Gauge Theories and Associatives

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UNIVERSITY OF OXFORD

European Research Council
Established by the European Commission

Simons Foundation Meeting, September 2018

Associatives in G_2-manifolds

G_2-holonomy manifolds are 7d admitting a Ricci-flat metric with holonomy G_2.

M/superstring theory on G_2 manifolds preserves supersymmetry as this gives a solution to the Killing spinor equation $\langle \delta\psi \rangle = \nabla \epsilon = 0$

Metric specified by a three-form, the G_2-form, Φ

$$\mathrm{d}\Phi = \mathrm{d} \star \Phi = 0\,.$$

Calibrated submanifolds are 3d associatives M_3

$$\Phi|_{M_3} = \mathrm{vol}(M_3)\,.$$

i.e. volume minimising in their homology class, or 4d co-associatives, which are calibrated by $\star\Phi$.

M-/D-branes wrapped on these cycles preserve supersymmetry, if suitably topologically twisted.

Plan

Main focus in this talk will be associative three-cycles.

I. 7d SYM on Associatives and Higgs bundles

II. String Dualities, Associatives and M2-instanton

III. 3d $\mathcal{N} = 1$ from M5-branes on Associatives and a 3d-3d Correspondence

- Witten Index
- S^3-partition function

Associatives in M-theory Compactifications

M-theory on G_2 Holonomy

M-theory on a compact G_2 manifold preserves 4d $\mathcal{N} = 1$ supersymmetry:
4d $\mathcal{N} = 1$ Super-Yang Mills (SYM) + matter coupled to supergravity.

Two reasons why this is interesting:

- Phenomenological applications: realizing compact G_2 manifolds, with suitable singularities in codim 4 and 7 that yield chiral 4d theories with non-abelian gauge symmetry.
  $\Rightarrow$ familiar from the 2000s.
  Mathematical setup: compact G_2 manifolds

- Field-theoretic motivations: structure of minimally supersymmetric gauge theories. With minimal, i.e. $\mathcal{N} = 1$ supersymmetry, the theory is much less under control, e.g. non-perturbative corrections.
  Mathematical setup: non-compact G_2 manifolds.

Associatives in M-theory: a Trifecta

M-theory on G_2:

I. 4d $\mathcal{N} = 1$ gauge sector of M/G_2 characterized by a Higgs-bundle on an associative cycle: ALE-fibration over an associative M_3
$\Rightarrow$ partial topological twist of 7d SYM on $M_3 \times \mathbb{R}^{1,3}$

[work in progress: Braun, Cizel, Hubner, SSN]

In M-theory there are two types of branes: M2 and M5. Their low energy description is not a SYM theory, however:

II. M2-branes on associatives:
Instantons (non-perturbative) corrections to 4d $\mathcal{N} = 1$ theory

[Braun, SSN 2017][Braun, del Zotto, Halverson, Larfors, Morrison, SSN, 2018]

III. M5-branes on associatives: 3d $\mathcal{N} = 1$ theories [Eckhard, SSN, Wong, 2018]

I. 7d SYM on Associatives

I. 7d SYM on M_3

ADE-singularity over M_3 gives a local description of the 4d $\mathcal{N}=1$ gauge sector of M/G_2 [see Acharya, Witten,..., Pantev, Wijnholt]

Field theoretic description:
1. start with M-theory on $\mathbb{C}^2/\Gamma_{ADE} \Rightarrow$ 7d SYM with $G=$ADE.
2. Dimensionally reduce 7d SYM on M_3 with a partial topological twist:

$$\begin{array}{ccc} SO(1,6)_L \times SU(2)_R & \rightarrow & SO(1,3)_L \times SO(3)_M \times SU(2)_R \\ (\mathbf{8},\mathbf{2}) & \rightarrow & (\mathbf{2},\mathbf{1};\underline{\mathbf{2},\mathbf{2}}) \oplus (\mathbf{1},\mathbf{2};\underline{\mathbf{2},\mathbf{2}})\,. \end{array}$$

To preserve 4d supersymmetry, twist $SO(3)_M$ with the R-symmetry $SU(2)_R$ of 7d SYM: under this twisted local Lorentz group $\underline{(\mathbf{2},\mathbf{2})} \rightarrow \mathbf{1} \oplus \mathbf{3}$. Thus giving rise to 4 supercharges in 4d: $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$.

The supersymmetric field configurations on M_3 are characterized by the BPS equations $\langle \delta\psi \rangle = 0$, constraining ϕ (twisted scalars) and $\mathcal{A}$ (gauge field components along M_3), both in the **3** of $SO(3)_{\text{twist}}$:

$$0 = F_{\mathcal{A}} + i[\phi, \phi] \,, \qquad 0 = D_{\mathcal{A}}\phi \,, \qquad 0 \quad = D^{\dagger}_{\mathcal{A}}\phi \,.$$

For $[\phi, \phi] = 0$ and ϕ regular, non-trivial solutions only exist for $\pi_1(M_3) \neq 0$.

Consider M_3 with boundaries, which maps this problem to an electro-statics problem, with ϕ the interpretation of the potential and $\phi = df$ with $\Delta f = 0$ on M_3 with $\partial M_3 \neq \emptyset$. This system can be studied using Morse theory for critical loci given by points ([Pantev, Wijnholt]) or Morse-Bott theory for more general critical loci, e.g. as in twisted connected sum constructions of G_2s manifolds [wip: Braun, Cizel, Hubner, SSN]

Open questions:

1. Compact models realizing local models with chiral matter
2. Generalized matter, e.g. conformal matter theories from local model.

II. M2-branes on Associatives

M2-branes and Dualities

Euclidean M2-branes on associative three-cycles in G_2 give rise to non-perturbative corrections the 4d effective theory of M$/G_2$ – hard problem, in particular, it's hard to identify associatives.

