Higgs Bundles for M-theory on $G_2$-Manifolds

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Motivation

M-theory on $G_2$-manifolds is in theory a perfect place to construct 4d $\mathcal{N} = 1$ SYM coupled to matter, with interactions, and coupling to (super-)gravity:

$$SO(7) \rightarrow G_2$$

$$8 \rightarrow 7 \oplus 1.$$  

- Interesting 4d gauge theories: non-compact $G_2$s with codimension 4 and 7 singularities [Acharya, Witten, Atiyah, Maldacena, Vafa...] ⇒ Main challenge: compact $G_2$s with codim 4 and 7 conical singularities

- Compact $G_2$-manifolds:
  - Joyce orbifolds $T^7/\Gamma$
  - $CY_3 \times S^1/\Omega$
  - Twisted Connected Sums: Kovalev; Corti, Haskins, Nordstrom, Pascini, ⇒ codim 4 & 6 singularities but not codim 7

A useful way to guide the search: Higgs bundles
Proposed first by [Pantev, Wijnholt, 2009]
Some Lessons from F-theory

The framework of choice in recent years for geometric engineering, e.g. $4d\ N = 1$, is F-theory (i.e. Type IIB with varying axio-dilaton $\tau$) on elliptic Calabi-Yau four-folds (CY4). Lessons we learned there:

- Start with ‘local’ models, i.e. Higgs bundles, encoding gauge sector of 7-branes on $M_4$ inside CY4:

$$7\text{-branes on } M_4 \times \mathbb{R}^{1,3} \equiv \{ (\phi, A) : \omega \wedge F_A + i[\phi, \bar{\phi}] = 0, \bar{\partial} \phi = 0, F^{(0,2)} = 0 \}$$

VEV for adjoint valued Higgs field $\langle \phi \rangle \neq 0$ breaks $\widetilde{G} \to G \times G_{\perp}$.

- Spectral cover description for $[\phi, \bar{\phi}] = 0$:

The local ALE-fibration over $M_4$ is encoded in the eigenvalues of $\phi \sim \text{diag}(\lambda_1, \cdots, \lambda_n)$.

- Most importantly: these spectral cover models opened up the systematic study of global F-theory compactifications. ⇒ Precise connection between elliptic fibrations (+ flux) and Higgs bundles
Higgs bundles/Hitchin systems ubiquitous in the description of the
gauge sectors in string theory.

$D_p$-branes on calibrated cycles $M_d$ in reduced holonomy manifolds $X$: partial topological twist of the $p + 1$ dimensional supersymmetric Yang-Mills theory on $M_d$ always yields an equation on $M_d$ of the type

$$F + [\phi, \phi] = 0, \quad D\phi = D^\dagger \phi = 0$$

The specific details of this depend on the characteristics of $X$ and $M_d$.

For the gauge sector of M-theory compactifications a similar argument holds, as we shall see, using the Super-Yang-Mills (SYM) arising from twisted dimensional reduction

M-theory on ALE-space $\mathbb{C}^2/\Gamma_{ADE} \Rightarrow$ 7d SYM with gauge group $G$

Further reduction from 7d to 4d $\Rightarrow$ Higgs bundle on $M_3$, which reconstructs ALE-fibration over $M_3$
Plan

1. Gauge sector of $G_2$-compactifications:
   Local Higgs bundles for $G_2$

2. Twisted Connected Sum (TCS) $G_2$

3. From TCS to chiral models.
4d $\mathcal{N} = 1$ Gauge Theories from $G_2$ Holonomy
Gauge Sector of M-theory on $G_2$ Manifolds

- M-theory on $\mathbb{C}^2/\Gamma_{ADE}$ gives 7d SYM with $G=ADE$: gauge connection $A$, adjoint scalars $\phi_i$, $i = 1, \cdots, 3$, and fermions $\lambda$

$$S = \frac{1}{g^2_7} \int d^7 x \left[ -\frac{1}{4} \text{Tr} F_{MN} F^{MN} - \frac{1}{2} \text{Tr} \left( D_M \phi_i D^M \phi^i \right) + \frac{1}{4} \text{Tr} \left( [\phi_i, \phi_j] [\phi^i, \phi^j] \right) \right]$$

$$+ \frac{1}{g^2_7} \int d^7 x \left[ + \frac{i}{2} \text{Tr} \left( \bar{\lambda}^{\alpha \dot{\alpha}} (\hat{\gamma}^M)_{\alpha \beta} D_M \lambda_{\beta \dot{\alpha}} \right) + i \frac{1}{2} \text{Tr} \left( \bar{\lambda}^{\alpha \dot{\alpha}} (\sigma^i)_{\alpha \dot{\alpha}} [\phi_i, \lambda_{\alpha \beta}] \right) \right],$$

- ADE-singularity fibered over a three-manifold:

$$\mathbb{C}^2/\Gamma_{ADE} \rightarrow M_3$$

This can be given a local $G_2$-structure.

