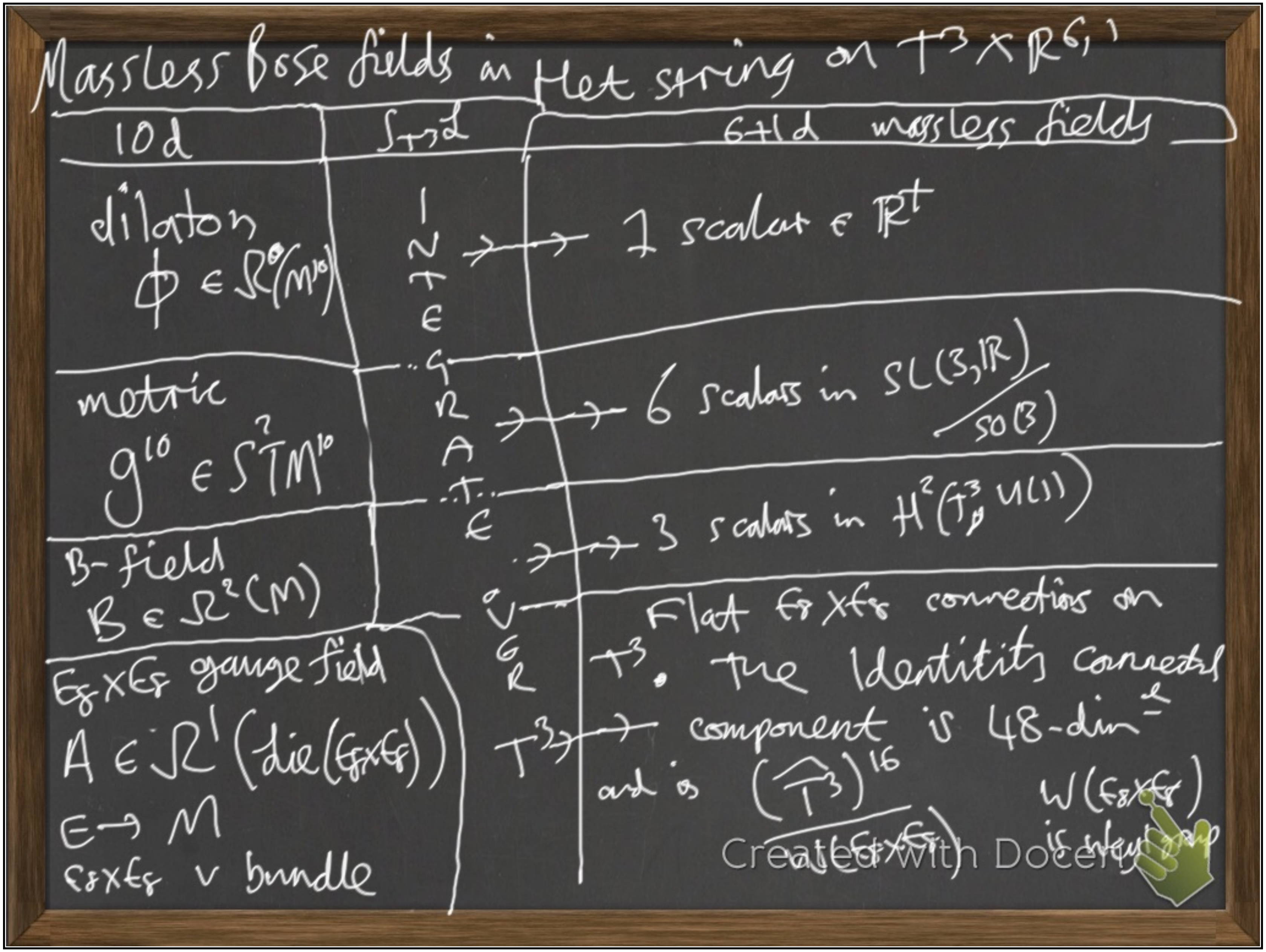
This looks v.similar to the theory obtained by considering $E_8\times E_8$ Heterotic superstring theory on $M^{9,1} = T^3 \times \mathbb{R}^{6,1}$.



Massless Bose fields in Het string on $T^3 \times \mathbb{R}^{6,1}$

10d	$\int_{T^3} d$	6+1d massless fields
dilaton $\phi \in \Omega^0(M^{10})$	I N T E $\to$	1 scalar $\in \mathbb{R}^+$
metric $g^{10} \in S^2 TM^{10}$	G R A T $\to$	6 scalars in $SL(3,\mathbb{R})/SO(3)$
B-field $B \in \Omega^2(M)$	E $\to$	3 scalars in $H^2(T^3; U(1))$
$E_8\times E_8$ gauge field $A \in \Omega^1(\mathrm{Lie}(E_8\times E_8))$ $E \to M$ $E_8\times E_8$ v bundle	O V E R $T^3 \to$	Flat $E_8\times E_8$ connections on T^3. The Identitits connected component is 48-dim$^{\underline{l}}$ and is $\dfrac{(\widehat{T^3})^{16}}{W(E_8\times E_8)}$ $W(E_8\times E_8)$ is Weyl group

So the moduli space is 58-dim^l

$$58 = \underset{\phi}{\overset{\uparrow}{1}} + \underset{SL(3)/SO(3)}{\overset{\uparrow}{6}} + \underset{H^2(T^3)}{\overset{\uparrow}{3}} + \underset{(H_1(T^3))^{\mathrm{rk}(E_8\times E_8)}}{\overset{\uparrow}{48}}$$

What about gauge bosons?

$g_{\mu\nu} \longrightarrow U(1)^3$ from the 3 Killing vectors on T^3.

$B \longrightarrow U(1)^3$ from the 3 harmonic 1 forms on T^3.

$A \longrightarrow U(1)^{16}$ at generic points in space of flat $E_8 \times E_8$ connections (1d comp)

So $U(1)^{22}$, as in M-theory on K3.

- In fact, the Heterotic moduli space on T^3 is also, locally $\mathbb{R}^+ \times \frac{SO(3,19)}{SO(3) \times SO(19)}$

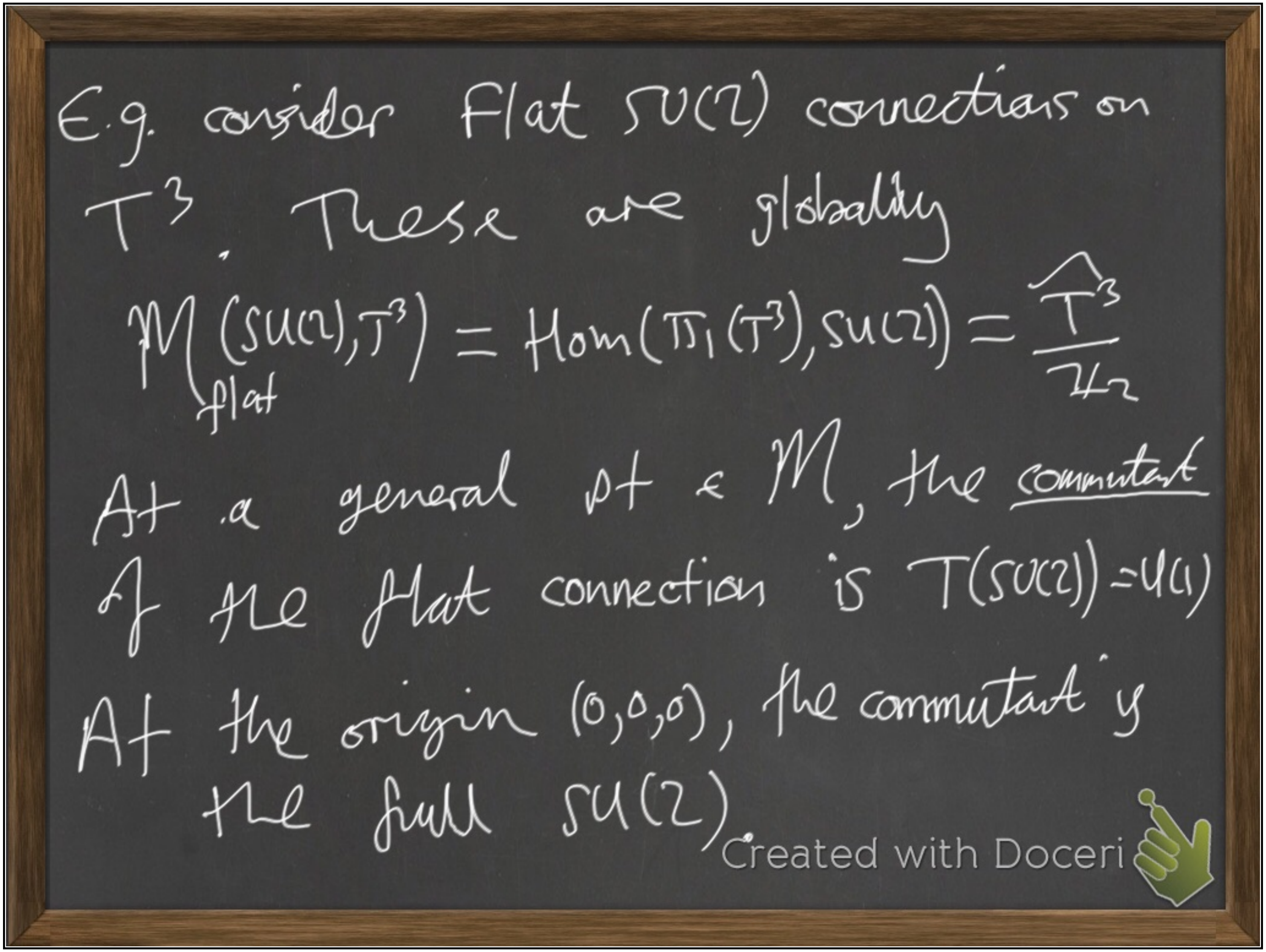
- String dualities assert that the heterotic string on T^3 is equivalent to M theory on K3.