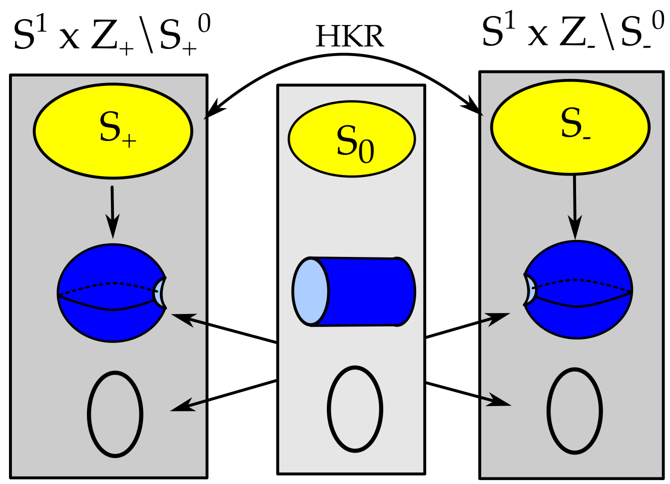
With Andreas Braun we identified a duality chain, mapping M-theory on TCS-G_2 to heterotic as well as F-theory duals. [Braun, SSN, 2017]

As part of the Simons Collaboration:
we used this duality to map known D3-instantons in F-theory to M2-instantons in M-theory on TCS G_2

[Braun, del Zotto, Halverson, Larfors, Morrison, SSN, 2018]

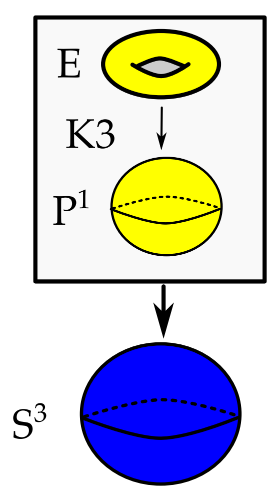
We thereby give evidence for an infinite number of M2-instanton corrections to the superpotential in twisted connected sum G_2 manifolds, and thereby conjecturaly a construction of infinitely many associatives in such TCS geometries.

TCS – Cartoon



Acyl CY3 building blocks that are K3-fibered $S_\pm$ over $\mathbb{P}^1$. Remove a fiber ($S_0^\pm$), take a product with S^1 and glue $S_\pm$ with a hyper-Kähler rotation

[Kovalev; Corti, Haskins, Nordström, Pacini]



Let $S_\pm$ to elliptically fibered K3 with sections, i.e. Weierstrass models over $\mathbb{P}^1$, and e.g.

S_+: smooth elliptic fibration

S_-: two II^* singular fibers

[Braun, SSN]

Elliptic building blocks enable application of duality between M-theory/heterotic/F-theory.

M-theory/Heterotic String Duality for TCS

Moduli space for both theories: $\Gamma \backslash SO(3,19)/(SO(3) \times SO(19)) \times \mathbb{R}^+$

M-theory on K3: moduli space of Einstein metrics on K3
Heterotic: Narain moduli space for T^3 compactification.
Specializing to elliptic K3s: 3 complex structures ω_i of the K3 are idenfied in the T^3 as follows:

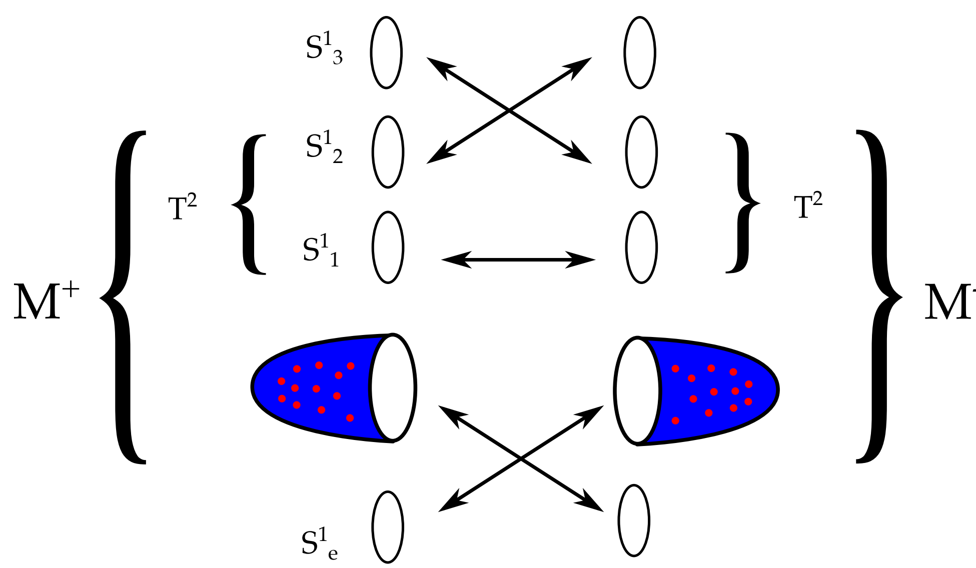
$$H^2(K3, \mathbb{Z}) = U_1 \oplus U_2 \oplus U_3 \oplus (-E_8)^{\oplus 2}$$

$$\begin{aligned} \text{Periods of } \omega_i \text{ along } U_i \quad &\leftrightarrow \quad \text{radii of the } S^1_i \\ \text{Periods of } \omega_i \text{ along } (-E_8)^2 \quad &\leftrightarrow \quad \text{Wilson lines along } S^1_i \end{aligned}$$

Fiber-wise duality for the TCS geometries with elliptic building blocks:
For an elliptic K3, additionally fibered over $\widehat{\mathbb{P}}^1$, only ω_1 and ω_2 vary.
By fiber-wise duality in heterotic only $T^2 \subset T^3$ varies over the base $\widehat{\mathbb{P}}^1$, and the total space of the heterotic compactification is an elliptic K3$\times S^1_3$.

M-theory/Heterotic String Duality for TCS

[Braun, SSN, 2017]



Apply same gluing, i.e. HK rotation to these building blocks:

$$S^1_{2+} = S^1_{3-}\,, \quad S^1_{1+} = S^1_{1-}\,, \quad S^1_{3+} = S^1_{2-}\,.$$

We find: $h^{1,1}(X_{\text{het}}) = 19 = h^{1,2}(X_{\text{het}})$ for any such TCS!