- Adiabatic picture: 7d SYM on $M_3$

$$SO(1, 6)_L \times SU(2)_R \rightarrow SO(1, 3)_L \times \underbrace{SO(3)_M \times SU(2)_R}$$

To retain susy in 4d, **topologically twist** $SO(3)_M$ with $SU(2)$

R-symmetry: $SO(3)_{\text{twist}} = \text{diag}(SO(3)_M \times SU(2)_R)$
Higgs bundle on $M_3$

The supersymmetric field configurations on $M_3$ are characterized by the BPS equations
\[ \langle \delta \lambda \rangle = 0 \]

where
\[ \delta \lambda_{\alpha \hat{\alpha}} = -\frac{1}{4} F_{MN}(\hat{\gamma}^{MN})_{\alpha} \beta \epsilon_{\beta \hat{\alpha}} + \frac{i}{2} D_M \phi_i (\hat{\gamma}^M)_{\alpha} (\sigma^i)_{\hat{\beta}} \epsilon_{\beta \hat{\beta}} - \frac{1}{4} [\phi_i, \phi_j] \epsilon_{ij}^k (\sigma^k)_{\alpha} \epsilon_{\alpha \hat{\beta}} \]

After the twist: background fields are one-forms $3$ of $SO(3)_{\text{twist}}$:

- $\phi$ twisted scalars are adjoint valued one-forms, i.e. $\Omega^1(M_3) \otimes \text{Ad}(G_{\perp})$
- A gauge field for principal $G_{\perp}$ bundle, components along $M_3$

\[ 0 = F_A - i[\phi, \phi], \quad 0 = D_A \phi \]
\[ 0 = D_A^\dagger \phi. \]

[Pantev, Wijnholt][Braun, Cizel, Huebner, SSN]

$\langle \phi \rangle \neq 0$ breaks $\tilde{G} \to G \times G_{\perp}$, e.g. $SU(N+1) \to SU(N) \times U(1)$. 
Solutions

Higgs bundle \((\phi, A)\):
\[ 0 = F_A - i[\phi, \phi], \quad 0 = D_A \phi \]
\[ 0 = D^\dagger_A \phi. \]

Consider first \([\phi, \phi] = 0\) and so \(F_A = 0\). If \(\phi\) regular:
- \(\pi_1(M_3) = 0\) then \(\phi = 0\)
- \(\pi_1(M_3) \neq 0\): \(\phi\) can have non-trivial solutions

Relax regularity: allow \(\phi\) to have poles. Model by electrostatics
\[ \phi = df, \quad \Delta f = \rho \]
\[ \rho = \text{charge distribution on } M_3 \]
\[ f = \text{electrostatics potential} \]
\( \phi \) singular along support of \( \rho: \Gamma \). Excise a tubular neighborhood \( T(\Gamma) \) and consider instead manifold with boundary

\[
M_3 \rightarrow M_3 = M_3 \setminus T(\Gamma)
\]

In summary: we consider solutions to the Hitchin equations on \( M_3 \) that satisfy:

- \( \partial M_3 \neq \emptyset \)
- \( \phi \in \Gamma(\Omega^1(M_3, \text{Ad}\tilde{G}) \) with non-trivial entries along \( G_\perp \)
- \( \phi \) regular, \( \phi = df \) and \( \Delta f = 0 \) and suitable boundary conditions on \( \partial M_3 \)

\( \phi = (\phi_1, \phi_2, \phi_3) \) vanishes generically in codim 3 in \( M_3 \), where gauge group is unhiggsed from \( G \) to \( \tilde{G} \).
ALE-fibration

As per usual: Higgs bundles define ALE-fibrations over the base, here $M_3$. Local geometry

$$\phi = \phi_{i,\alpha} dx^i T_\alpha, \quad T_\alpha = \text{generators of Lie } (G_\perp)$$

then the vevs of $\phi_{i,\alpha}$ give the volume of the rational curves in the ALE fiber with HK structure $\omega_1, \omega_2, \omega_3$

$$\phi_{i,\alpha} = \int_{\mathbb{P}^i_\alpha} \omega_i.$$ 

E.g. for $f = c + \sum_i x_i^2$ then $z_1^2 + z_2^2 + z_3^2 = \sum_i x_i^2$ in $\mathbb{C}^3 \times \mathbb{R}^3$ gives a local ALE fibration where fiber collapses at $x = 0$.