- Non-Abelian gauge symmetry is present in Het string from the start

Non-Abelian symmetries in Het on T^3

The $U(1)^{16}$ gauge group is identified as the commutant of a generic flat connection on T^3 inside $E_8 \times E_8$.

But: at special codim 3 and 3n subspaces, the commutant enhances to non-Abelian groups



E.g. consider flat SU(2) connections on T^3. These are globally

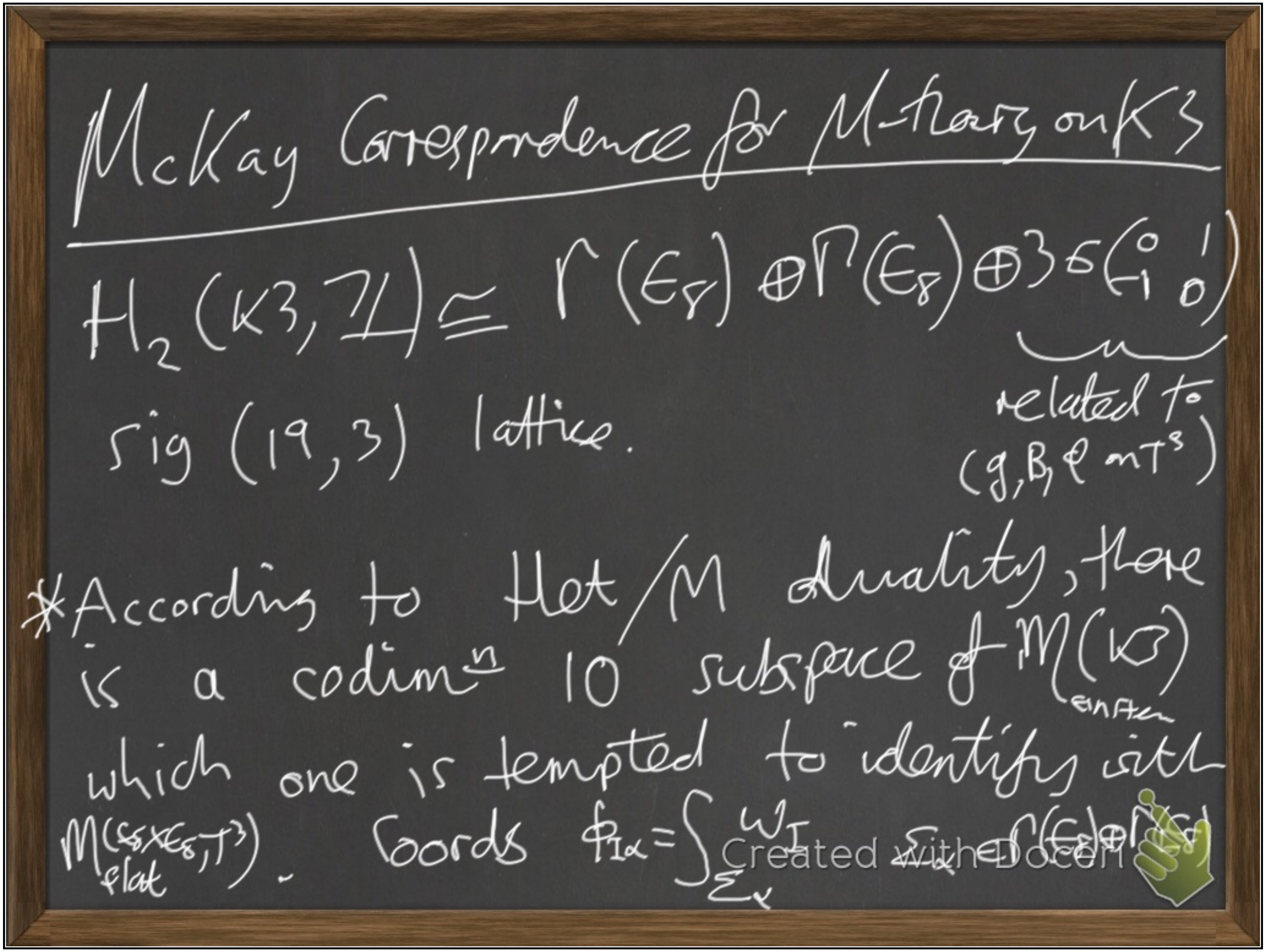
$$\mathcal{M}_{flat}(SU(2),T^3) = \mathrm{Hom}(\pi_1(T^3), SU(2)) = \frac{\widehat{T^3}}{\mathbb{Z}_2}$$

At a general pt $\in \mathcal{M}$, the commutant of the flat connection is $T(SU(2)) = U(1)$

At the origin $(0,0,0)$, the commutant is the full $SU(2)$.

In general for $\mathcal{M}_{flat}(E_8 \times E_8; T^3)$, one requires choosing at least one $U(1) \subset T(E_8 \times E_8)$ and the 3 holonomies of this connection must vanish in order to ENHANCE the GAUGE SYMMETRY.