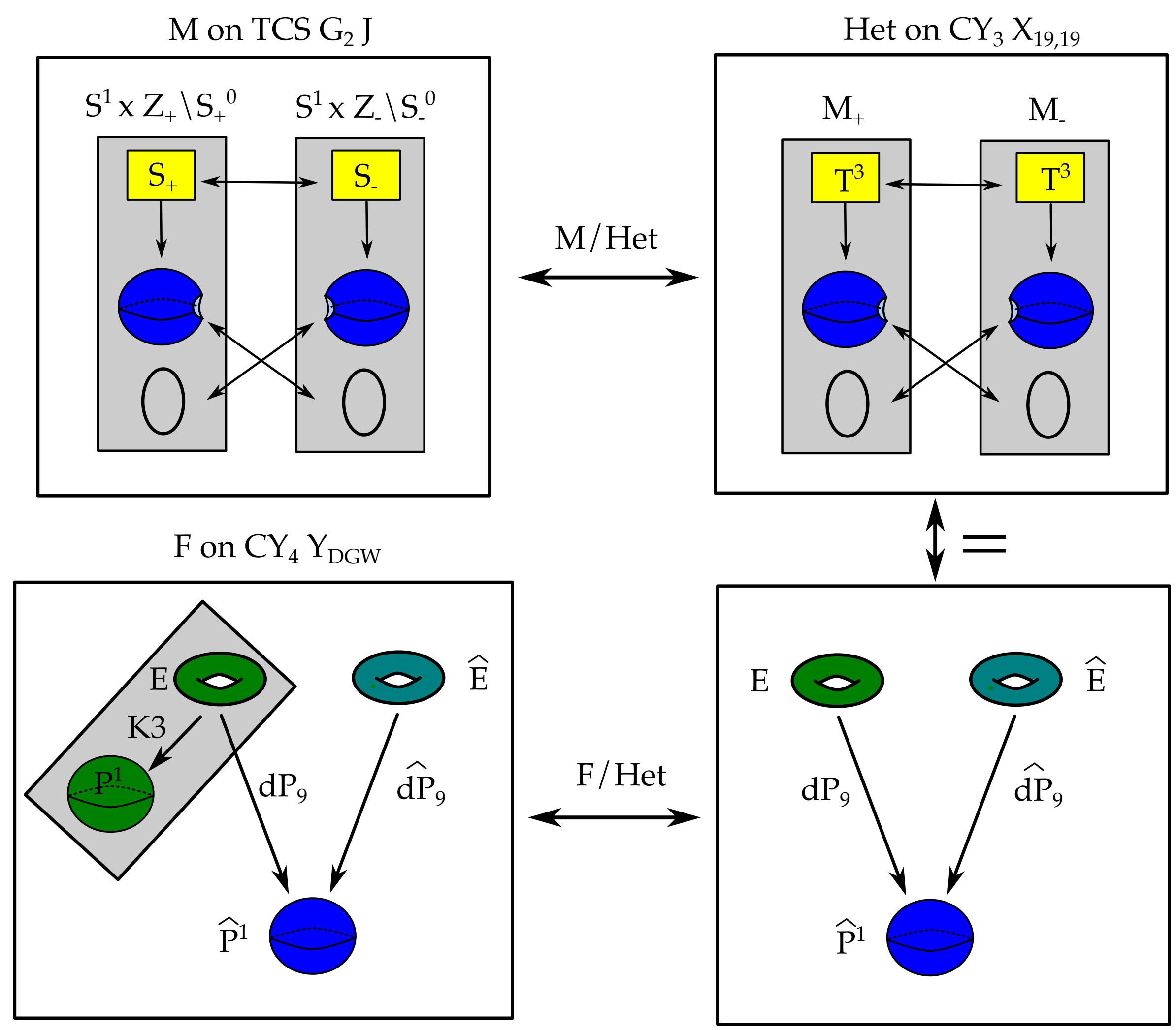
$\Rightarrow$ TCS-construction of SYZ-fibration of the Schoen CY3

$\Rightarrow$ All TCS with elliptic building blocks are dual to the Schoen CY3 with a choice of vector bundles.

Duality Chain for TCS G_2 Manifolds

[Braun, SSN, 2017]

Recap: M/K3 = Het/T^3 and Het/Elliptic CY3 = F-theory/K3-fiberedCY4.



Instantons in the Duality Chain for TCS G_2 Manifolds

[BdZHLMS, 2018]

F-theory on $\mathbb{E} \hookrightarrow Y_{\text{DGW}} \to (\mathbb{P}^1 \times \widehat{dP_9})$ has inftinitely many D3-instantons [Donagi, Grassi, Witten], wrapping surfaces D which satisfy $\chi(D, \mathcal{O}_D) = 1$: $D_\gamma = \sigma_\gamma \times \mathbb{P}^1$, where σ_γ are sections of $\widehat{dP_9}$: choose in $H^2(dP_9, \mathbb{Z}) = U \oplus (-E_8)$

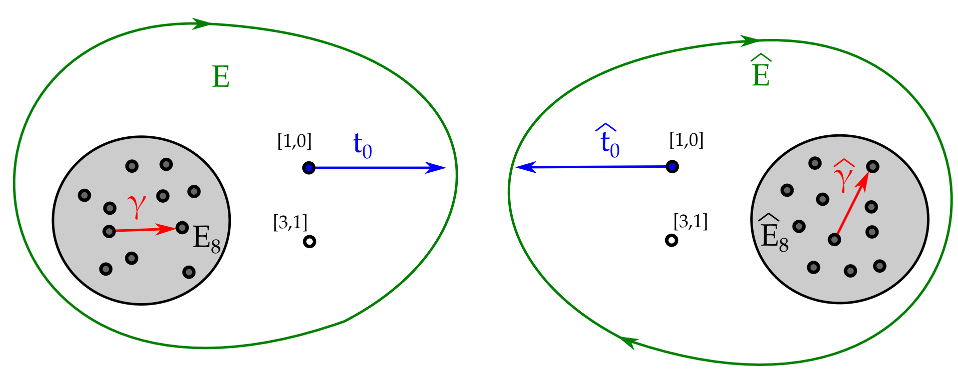
$$\sigma_\gamma = \sigma_0 + \gamma + n\hat{\mathbb{E}}$$

where $\sigma_0, \hat{\mathbb{E}} \in U$ are zero section and fiber class, $\gamma \in E_8$ with $\gamma^2 = -2n$. Then $\sigma_\gamma^2 = -1$ and $\sigma_\gamma \cdot \hat{\mathbb{E}} = 1$.

Heterotic string theory on the Schoen CY3 $X_{19,19}$: duality map allows to identify infinitely many world-sheet instantons. These can be identified in the SYZ-description using "string junctions":

We associative a "string junction", i.e. a worldsheet for the Euclidean heterotic string, $\mathfrak{t}_\gamma$ to each section σ_γ: Consider one building block in the TCS-description of the Schoen CY3: The T^2-fiber degenerates at 12 points: 10 realize the E_8 roots, whereas the remaining two correspond to asymptotic $[p, q]$ charges $[1, 0]$ and $[3, 1]$: the string junction is T^2 fibered over the paths in the base $\mathbb{P}^1$:

$$\mathfrak{t}_\gamma = \gamma + \mathfrak{t}_0 + nE$$



To construct the sections σ_γ we glue the thimbles from each building block together.

M-theory on the TCS J thereby has an $E_8 \oplus E_8$ worth of assocative three-cycles, which are homology three-spheres $\Sigma_{\gamma\hat{\gamma}}$.