$\Rightarrow$ Critical points of $f$ correspond to collapse of cycles in the fiber. Defines a local $G_2$: ALE-fibration over $M_3$. 

Spectrum

Consider $\phi U(1)$-Higgs field, Higgsing

$$\text{Ad}SU(N + 1) \to \text{Ad}SU(N) \oplus \text{Ad}U(1) \oplus \mathbb{R}_q \oplus \overline{\mathbb{R}}_{-q}.$$ 

Given background values "vevs" for $(\phi, \mathcal{A})$, i.e. a local $G_2$, what is 4d matter content? 7d SYM dimensionally reduced along $M_3$ yields:

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**Fermions:**

$$\chi_{\alpha} \in H_D^3(M_3) \quad \text{where} \quad D = d + [\varphi \wedge \cdot], \quad \varphi = \phi + i\mathcal{A}.$$ 

$$\psi_{\alpha} \in H_D^1(M_3)$$

---

Compute twisted cohomology for $D = d + [\varphi \wedge \cdot]$ and $D^\dagger = d - [\bar{\varphi} \wedge \cdot]$ with $\phi = df$, or harmonic forms for twisted Laplacian

$$\Delta_f = DD^\dagger + D^\dagger D = d^\dagger d + dd^\dagger + q^2 |df|^2 + q \sum_{i,j=1}^{3} (H_f)_{ij} [((a^i)^\dagger, a^j)].$$

where $H_f =$ Hessian of $f$, $(a^i)^\dagger = dx^i \wedge$ and $a^i = \iota_{\partial_i}$. 

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Zero-Modes

Boundary conditions: \( \mathcal{D} \) and \( \mathcal{D}^\dagger \) acting on forms are not adjoints unless we impose on the boundary

\[
\int_{\partial M_3} \bar{\alpha} \wedge \star \beta = 0
\]

\( \alpha_{t,n} \) be the tangent (i.e. pullback of \( \alpha \) to the boundary) and normal components \( \alpha = \alpha_t + \alpha_n \), of the forms and \( \partial M_3 = \Sigma_+ \cup \Sigma_- \):

- Dirichlet b.c. on \( \Sigma_- \): \( \alpha_t |_{\Sigma_-} = 0 \)
- Neumann b.c. on \( \Sigma_+ \): \( \star \alpha_n |_{\Sigma_+} = 0 \)

Then the twisted cohomologies are computed by the relative cohomology wrt \( \Sigma_- \)

\[
H^*_\mathcal{D}(M_3) = H^*(M_3, \Sigma_-)
\]
**Example**

\[ M_3 = S^3 \setminus T(\Gamma), \text{ where } \Gamma = \text{points, links.} \]

\[ n_\pm = \# \text{components with charge } \pm \]

\[ \ell_\pm = \# \text{loops with charge } \pm \]

\[ r = \# \text{- charged loops that are independent in homology in } S^3 \setminus \Gamma_+ \]

Then the zero-mode spectrum is

\[ b^1(M_3, \Sigma_-) = \ell_+ + n_- - r - 1, \quad b^2(M_3, \Sigma_-) = \ell_- + n_+ - r - 1, \]

and the chiral index is simply

\[ \chi = (n_+ - \ell_+) - (n_- - \ell_-) \]
Next: Interactions

However to describe the interactions we first need to take an alternative, but equivalent description, of the zero-mode spectrum, using Super-Quantum Mechanics (SQM) and Morse/Morse-Bott theory (cf. Witten)

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For $U(1)$ Higgs field and $f = c + \frac{1}{2} \sum_{i=1}^{3} c_i (x^i)^2 + \cdots$ with isolated critical points

$\Rightarrow f$ Morse.