ie codim 3 singularities

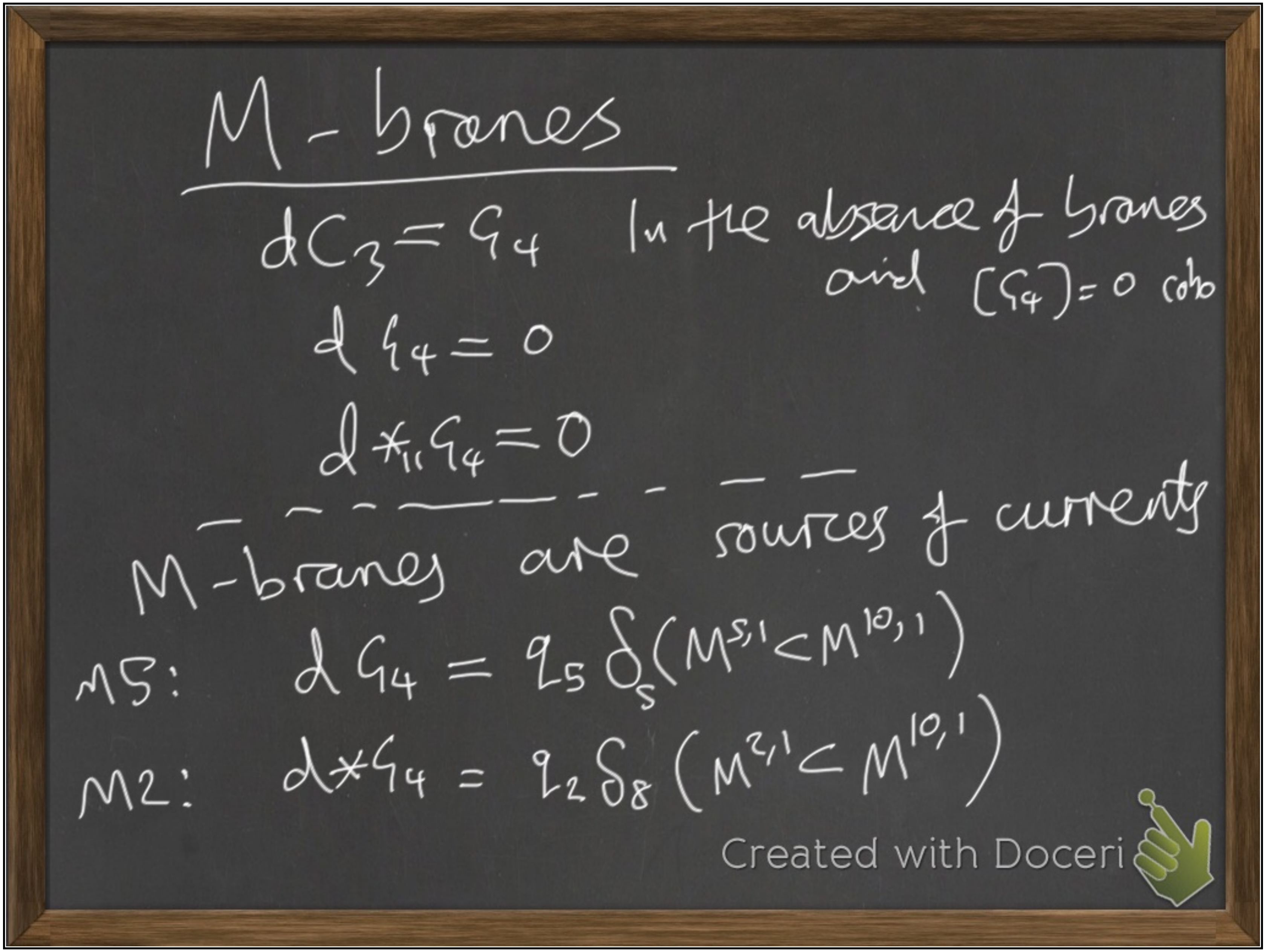


McKay Correspondence for M-theory on K3

$$H_2(K3,\mathbb{Z}) \cong \Gamma(E_8)\oplus\Gamma(E_8)\oplus \underbrace{3\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\substack{\text{related to}\\(g,B,\phi\ \text{on}\ T^3)}}$$

sig (19, 3) lattice.

*According to Het/M duality, there is a codim$^{\underline{n}}$ 10 subspace of $\mathcal{M}(K3)_{\text{einstein}}$ which one is tempted to identify with $\mathcal{M}^{\text{flat}}(E_8\times E_8, T^3)$. Coords $\phi_{I\alpha} = \int_{\Sigma_\alpha}\omega_I$ $\Sigma_\alpha \in \Gamma(E_8)\oplus\Gamma(E_8)$



M - branes

$dC_3 = G_4$ In the absence of branes and $[G_4] = 0$ coh

$d G_4 = 0$

$d *_{11} G_4 = 0$

M-branes are sources of currents

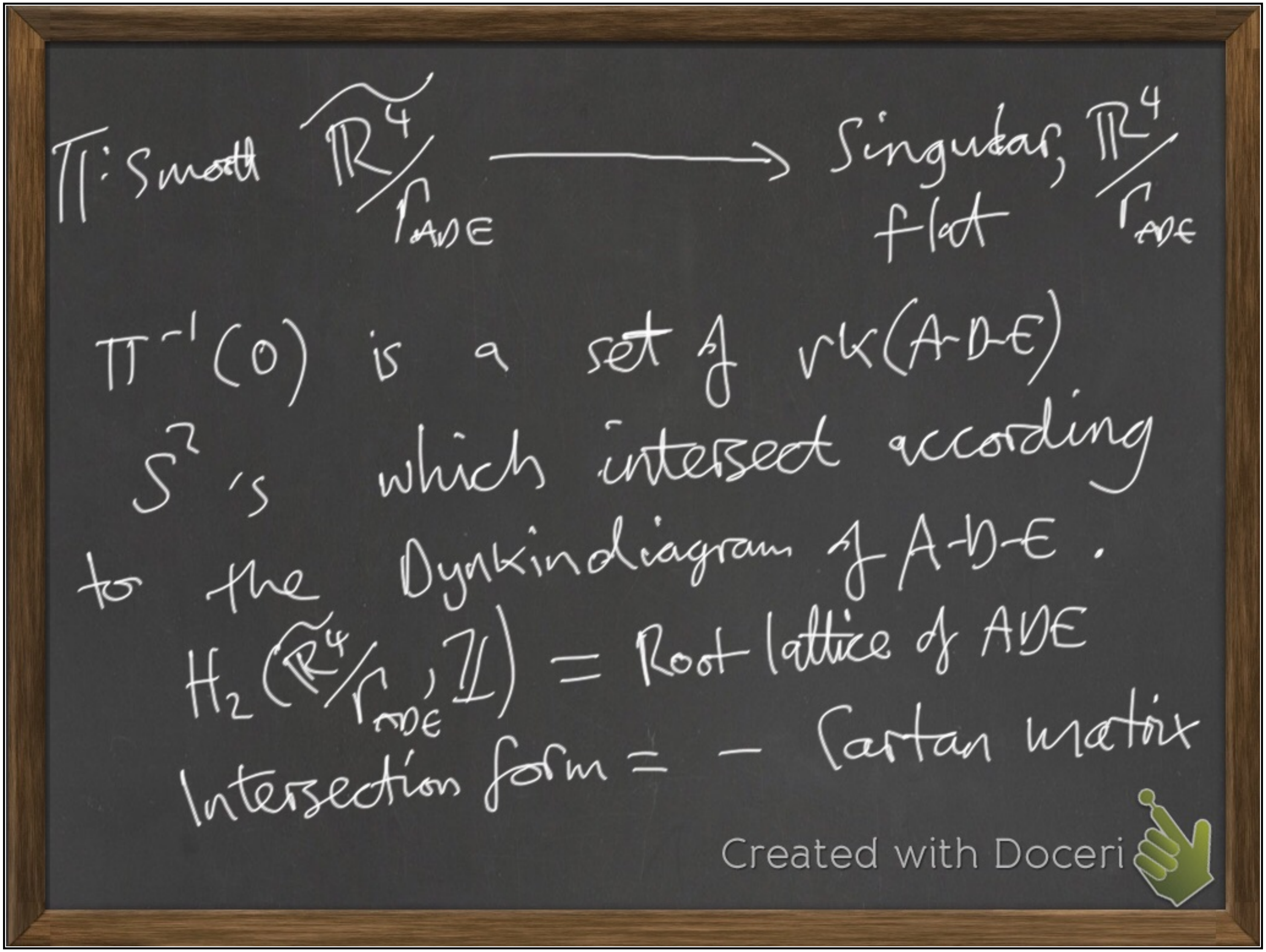
M5: $d G_4 = q_5 \delta_5 (M^{5,1} \subset M^{10,1})$

M2: $d * G_4 = q_2 \delta_8 (M^{2,1} \subset M^{10,1})$

Near singularities of $\mathcal{M}^{(K3)}_{\text{EINSTEIN}}$,

The K3 becomes singular and has codim 4 orbifold singularities. We can model these on orbifold singularities of the form $\mathbb{R}^4/\Gamma_{ADE}$ where Γ_{ADE} is a finite subgroup of $SU(2) \subset SO(4)$ acting on $\mathbb{C}^2 \cong \mathbb{R}^4$ in the fundamental rep$^{\underline{n}}$.

The flat metric on $\mathbb{R}^4/\Gamma_{ADE}$ admits a desingularisation with a smooth hyperkähler metric



Π: Smooth $\widetilde{\mathbb{R}^4}/\Gamma_{ADE} \longrightarrow$ Singular, flat $\mathbb{R}^4/\Gamma_{ADE}$

$\Pi^{-1}(0)$ is a set of rk(A-D-E) S^2's which intersect according to the Dynkindiagram of A-D-E.

$H_2(\widetilde{\mathbb{R}^4}/\Gamma_{ADE}, \mathbb{Z})$ = Root lattice of ADE

Intersection form = − Cartan matrix

$$A_\alpha = \int_{\Sigma_\alpha \subset H_2\left(\widetilde{\mathbb{R}^4/\Gamma_{ADE}}\right)} C_3$$

are a set of $rk(ADE)$ $U(1)$ gauge fields. The gauge group, $U(1)^{rk(ADE)}$ is naturally identified with T(ADE group).

max torus

Moreover, since $d\star G_4 = q_2\,\delta_8(M^{2+1})$,

M2-branes wrapped on the $\Sigma_\alpha \subset H_2$

are like charged particles in $\mathbb{R}^{6,1}$:

$$d\star G_4 = \widetilde{\Sigma}_\alpha \wedge d\star_{6+1} dA_\alpha = q_2\,\delta_6(\text{pt}) \wedge [\Sigma_\alpha]$$

$\widetilde{\Sigma}_\alpha$ = Poincaré dual of Σ_α.