Expanding $C_3 + i\Phi$ in terms of these $H^3(J,\mathbb{Z})$ cycles (coefficients given by ω_i) the superpotential correction by M2-instantons is then [BdZHLMS]

$$\begin{aligned}\Delta W^{\mathrm{M2}} &= \sum_{\Sigma_{\gamma\hat{\gamma}}} G(\gamma\hat{\gamma}) \exp\left[2\pi i \int_{\Sigma_{\gamma\hat{\gamma}}} C + i\Phi_3\right] \\ &= \sum_{m,\hat{m}\in\mathbb{Z}^8\times\mathbb{Z}^8} G(\gamma\hat{\gamma}) \exp 2\pi i \left[z + n\tau + \hat{n}\hat{\tau} + \sum_i m_i\varsigma_i + \hat{m}_i\hat{\varsigma}_i\right] ,\end{aligned}$$

For $G(\gamma\hat{\gamma}) = 1$ this just becomes a product of two E_8 θ-functions.

$\Rightarrow$ Using M/het/F duality applied to the TCS-construction with elliptic K3-building blocks as proposed in [Braun, SSN].

Conjecture:
For every element $(\gamma,\hat{\gamma}) \in E_8 \oplus E_8$ there is a pair of three-chains Σ_γ^+ in Z_+ and Σ_γ^- in Z_-, with boundary a (-2) curve in the transcendental lattice of the asymptotic K3 S_0, which can be glued together to a $\Sigma_{\gamma\hat{\gamma}} \in H^3(J)$ We conjecture that the class of this three-cycle contains a unique associative representative that has the topology of a three-sphere.

III. M5-branes on Associatives

M5-branes are 6d membranes in M-theory. The effective theory is not a SYM theory (unlike D-branes) and most likely is non-Lagrangian, but is known to be the unique 6d $\mathcal{N} = (2, 0)$ superconformal field theory with gauge group ADE. Whatever can be learned about M5-branes should be, as they form one of the key missing pieces in our understanding of M-theory.

Recently a whole class of correspondences have been determined from M5-branes wrapped on supersymmetric cycles. The basic idea is:

- M5-branes on M_d yields a supersymmetric theory in $6 - d$ dimensions: $T[M_d]$
- Observables such as partition functions on S^{6-d} or indices of $T[M_d]$ can be computed by considering a 'dual' theory obtained from M5-branes on S^{6-d}. This d dimensional theory is usually not supersymmetric, but a conformal or TQFT.
- Conjecture: TQFT partition function on M_d computes the supersymmetric partition function of $T[M_d]$.

M5-brane Correspondences: $\mathcal{N} = 2$ SUSY

The sphere-partition functions for the $T[M_d]$ theories are computed by the following d-dimensional theories:

- d=2: AGT correspondence between 4d $\mathcal{N} = 2$ theories and 2d Toda theories on M_2 [Alday, Gaiotto, Tachikawa]
  $\Rightarrow M_2$ is a curve in CY3
- d=3: 3d–3d correspondence between 3d $\mathcal{N} = 2$ theories and complex Chern-Simons on M_3 [Gaiotto, Gukov, Dimofte]
  $\Rightarrow M_3$ is a Slag in a CY3
- d=4: 4d–2d correspondence between 2d $\mathcal{N} = (0, 2)$ and topological sigma-model from M_4 into the Nahm moduli space [Assel, SSN, Wong]
  $\Rightarrow M_4$ is a Coassociative in G_2

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M5-brane Correspondences: $\mathcal{N}=1$ SUSY

The sphere-partition functions for the $T[M_d]$ theories are computed by the following d-dimensional theories:

- d=2: AGT correspondence between 4d $\mathcal{N}=2$ theories and 2d Toda theories on M_2 [Alday, Gaiotto, Tachikawa]
  $\Rightarrow M_2$ is a curve in CY3
- d=3: 3d–3d correspondence between 3d $\mathcal{N}=2$ theories and complex Chern-Simons on M_3 [Gaiotto, Gukov, Dimofte]
  $\Rightarrow M_3$ is a Slag in a CY3
- d=3: $\mathcal{N}=1$ 3d–3d correspondence between 3d $\mathcal{N}=1$ theories and Chern-Simons-Dirac on M_3 [Eckhard, SSN, Wong, 2018]
  $\Rightarrow M_3$ is an associative in G_2
- d=4: 4d–2d correspondence between 2d $\mathcal{N}=(0,2)$ and topological sigma-model from M_4 into the Nahm moduli space [Assel, SSN, Wong]
  $\Rightarrow M_4$ is a Coassociative in G_2

In the 4d $\mathcal{N}=1$ theory from M/G_2 M5-branes on asssocatives M_3 correspond to domain walls. For SQCD this was studied in [Acharya, Vafa].

Complementary motivation to study such theories: partial topological twist results in 3d $\mathcal{N}=1$ theories: $T_{\mathcal{N}=1}[M_3]$ ($G=SU(N)$, but more generally can be any ADE). [Eckhard, SSN, Wong]

Questions:

How does the geometry of M_3 enter the 3d theory?

T^3 and S^3 partition functions for $T[M_3]$ via TQFTs and compute observables of the 3d theory from a dual topological theory

Recent progress in understanding of partition functions and generalized dualities in 3d $\mathcal{N}=1$ theories [Gaiotto, Gomis, Komargodski, Seiberg, Witten, Benini, Benvenuti,...]. What is the counterpart in the TQFT dual?

III.1. 3d $\mathcal{N} = 1$ Gauge Theories from M5-branes on Associatives

M5-branes

Nahm's classification of Superconformal theories implies that there is a unique up to choice of ADE-gauge group 6d $\mathcal{N}=(2,0)$ superconformal theory with superconformal algebra $OSp(6|4) \supset SO(6)_L \times Sp(4)_R$. For $G = A_N$ this is the effective theory on a stack of M5-branes. Single M5-brane has $G = U(1)$.

Dimensional reduction on a three-cycle:

$$
\begin{array}{ccll}
SO(1,5)_L & \rightarrow & SO(1,2)_L \times \underline{SO(3)_M} & \\
Sp(4)_R & \rightarrow & \begin{cases} \underline{SU(2)_R} \times U(1)_R & \text{3d } \mathcal{N}=2;\ M_3 = \text{ sLag in CY}_3 \\ \underline{SU(2)_r} \times SU(2)_\ell & \text{3d } \mathcal{N}=1;\ M_3 = \text{Associative in } G_2\,. \end{cases}
\end{array}
$$

The main challenge is: we have absolutely no idea what the theory is for $G \neq U(1)$!