Let $\mu(p)$ be the Morse index of the critical point $p$, i.e. $\#c_i < 0$. Then

$$
\Delta_f = d^{\dagger}d + dd^{\dagger} + q^2 |df|^2 + q\{d, \iota_{\text{grad}f}\} + q\{d^{\dagger}, df \wedge\} \\
= \sum_{i=1}^{3} -\frac{\partial^2}{\partial (x^i)^2} + q^2 c_i^2 (x^i)^2 + qc_i [dx^i, \iota_{\partial/\partial x^i}] + \cdots
$$

So that zero modes are to this order ("perturbative zero-modes") are essentially harmonic oscillator wave-functions:

$$
\mu(p) = 1 : \psi_{(p,q)} = \psi \exp \left( -q \sum_{i=1}^{3} |c_i|(x^i)^2 \right) dx^1 \\
\mu(p) = 2 : \bar{\psi}_{(p,q)} = \bar{\psi} \exp \left( -q \sum_{i=1}^{3} |c_i|(x^i)^2 \right) dx^1 \wedge dx^2
$$

$\psi$ take care of the spinor nature of the fields.
Instanton Corrections

In the 7d SYM: \(\text{Tr}(\psi \wedge D\psi)\) coupling, which descends to a mass term \((p_a, p_b\text{ critical points of } f)\)

\[
M^{ab} = \langle \psi(p_a, q) | D\psi(p_b, q) \rangle \\
= \frac{1}{q f(p_a) - q f(p_b)} \int_{\gamma(-\infty)=p_b}^{\gamma(+\infty)=p_a} D\gamma D\eta D\bar{\eta} [D, f] e^{-S_{\text{SQM}}},
\]

where the action for the SQM is the sigma model into \(M_3\), with the fields being paths \(\gamma : p_a \rightarrow p_b\)

\[
S_{\text{SQM}} = \int_{\mathbb{R}} ds \left( \frac{1}{2} g_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} + \frac{q^2 t^2}{2} g^{ij} \partial_i f \partial_j f \\
+ g_{ij} \bar{\eta}^i D_s \eta^j + q D_i \partial_j f \bar{\eta}^i \eta^j + \frac{1}{2} R_{ijkl} \eta^i \bar{\eta}^j \eta^k \bar{\eta}^l \right),
\]

This localizes on gradient flow trajectories for \(f\)

\[
\frac{d\gamma^i}{ds} = q g^{ij} \partial_j f
\]
Zero-mode counting gets correct by

$$M^{ab} = \sum_{\text{gradientflow } \gamma : p_a \rightarrow p_b} n_\gamma e^{-q(f(p_a) - f(p_b))}$$

where $n_\gamma = \pm 1$ depending on orientation on the moduli space of gradient flows.

$S_3^\gamma$ are associatives iff $\gamma$ is a gradient flow line
$\Rightarrow$ non-trivial M2-instanton contributions from associatives in $G_2$ (cf. [Harvey, Moore]), depending on # of $\gamma$ from $p_a$ to $p_b$, and $n_\gamma$.

Upshot: This reproduces $H^*(M_3, \Sigma_-)$. 
Spectral Cover

Consider $[\phi, \phi] = 0$, diagonalizable $\phi$ in $U(1)^n$

$$C : \quad 0 = \det(\phi - s) = \sum_{i=0}^{n} b_{n-i}s^i = b_0 \prod_{i=1}^{n} (s - \lambda_i)$$

$\phi = df = 0$ becomes $\lambda_i = 0$ loci, i.e. when one of the covers intersects the zero-section $M_3$.

If $p$ is connected by a flow line to another critical point, there is a corresponding associative three-cycle which is built by fibering the collapsing $S^2$ (blue) over the flow line.
Couplings

From the 7d SYM the following coupling decends:

\[ Y^{abc}_{pqr} = \int_{M_3} \psi^{(a,p_1)} \wedge \varphi^{(b,p_2)} \wedge \psi^{(c,p_3)}, \quad Q_1 + Q_2 + Q_3 = 0 \]

\( p_i \) are the points where matter is localized; \( a, b, c \) labels the modes.