Because $H_2\left(\widetilde{\mathbb{R}^4/\Gamma_{ADE}}\right)$ = root lattice ADE,

These M2-branes, plus the rk(ADE) "photons" have charges of the adjoint rep^n^ of ADE.

We also get $3 \times \text{rk}(ADE)$ moduli fields from the moduli space of Einstein metrics.

$$\int_{\Sigma_\alpha} \omega_I \sim \phi_{I\alpha} \qquad I = 1,2,3$$

The wrapped M2-branes are "BPS" states whose masses are *exactly* given by the volumes of the exceptional S^2's

i.e. $\text{Mass}_2 = |\phi_{I\alpha}|$

$\therefore$ At the origin of moduli space, we have a copy of the adjoint rep^n of ADE which is MASSLESS .

- The whole system is also supersymmetric
- How do we describe this?

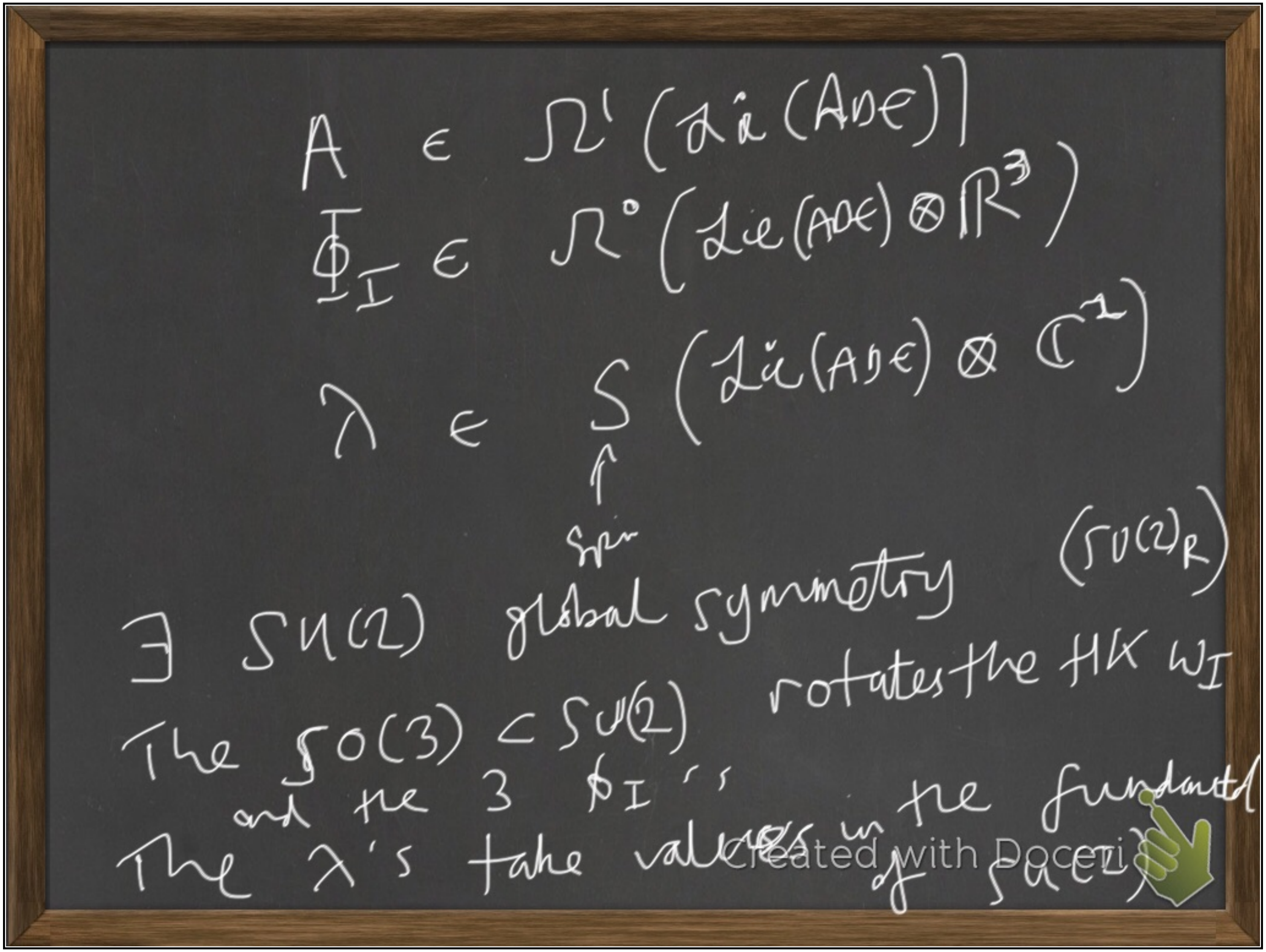
☆ At the two derivative level, ∃ a unique supersymmetric $\mathcal{L}$agrangian theory in $\mathbb{R}^{6,1}$ with non-Abelian gauge symmetry : 6+1 d Super Yang-Mills. $\mathcal{N}$

∴ The physics of M-theory on

$$\left(\widetilde{\frac{\mathbb{R}^4}{\Gamma_{ADE}}} \times \mathbb{R}^{6,1}, \underset{\overset{\uparrow}{H\text{-}K}}{\omega_I}\right)$$

is described by (6+1) d Super-Yang-Mills theory with $G =$ A-D-E gauge symmetry:

$$\mathcal{L}^{6+1d} = -\frac{1}{4}\operatorname{tr} F_A * F_A + i\bar{\lambda}\not{D}_A\lambda + \sum_I D_A\phi_I * D_A\phi_I - \operatorname{tr}[\phi_I,\phi_J]^2$$



$$A \in \Omega^1(\mathcal{L}ie(ADE))]$$

$$\Phi_I \in \Omega^0(\mathcal{L}ie(ADE)\otimes\mathbb{R}^3)$$

$$\lambda \in \underset{\substack{\uparrow\\ \text{spin}}}{S}(\mathcal{L}ie(ADE)\otimes\mathbb{C}^2)$$

$\exists$ $SU(2)$ global symmetry $(SU(2)_R)$

The $SO(3) \subset SU(2)$ rotates the HK ω_I and the 3 ϕ_I's

The λ's take values in the fundmtl of $SU(2)$

The vacua are described by the set of

$$\Phi_I \in \Omega^0(\mathcal{L}ie(ADE) \otimes \mathbb{R}^3)$$

with $$[\Phi_I, \Phi_J] = 0$$

modulo gauge transforms.

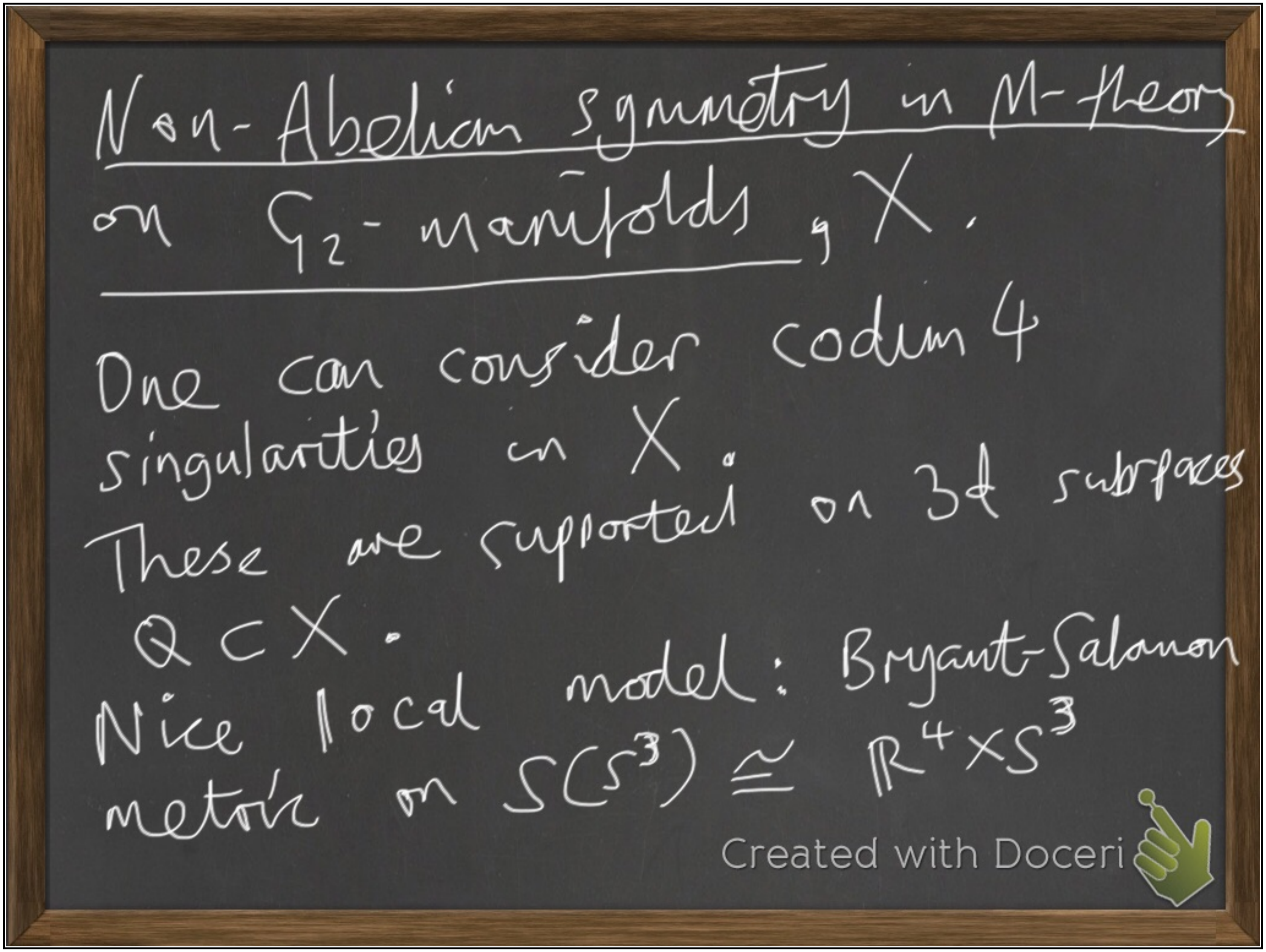
ie a local description of a flat ADE connection on T^3.