Associatives in G_2-manifold

Normal bundle of M_3 is the spin-bundle twisted with $SU(2)$-bundle

$$N_{M_3} = \mathbb{S} \otimes V$$

Linear deformations parametrised by twisted harmonic spinors satisfying

$$\not{D}_{\mathfrak{V}} \phi = 0$$

on M_3. Moduli space of solutions $\mathcal{H}_{\mathcal{D}}$ metric dependent!

VitualDim($\mathcal{H}_{\mathcal{D}}$) = 0 $\Rightarrow$ dim(Ker$\not{D}_{\mathfrak{V}}$) $=$ dim(Coker$\not{D}_{\mathfrak{V}}$).

So there can be obstructions. However, generically $d_{\not{D}} \equiv$ dim(Ker$\not{D}_{\mathfrak{V}}$) vanishes. [McLean]

Harmonic Spinors

When V is trivial i.e. $\mathfrak{V} = 0$ there are three distinct cases:

$$(\not{D})^2\psi = \nabla^*\nabla\psi + \frac{R}{4}\psi$$

- $R > 0$: $d_{\not{D}} = 0$ and the associative is rigid
- $R = 0$: $M_3 = T^3$ and harmonic spinors coincide with parallel spinors
- $R < 0$: Every closed spin manifold admits a metric with $d_{\not{D}} \geq 1$

Space of linear deformations depends on induced metric on M_3

Theory of a single M5-brane

Lorentz and R-symmetry:

$$SO(6)_L \times Sp(4)_R \subset OSp(6|4)$$

Tensor multiplet:

$$\begin{aligned}
B_{\underline{ab}} : &\qquad (\mathbf{15}, \mathbf{1}) \qquad \text{with selfduality } H = dB = *_6 H \\
\Phi^{\underline{\hat{m}\hat{n}}} : &\qquad (\mathbf{1}, \mathbf{5}) \\
\varrho^{\underline{\alpha\hat{m}}} : &\qquad (\bar{\mathbf{4}}, \mathbf{4})
\end{aligned}$$

EOMs:

$$H^- = dH = 0\,, \qquad \partial^2 \Phi^{\underline{\hat{m}\hat{n}}} = 0\,, \qquad \not{\partial}\rho = 0\,.$$

An M5-brane on an Associative

Recall: partial topological twist along M_3:

$$SU(2)_{\text{twist}} = \text{diag}(SU(2)_M, SU(2)_r)\,.$$

$$\begin{array}{rcccl} & SO(6)_L \times Sp(4)_R & \to & SO(3)_L \times SU(2)_{\text{twist}} \times SU(2)_\ell \\ \Phi^{\underline{\hat{m}\hat{n}}}: & (\mathbf{1},\mathbf{5}) & \to & (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1}) \equiv (\phi^{\alpha\hat{\alpha}}, \varphi) \\ H_{\underline{abc}}: & (\mathbf{10},\mathbf{1}) & \to & (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{3},\mathbf{1}) \equiv (h, H_{axy}) \\ \varrho^{\underline{\alpha\hat{m}}}: & (\bar{\mathbf{4}},\mathbf{4}) & \to & (\mathbf{2},\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{3},\mathbf{1}) \equiv (\rho^{\sigma\alpha\hat{\alpha}}, \lambda^\sigma, \xi^\sigma_a)\,. \end{array}$$

$SU(2)_\ell$ identified with the structure group of V, and ϕ a section of N_{M_3}.
The zero-mode spectrum depends on

$$H_1(M_3, \mathbb{Z}) \cong \mathbb{Z}^{b_1(M_3)} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$$

$$d_{\not{D}}(M_3, g) = \#\text{ of twisted harmonic spinors on } M_3 \text{ wrt metric } g$$

$$T[M_3, U(1)]$$

The theory $T[M_3, U(1)]$ enjoys $\mathcal{N} = 1$ supersymmetry and is a supersymmetric CS-theory coupled to scalar multiplets:

1. A single scalar multiplet $\mathcal{A}_\varphi \ni \{\varphi, \lambda^\sigma, h\}$. If we view $T_{\mathcal{N}=1}[M_3, U(1)]$ as a domain wall in the 4d $\mathcal{N} = 1$ bulk theory, obtained by compactifying M-theory on the G_2-holonomy manifold, this multiplet describes the center of mass.
2. $b_1(M_3)$ massless scalar multiplets $\mathcal{A}_\alpha^I \ni \{\alpha^I, \xi^{\sigma I}\}$ coming from the free part of the first homology group of M_3.
3. $d_{\not{D}}(M_3, g)$ massless scalar multiplets $\mathcal{A}_\phi^i \ni \{\phi^i, \rho^{\sigma i}\}$ which describe the deformations of the associative M_3 inside the G_2-holonomy manifold. These explicitly depend on the G_2-holonomy metric g restricted to the associative cycle M_3.
4. A set of r massive gauge multiplets $\mathcal{V}_A^m \ni \{A^m, \xi^{\sigma m}\}$ whose masses are generated by Chern-Simons terms at levels p_m. Each multiplet $\mathcal{V}_A^m$ is induced by a factor in the torsion part of $H^1(M_3, \mathbb{Z})$

Non-abelian Generalization

In general this is unknown. However we can use a key fact about the M5-brane theory:

6d (2,0) Theory on S^1 with gauge group G
= 5d Super-Yang Mills with gauge group G

In particular, if one wishes to compactify M5-brane on circle-fibration we can infer the non-abelian generalization by defining the 5d SYM theory in a suitable ”supergravity background”.

Examples:

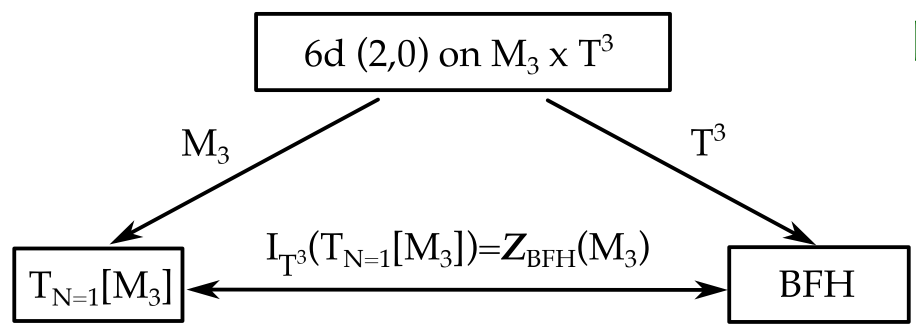
- $M_3 = L(p, 1)$.
- S^3- or $L(p, 1)$ partition function, via 5d SYM on S^2 + graviphoton background that models the Hopf fibration.

I will discuss this in detail in the next part of the talk.