[cover with triple

This is exactly the moduli space of hyperkähler metrics on $\dfrac{\widetilde{\mathbb{R}^4}}{\Gamma_{ADE}}$. (Kronheimer)

To obtain an identity connected flat A-D-E connection on T^3, one simply exponentiates the 3 ϕ_I's

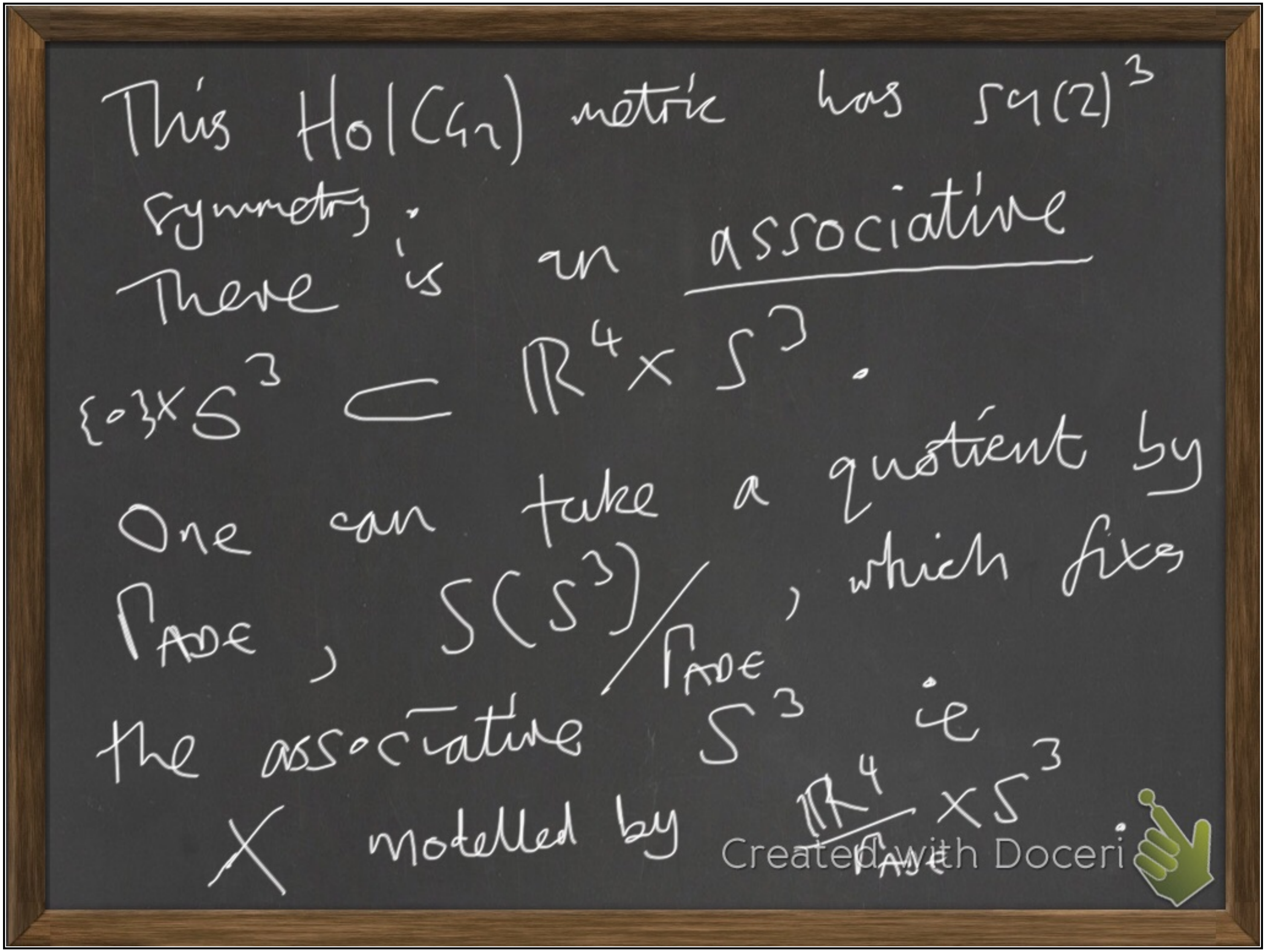
$$g_I = e^{i\Phi_I} \quad \left(e^{i\int_{\Sigma}\omega_I}\right)$$



Non-Abelian symmetry in M-theory on G_2-manifolds, X.

One can consider codim 4 singularities in X. These are supported on 3d subspaces $Q \subset X$.

Nice local model: Bryant-Salamon metric on $S(S^3) \cong \mathbb{R}^4 \times S^3$



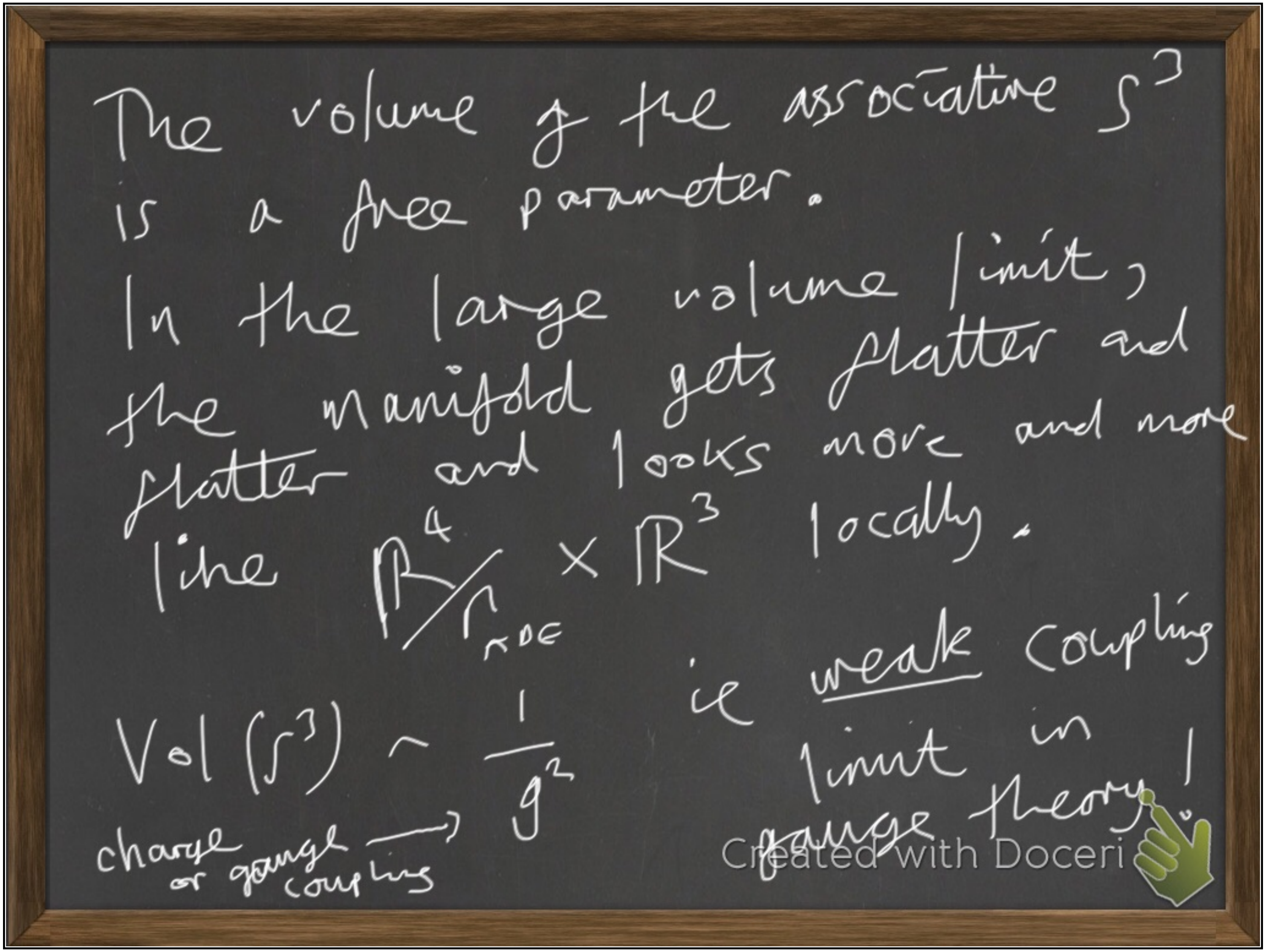
This Hol(G_2) metric has $SU(2)^3$ symmetry.

There is an associative

$$\{0\}\times S^3 \subset \mathbb{R}^4 \times S^3 .$$

One can take a quotient by Γ_{ADE}, $S(S^3)/\Gamma_{ADE}$, which fixes the associative S^3 ie

X modelled by $\frac{\mathbb{R}^4}{\Gamma_{ADE}} \times S^3$.



The volume of the associative S^3 is a free parameter.

In the large volume limit, the manifold gets flatter and flatter and looks more and more like $\mathbb{R}^4/\Gamma_{ADE} \times \mathbb{R}^3$ locally.

$$\mathrm{Vol}(S^3) \sim \frac{1}{g^2}$$

ie weak coupling limit in gauge theory!

charge or gauge coupling $\longrightarrow$

So we can describe the physics of the singularity by taking the 6+1d Super Yang-Mills on $\{0\} \times S^3$, and any suitably large associative Q supported at a codim 4 singularity.

Expanding in "harmonics",
keeping only zero modes and
integrating over Q, we find that:

low energy physics of a
codim 4 ADE singularity on Q
is given by
3+1 d Super Yang-Mills theory
plus $b_1(Q)$ "chiral" multiplets
$(\Omega^0(\mathrm{Lie}(ADE)), S^+(\mathrm{Lie}(ADE)))$

Actually, precise statement is chiral mutiplets are the moduli space of Complex flat $ADE_{\mathbb{C}}$ connections on $\bar{Q}$.

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Essentially, when we consider the 6+1 d Super Yang-Mills theory on $Q \times \mathbb{R}^{3,1} \subset X_7 \times \mathbb{R}^{3,1}$, the *three* scalars of the 6+1 d theory in flat spacetime, Φ_I, *become components* of a 1-form, Φ on Q. $\Phi \in \Omega^1(Q, \mathcal{L}ie(ADE))$

The components of the gauge field A on Q then combine with Φ to give a complex ADE connection

$$\mathcal{A}_\sigma = A + i\Phi$$

The "BPS" or supersymmetry conditions then assert that $F_{\mathcal{A}_\sigma}$ is flat

and $d_A{*}\Phi = 0$

i.e., stable, complex, flat connection

One expects to find situations in which there is a coassociative fibration over Q, where the fibers are diffeomorphic to the $\widetilde{\mathbb{R}^4/\Gamma_{ADE}}$.

(cf S. Donaldson)

If we assume the generic fiber is smooth, then we have a $U(1)^{rk(ADE)}$ gauge theory on Q

There are rk(ADE) complex gauge fields A^α_4 which are essentially the integrals of the complexified 3-form $C_3 + i\varphi$ over the 2-cycles in $\widetilde{\mathbb{R}^4}/\Gamma_{ADE}$

$$A^\alpha_4 \sim \int_{\Sigma_\alpha} (C_3 + i\varphi)$$

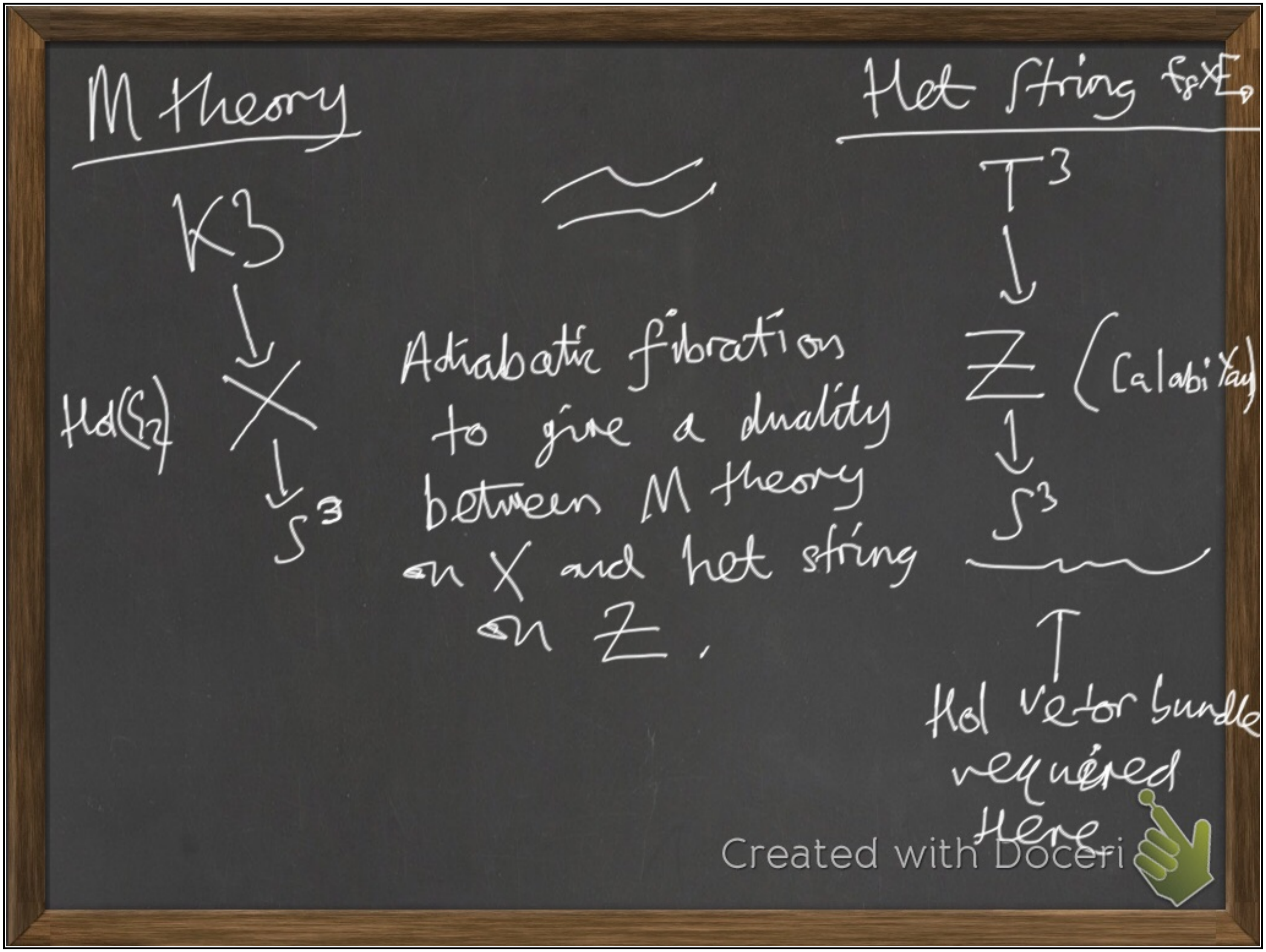
Note that for the Cartan directions in Lie(ADE), the one form Φ_α is both closed and co-closed and is presumably related to the 1-form discussed by Donaldson recently in "Adiabatic Co-associative Fibrations".