III.2. A 3d–3d Correspondence: TQFT Dual to 3d $\mathcal{N} = 1$

Witten-Index: 3d-3d Correspondence

[Eckhard, SSN, Wong]



BFH = BF-model coupled to a spinorial hypermultiplet. The Witten index $\mathrm{Tr}(-1)^F$ is

$$I_{T^3}\left(T_{\mathcal{N}=1}[M_3]\right) = Z_{\text{BFH}}(M_3)\,.$$

BPS equations for $(\phi^{\alpha\hat{\alpha}}, A)$ fields of BFH on M_3 are generalized Seiberg Witten equations:

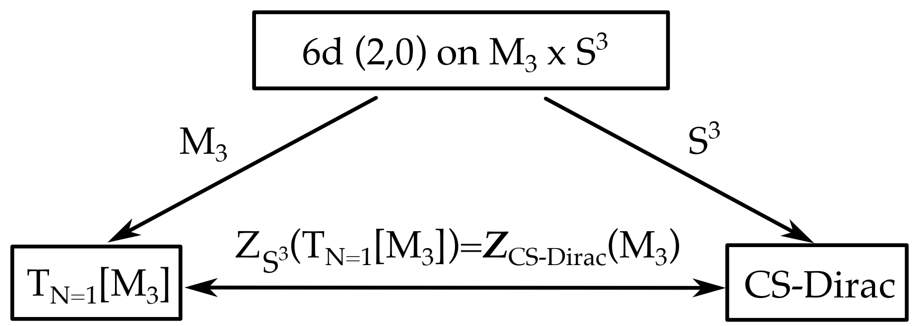
$$(\text{gSW}_{M_3}): \quad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0\,. \end{aligned}$$

and

$$Z_{\text{BFH}}(M_3) = \chi\left(\mathcal{M}_{\text{gSW}_{M_3}}\right)$$

S^3-partition Function: 3d-3d Correspondence

[Eckhard, SSN, Wong]



CS-Dirac= level 1 CS coupled to a twisted harmonic spinor M_3, eom = gSW equations. S^3-partition function is computed by:

$$Z_{S^3}\left(T_{\mathcal{N}=1}[M_3, G]\right) = \mathcal{Z}_{\mathrm{CS}_1-\mathrm{Dirac},G}(M_3)$$

No twisted harmonic spinors for a given metric g induced from the G_2:

$$d_{\not{D}}(M_3, g) = 0: \qquad Z_{S^3}\left(T_{\mathcal{N}=1}[M_3, G]\right) = \mathrm{WRT}(\mathrm{M}_3)$$

Generalization: $L(p,1)$ reduction instead of S^3:

$$Z_{L(p,1)}\left(T_{\mathcal{N}=1}[M_3, G]\right) = \mathcal{Z}_{\mathrm{CS}_p-\mathrm{Dirac},G}(M_3)$$

Summary of the $\mathcal{N} = 1$ 3d–3d Correspondence

Observable | TQFT Dual

$$\begin{tikzcd}
 & \text{Witten Index} \arrow[r, leftrightarrow, "N = 1\ \text{3d-3d}"] & Z_{\text{BFH}}[M_3] \\
T_{N=1}[M_3] \arrow[ur, "T^3"] \arrow[dr, "S^3"'] & & \\
 & S^3\ \text{Partition Function} \arrow[r, leftrightarrow, "N = 1\ \text{3d-3d}"] & Z_{\text{CS-Dirac}}[M_3]
\end{tikzcd}$$

BFH: supersymmetric BF model coupled to spinorial hypermultiplet

CS-Dirac: Chern-Simons-Dirac theory

Witten Index: Derivation

M5-branes compactified on $T^3 \Rightarrow$ 3d $\mathcal{N} = 8$ SYM

Two topological twists of 3d $\mathcal{N} = 8$ SYM, both preserving two topological supercharges

$$\begin{array}{c}\text{Twist acting on}\\ \text{scalars in hyper-}\\ \text{multiplet}\end{array} \xleftarrow{SU(2)_r} \begin{array}{c}\text{3d N = 8 SYM}\\ (SU(2)_l \times SU(2)_r) \times SU(2)_N\end{array} \xrightarrow{SU(2)_N} \begin{array}{c}\text{Twist acting on}\\ \text{scalars in vector}\\ \text{multiplet}\end{array}$$

$SU(2)_r$ twist: scalars $\phi^{\alpha\hat{\alpha}}$ in $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ twisted into 'bispinors' under twisted Lorentz group and $SU(2)_\ell$

$\Rightarrow$ sections of N_{M_3}, where $SU(2)_\ell$ identified with structure group of V

BFH-Model

[Eckhard, SSN, Wong]

BF-model coupled to spinorial Hypermultiplet preserving two topological supercharges

$$\mathcal{L}_{\mathrm{BFH}} = B^a(B_a - \varepsilon_{abc}F^{bc} + \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^\alpha{}_\beta\phi^{\beta\hat{\alpha}}]) + \frac{1}{2}W_{\alpha\hat{\alpha}}(W^{\alpha\hat{\alpha}} - 2i\not{D}^\alpha_\beta\phi^{\beta\hat{\alpha}}) + \cdots$$

where B_a, $W_{\alpha\hat{\alpha}}$ are auxiliary fields, whose eoms are

$$B_a = \frac{1}{2}\left(\varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, \phi^{\beta\hat{\alpha}}](\sigma_a)^\alpha{}_\beta\right)$$
$$W^{\alpha\hat{\alpha}} = i\not{D}^\alpha{}_\beta\phi^{\beta\hat{\alpha}},$$

The action can be written as

$$S_{\mathrm{BFH}} = \varepsilon_{\sigma\tau}Q^\sigma Q^\tau V_{\mathrm{BFH}}$$

and the energy-momentum tensor is Q-exact, however partition function depends on the metric, due to the dependence of the bispinors on g.