Chiral Fermions from Codim 7 Singularities

- We have seen that a key ingredient of the SM of particles, ie non-Abelian gauge fields, reside at codim 4 orbifold singularities.
- The other key ingredient are chiral fermions: fermions whose left and right handed components transform in different complex represantations of G_{SM}.
- These have to arise from higher codimension singularities.

- One expects this to be codim 7, because, lower codimension singularities will effectively give L and R spinors in the same rep$^{\underline{n}}$ of G_{SM}.

- We will motivate this study via duality with the heterotic string on T^3



M theory

Het String $E_8 \times E_8$

$$\begin{tikzcd}
K3 \arrow[d] & & T^3 \arrow[d] \\
X \arrow[d] & \approx & Z \arrow[d] \\
S^3 & & S^3
\end{tikzcd}$$

Hol(G_2)

(Calabi Yau)

Adiabatic fibration to give a duality between M theory on X and het string on Z.

Hol vector bundle required here

Chiral Fermions in Het String on Z

- Data for the background:

(Z,g) – Calabi-Yau; $(E \to Z)$ $E_8 \times E_8$ hol vector bundle

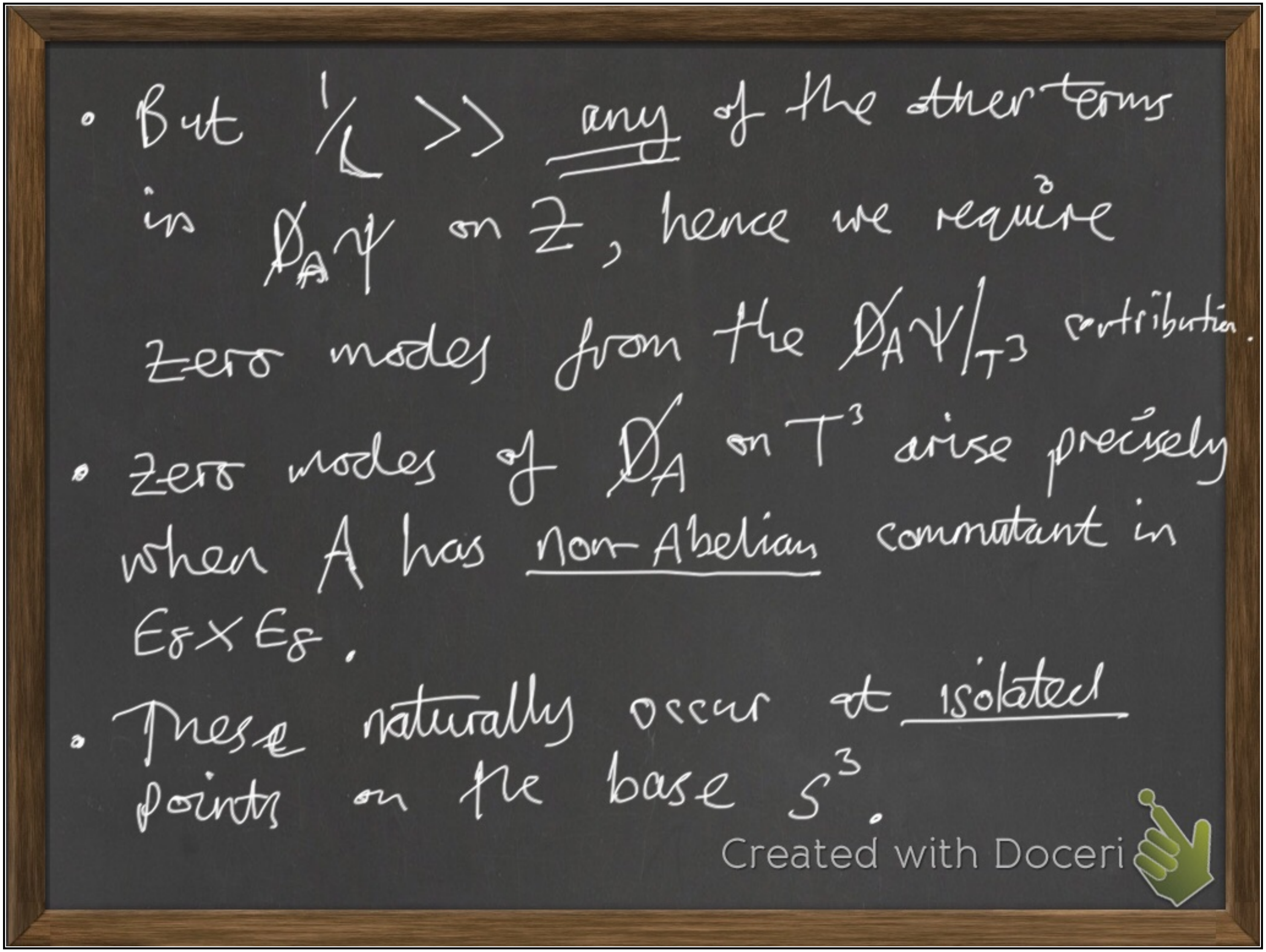
A – a Hermitian Yang-Mills connection on E with $c_2(E) = c_2(TZ)$.

- Chiral fermions arise from zero modes of the Dirac operator on Z, coupled to the connection A.

$$\not{D}_A \psi = 0.$$

- We suppose Z is T^3-fibered over a large S^3, with small fibers, of length L.
- The Dirac equation on Z will have a contribution from $\not{D}_A \Psi\big|_{T^3}$ ie the Dirac operator on the fibers, coupled to the connection restricted to the fibers.
- We do not have to worry about singular fibers in what follows!

- The restriction of the HYM eq^{ns} to the fibers, require the connection A to be flat on each fiber.
- This is expected from our discussion of M on K3 vs flat on T^3.
- If $\tilde{A}$ is a generic flat connection on T^3, $\not{D}_{\tilde{A}}\psi$ has no zero modes, hence will have eigenvalues of order $1/L$



- But $1/L \gg$ any of the other terms in $\not{D}_A\psi$ on Z, hence we require zero modes from the $\not{D}_A\psi|_{T^3}$ contribution.
- Zero modes of $\not{D}_A$ on T^3 arise precisely when A has non-Abelian commutant in $E_8 \times E_8$.
- These naturally occur at isolated points on the base S^3.

So chiral fermions are localised at pts on S^3 above which the T^3 has a connection with "enhanced symmetry".

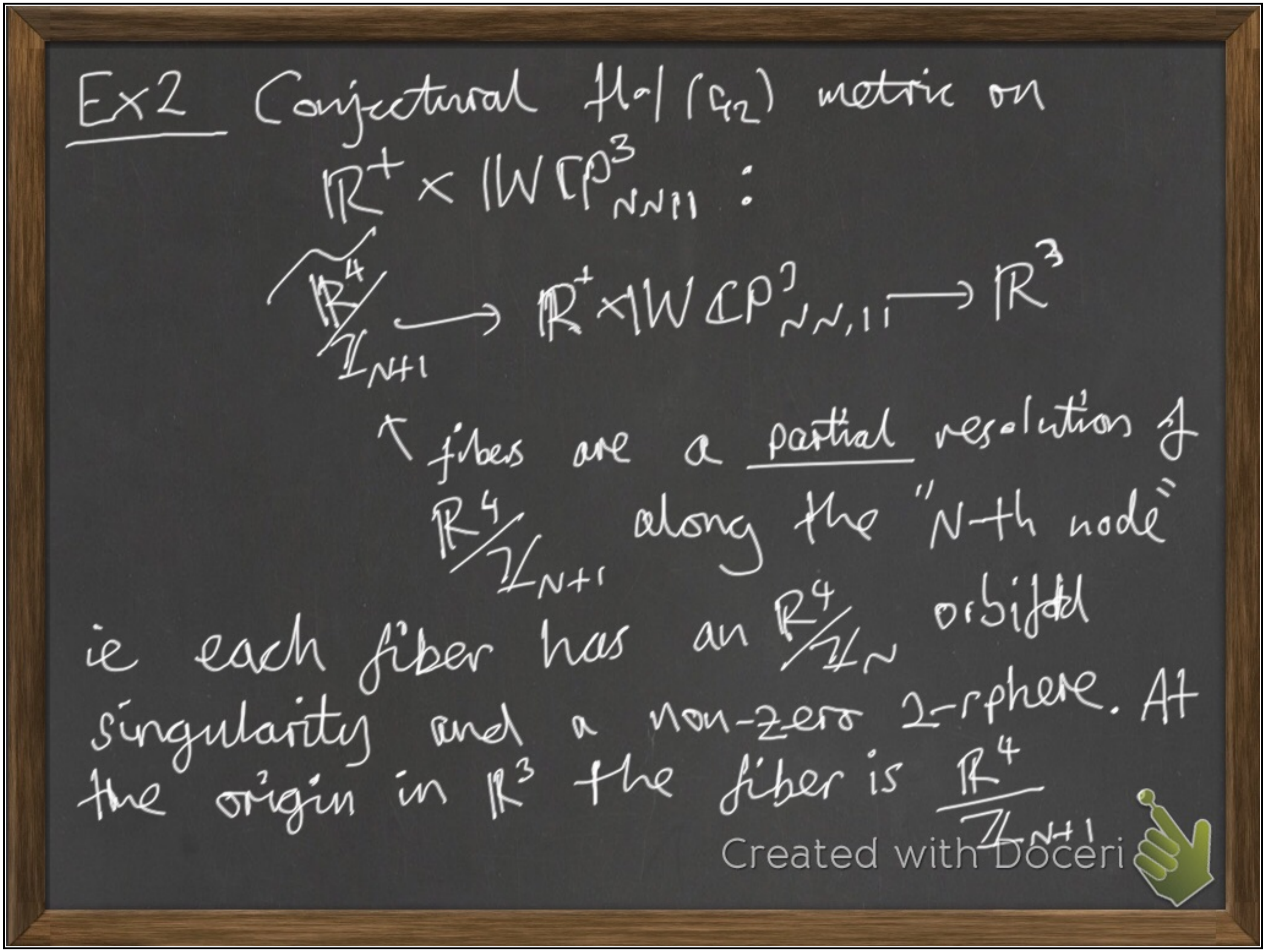
- In M theory, these are precisely the K3 fibers with ADE singularities.