BFH Partition Function $Z_{\text{BFH}}(M_3)$

[Eckhard, SSN, Wong]

BPS equations given by generalised Seiberg-Witten equations

$$(\text{gSW}_{M_3}): \qquad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0 \end{aligned}$$

Partition function of $N_T = 2$ TQFTs computes $\chi(\mathcal{M}_{\text{BPS}})$ [Blau, Thompson][Dijkgraaf, Moore]. Applied to this theory, we expect:

$$Z_{\text{BFH}}(M_3) = \chi(\mathcal{M}_{\text{gSW}_{M_3}})$$

Checks: Abelian Theory

Abelian spectrum depends on first integral homology group

$$H_1(M_3, \mathbb{Z}) \cong \mathbb{Z}^{b_1(M_3)} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$$

Reduction of topologically twisted 6d EoMs yielded:

- Centre of mass scalar multiplet
- $b_1(M_3)$ scalar multiplets
- $d_{\not{D}}(M_3, g)$ scalar multiplets
- r vector multiplets with Chern-Simons interactions at level p_m

Checks: Abelian Theory

Witten index: $I = \mathrm{Tr}(-1)^F$

Multiplet	Contribution to $I(T_{\mathcal{N}=1}[M_3, U(1)])$
Chern-Simons at level k	k
Free scalar multiplet	0

Combining with spectrum of the abelian theory:

$$I(T_{\mathcal{N}=1}[M_3, U(1)]) = \begin{cases} \prod_{m=1}^{r} p_m & b_1 = d_{\not{D}} = 0 \\ 0 & \text{else} \end{cases}$$

Checks: Abelian Theory

$\mathcal{N} = 1$ 3d–3d correspondence implies

$$I(T_{\mathcal{N}=1}[M_3, U(1)]) = \chi(\mathcal{M}_{U(1)\text{-Flat}})\chi(\mathcal{H}_{\not{D}})$$

$U(1)$-flat connections: $\mathrm{Hom}(\pi_1(M_3), U(1))$

Topologically, $\mathcal{M}_{F=0} = T^{b_1} \times (\prod_{m=1}^{r} p_m)$pts so for generic embeddings of M_3

$$d_{\not{D}} = 0 : \quad Z_{\mathrm{BFH},U(1)}(M_3) = \begin{cases} \prod_{m=1}^{r} p_m & b_1 = 0 \\ 0 & \text{else} \end{cases}$$

Matches abelian Witten index when associative is obstructed

Jump in Witten Index

Conjecture

$$d_{\not{D}} \neq 0 : \quad I(T_{\mathcal{N}=1}[M_3, U(1)]) \Rightarrow \chi(\mathcal{H}_{\not{D}}) = 0$$

Consider deforming metric on M_3 such that $d_{\not{D}} \neq 0$

$T_{\mathcal{N}=1}[M_3, U(1)]$ now has $d_{\not{D}}$ additional scalar multiplets

$$\Rightarrow I(T_{\mathcal{N}=1}[M_3, U(1)]) = 0$$

$\Rightarrow$ Witten index for abelian theory is not a metric independent quantity, but jumps when M_3 admits twisted harmonic spinors.

Checks: Lens-Space Theories

Consider G_2-manifolds $X_7 = (S^3 \times \mathbb{R}^4)/\mathbb{Z}_p$, where action on S^3 is free. Associative is a Lens spaces $L(p,1)$, and is embedded with $\mathfrak{V} = 0$

$$T_{\mathcal{N}=1}[L(p,1), U(N)] = \begin{cases} \text{3d } \mathcal{N}=1 \text{ Chern-Simons-Yang-Mills at level} \\ p \text{ coupled to adjoint scalar multiplet} \end{cases}$$

Witten index computed by considering

$$U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N}$$

and discarding fermion zero mode from centre of mass $U(1)$ factor

[Acharya, Vafa]

$$I(T_{\mathcal{N}=1}[L(p,1), U(N)]) = p \times \frac{(p-1)!}{(N-1)!(p-N)!} \times \frac{1}{N} = \binom{p}{N}$$

Check via 3d-3d Correspondence

Metric on $L(p,1)$ does not admit harmonic spinors

$$d_{\not{D}} = 0: \quad Z_{\text{BFH},U(N)} = \chi(\mathcal{M}_{U(N)\text{-Flat}})$$

Flat connections correspond to $\mathrm{Hom}(\pi_1(M_3), U(N))$

Moduli space consists of N-dimensional representations of $\mathbb{Z}_p$

Abelian flat connections $\Leftrightarrow$ Irreducible representations of $\mathbb{Z}_p$

$$\Rightarrow \chi(\mathcal{M}_{U(N)}) = \binom{p}{N}$$

Extension

[work in progress: Julius Eckhard, Heeyeon Kim, SSN]

[Bashmakov , Gomis, Komargodski, Sharon, 2018] observed that the Witten index for 3d $\mathcal{N} = 1$ $SU(N)_p$+ adjoint multiplet of mass M, has in fact some more subtle behavior, including phase transitions, as a function of the mass M. Using our approach this mass deformation is also realizable:

$$T_{\mathcal{N}=1}[L(p,1), U(N), M] = U(N)_p + \text{adjoint scalar multiplet of mass } M$$

Consider: $p > N$. For $|M| >> 0$ we can integrate out both the gaugino (which has negative) mass and the massive adjoint fermion. Each massive fermion shifts the $SU(N)$ level by sign$(m)\frac{N}{2}$ depending on the sign of its mass term, while the $U(1)$ level is unchanged.

Thus, the theory admits a single vacuum TQFT for parametrically large mass M:

$$M \gg 0: \qquad U(N)_{p-N,p} \quad \Rightarrow I_+ = \begin{pmatrix} p \\ N \end{pmatrix}$$

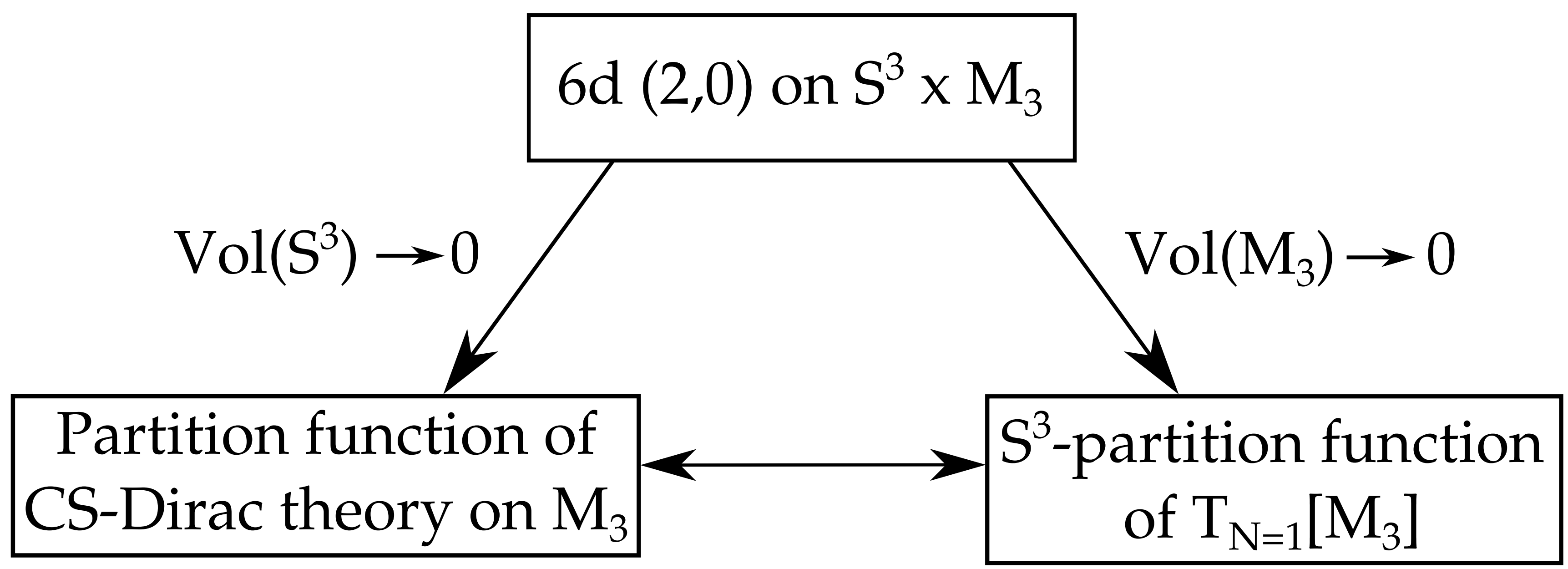
$$M \ll 0: \qquad U(N)_{p,p} \quad \Rightarrow I_- = \begin{pmatrix} p+N-1 \\ N \end{pmatrix} .$$