- Lets describe the local models for these codim 7 singularities.

Ex 1 : Bryant-Salamon Cone on $\mathbb{R}^+ \times \mathbb{CP}^3$

$$\widetilde{\mathbb{R}^4/\mathbb{Z}_2} \hookrightarrow \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^3$$

$\mathbb{R}^+ \times \mathbb{CP}^3$ is a 3d family of "Eguchi Hanson" 4-mflds. At the origin of $\mathbb{R}^3$, the S^2 in $\widetilde{\mathbb{R}^4/\mathbb{Z}_2}$ collapses so the fiber over this point is singular, $\frac{\mathbb{R}^4}{\mathbb{Z}_2}$, and the total space has a conical singularity.



Ex2 Conjectural $\mathrm{Hol}(G_2)$ metric on

$$\mathbb{R}^+ \times \mathbb{WCP}^3_{N,N,1,1} :$$

$$\widetilde{\mathbb{R}^4/\mathbb{Z}_{N+1}} \hookrightarrow \mathbb{R}^+ \times \mathbb{WCP}^3_{N,N,1,1} \longrightarrow \mathbb{R}^3$$

↑ fibers are a partial resolution of $\mathbb{R}^4/\mathbb{Z}_{N+1}$ along the "N-th node"

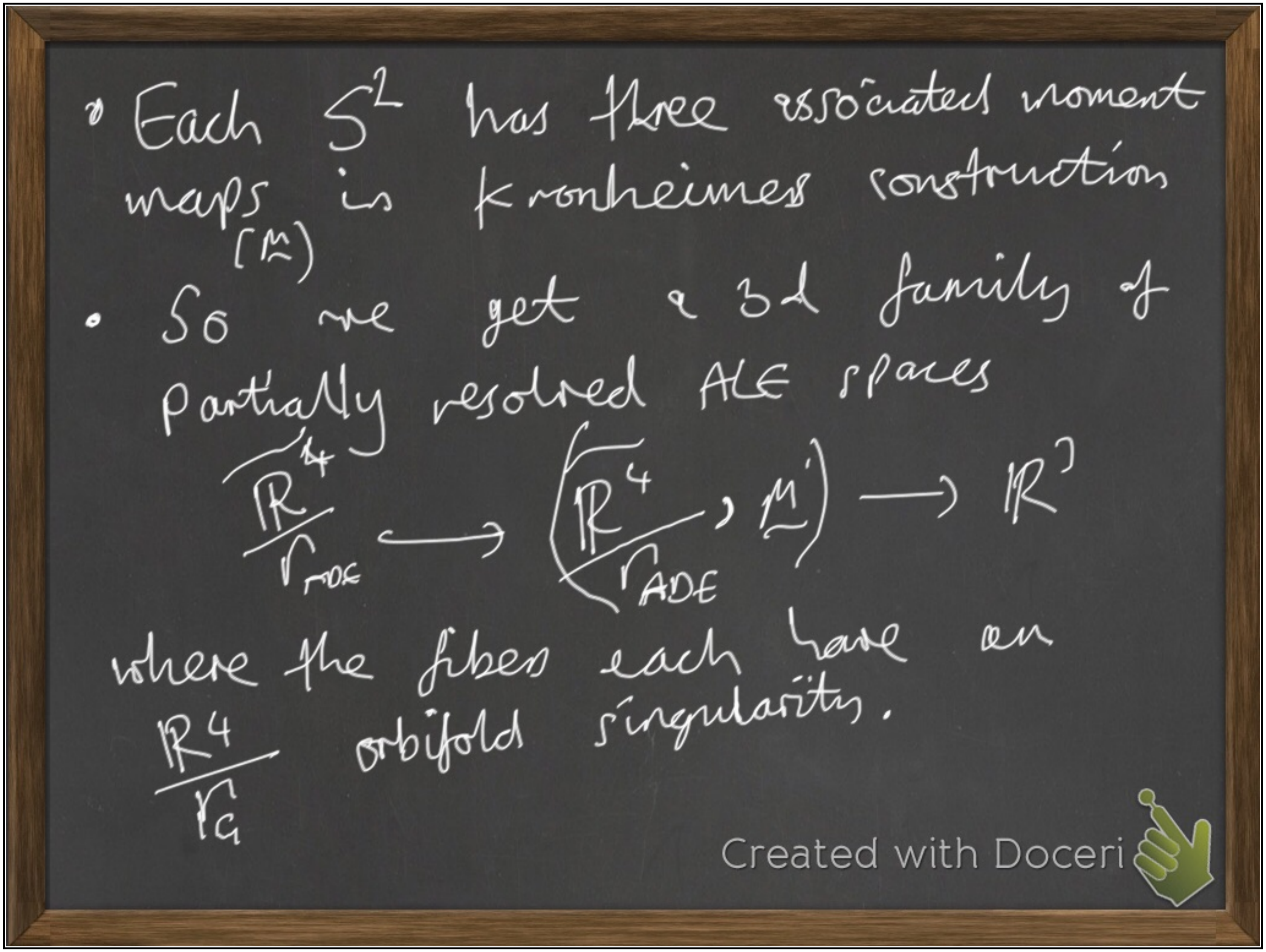
ie each fiber has an $\mathbb{R}^4/\mathbb{Z}_N$ orbifold singularity and a non-zero 2-sphere. At the origin in $\mathbb{R}^3$ the fiber is $\frac{\mathbb{R}^4}{\mathbb{Z}_{N+1}}$

In general, expect $\mathrm{Hol}(G_2)$ metrics on:

1) Start with any flat orbifold

$$\mathbb{R}^4/\Gamma_{ADE} \quad \text{with } \mathrm{rk}(ADE) = K+1$$

2) The nodes on the "boundary" of the ADE Dynkin diagram correspond to 2-spheres in partial resolutions of $\mathbb{R}^4/\Gamma_{ADE}$. Each of these nodes breaks ADE to a rk(K) subgroup



- Each S^2 has three associated moment maps $(\underline{\mu})$ in Kronheimers construction
- So we get a 3d family of partially resolved ALE spaces

$$\widetilde{\frac{\mathbb{R}^4}{\Gamma_{ADE}}} \longrightarrow \left(\frac{\mathbb{R}^4}{\Gamma_{ADE}}, \underline{\mu}\right) \longrightarrow \mathbb{R}^3$$

where the fibers each have an $\frac{\mathbb{R}^4}{\Gamma_G}$ orbifold singularity.

- At the origin, the fiber becomes

$$\frac{\mathbb{R}^4}{\Gamma_{ADE}} \text{ of } rk(k+1).$$

These 7d spaces are conjectured to have holonomy G_2 metrics,

(BSA, Witten)