Note: $U(N)_{p,q} = \frac{SU(N)_p \times U(1)_{Nq}}{\mathbb{Z}_N}$ has $I = \begin{pmatrix} p+N-1 \\ N-1 \end{pmatrix} \cdot \frac{q}{N}$.

Note that the index for $M \gg 0$ agrees with the index of $\mathcal{N} = 2\ U(N)_p$. The reason for this is that at $M = \frac{pg^2}{4\pi}$ supersymmetry enhances to $\mathcal{N} = 2$. g-independence then implies that the index will only depend on the sign of M.

Work in progress: show this phase transition as a function of M from the dual M-deformed TQFT.

S^3 Partition Function



Derivation of Theory on M_3

In $\mathcal{N}=2$ 3d–3d correspondence, complex Chern-Simons was determined from explicit reduction from 6d (2,0) on S^3
Key Observations:

- M5-branes on $S^1 \Rightarrow$ 5d $\mathcal{N}=2$ Super-Yang-Mills
- Hopf fibration

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

- Non-abelianise going via 5d

Strategy

1) Couple to conformal supergravity

2) Reduce on S^1 fiber

3) Non-abelianise 5d action

4) Complete reduction on S^2

Couple 6d EoMs to off-shell conformal supergravity to preserve supersymmetry on S^3

Turn on background auxiliary fields compatible with topological twist on M_3 e.g. R-symmetry gauge field: ${V_A}^{\hat{m}}{}_{\hat{n}} \supset v\varepsilon_{ABC}(\Gamma^{[BC]^+})^{\hat{m}}{}_{\hat{n}}$
Solution to Killing spinor equations are a one parameter family in v.

5d SYM on $M_3 \times S^2$

Curvature of S^3 and non-trivial background fields induce mass terms for fields.

Masses are dependent on background parameter v. For spinor $\phi^{\alpha\hat{\alpha}}$, which parametrises deformations of M_3, the mass term takes the form

$$M_\phi^2 \sim \frac{1}{r^2}(v+2)(v-2)$$

Massless in 5d if $v = \pm 2$.

Reduction on S^2: expand in harmonics on S^2

$$\phi^{\alpha\hat{\alpha}} = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \phi^{\alpha\hat{\alpha}}_{(k,m)} Y_k^m(\theta, \phi)$$

$Y_k^m(\theta, \phi)$ - spherical harmonics on S^2.

Massless field content in 3d depends on choice of v. Massless $\phi^{\alpha\hat{\alpha}}_{(k,m)}$ corresponds to

$$v = \pm(4k+2)$$

$v=0$: Real Chern-Simons Theory

Spinor $\phi^{\alpha\hat{\alpha}}$ is massive. The 3d Lagrangian of massless fields is

$$\mathcal{L} = \frac{r}{8\pi}\left(F\wedge\star F + \mathcal{D}_a\varphi\mathcal{D}^a\varphi + \frac{i}{2}\mathcal{D}_a\lambda^+\mathcal{D}^a\lambda^- - \frac{i}{2}[\varphi,\lambda^+][\varphi,\lambda^-]\right) + \frac{i}{4\pi}\mathrm{CS}(A)$$

(φ,λ) - ghost fields which gauge fix Chern-Simons action

Chern-Simons theory is topological, whereas spectrum of $T_{\mathcal{N}=1}[M_3,U(1)]$ depends on metric!

$v = 2$: Chern-Simons-Dirac Theory

Captures metric dependence expected from S^3-partition function

$$\mathcal{L} = \frac{r}{8\pi}\left(F \wedge \star F - \frac{1}{2}\phi_{\alpha\hat{\alpha}}(\not{D}^2\phi)^{\alpha\hat{\alpha}}\right) + \frac{i}{4\pi}\left(\mathrm{CS}(A) + \frac{i}{2}\phi_{\alpha\hat{\alpha}}(\not{D}\phi)^{\alpha\hat{\alpha}}\right)$$

In the limit $r \to 0$ we obtain CS coupled to 'bispinor' $\phi^{\alpha\hat{\alpha}}$ i.e. Chern-Simons-Dirac theory

EoMs given by the gSW equations on M_3

$$(\mathrm{gSW}_{M_3}): \quad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0 \end{aligned}$$

Summary and Outlook

I. Topologically twisted 7d SYM on associatives for gauge sector of M/G_2

II. String duality mapped instantons: conjectured associatives in TCS G_2

III. Associatives wrapped by M5-branes: new 3d–3d correspondence between TQFTs and observables of the 3d $\mathcal{N}=1$ theories $T_{\mathcal{N}=1}[M_3]$

Future directions:

Ad I. Complete Higgs-bundle description of the gauge sector of M/G_2

Ad II. Mathematical proof of 'associativity' of three-cycles

Ad III. Moduli space of solutions to gSW equations
Dualities in 3d: interpretation in terms of the TQFT duals; Phase transitions in TQFTs

Ad III.′ Alternative way to construct 3d $\mathcal{N}=1$ theories: M-theory on Spin(7)-holonomy. New generalized connected sum construction with $G_2 \times S^1$ and CY4 building blocks [Braun, SSN, 2018]