This localizes along gradient flows \( \gamma(f) \)

\[
\frac{d\gamma(f)^i}{ds} = qg^{ij} \partial_j f
\]

which emanuate from each critical point. The \( S^2 \)'s in ALE-fiber fibered over the gradient flow tree gives rise to a supersymmetric three-cycle

\[ \Rightarrow \text{M2-instanton contribution.} \]
Building of Models

\[ \tilde{G} \to G \times U(1)^n, \; t^i \text{ generate } U(1)\text{s}, \text{ and consider a charge configuration} \]

\[ i = 1, \ldots, n : \quad \phi = t^i df_i, \quad \rho = t^i \rho_i, \quad \Delta f_i = \rho_i, \quad \int_{M_3} \rho_i = 0. \]

Then for \( Q = (q_1, \ldots, q_n) \)

\[ \rho_Q = \sum_{i=1}^{n} q_i \rho_i, \quad f_Q = \sum_{i=1}^{n} q_i f_i \]

At every point in \( M_3 \) where \( df_Q = 0 \), there is a localized chiral multiplet transforming in \( R_Q \).
Example: Top Yukawa

\[ E_6 \rightarrow SU(5) \times U(1)_a \times U(1)_b , \]

Let the matter be localized along the critical loci of the following Morse functions, i.e. \( f \):

\[ 5_{-3,3} : \quad f_5 = -3f_a + 3f_b , \]
\[ 10_{-1,-3} : \quad f_{10}^{(1)} = -f_a - 3f_b , \]
\[ 10_{4,0} : \quad f_{10}^{(2)} = 4f_a . \]
2. Local Models for TCS $G_2$-Manifolds
Twisted Connected Sums

Building blocks: Calabi-Yau three-folds = K3s $S_{\pm}$ over $\mathbb{P}^1$. Remove a fiber ($S_{0}^{\pm}$), take a product with $S^1$ and glue $S_{\pm}$ with a hyper-Kähler rotation (HKR)

$$\omega_{\pm} \leftrightarrow \text{Re} \Omega^{(2,0)}_{\mp}, \quad \text{Im} \Omega^{(2,0)} \leftrightarrow -\text{Im} \Omega^{(2,0)}$$

[Kovalev; Corti, Haskins, Nordström, Pacini]

Let $S_{\pm}$ be elliptically fibered K3 with sections, i.e. Weierstrass models over $\mathbb{P}^1$, and e.g.

$S_{+}$: smooth elliptic fibration

$S_{-}$: two $II^*$ singular fibers

Singular K3-fibers result in non-abelian gauge groups, e.g. $E_n$

[Braun, SSN]
Field Theoretic Interpretation of TCS

- M-theory on Calabi-Yau $Z_\pm \times S^1$ preserves $\mathcal{N} = 2$ in 4d.
- Central region: $K3 \times T^2 \times \text{interval}$ preserves $\mathcal{N} = 4$ in 4d.
- HyperKähler rotation and gluing retains only a common $\mathcal{N} = 1$ susy.
- Key: building blocks have algebraic models.
- TCS are globally $K3 \to S^3$. Apply M on K3/het on $T^3$ duality; and even het/F-theory duality to e.g. understand instantons [Braun, SSN; Braun, del Zotto, Halverson, Larfors, SSN; Acharya, Braun, Svanes, Valandro]
TCS Higgs-Bundle

Local Higgs bundle model for Calabi-Yau threefolds in each building block is a spectral cover model over $\mathbb{P}^1$ (with charge loci excised). Charges: circles (red/blue), and critical loci are circles (yellow).

Due to product structure of each building block the critical loci of $f$, and so matter loci, are always 1d! Requires generalization to Morse-Bott theory. Upshot: Matter Spectrum is always non-chiral.
Morse-Bott generalization for TCS

Example: \( f(x, y, z) = z^2 \): two critical points and one critical line.

Gradient "curves", connect the critical loci (black lines)

SQM analysis generalizes to gradient trajectories between \( N_\mu \)-critical submanifolds of Morse index \( \mu \)

\[
\mathcal{M}(N_m, N_n) = \left\{ \gamma : \mathbb{R} \to M \left| \lim_{t \to \pm \infty} \gamma(t) \in N_{n,m}, \quad \frac{d\gamma^i}{ds} = t q g^{ij} \partial_j f \right\} \right/ \mathbb{R}.
\]

Applied to \( M_3 \) we have \( N_1, N_2 \) only. The Morse-Bott complex is built from

\[
C^1 = \Omega^0(N_1), \quad C^2 = \Omega^1(N_1) \oplus \Omega^0(N_2).
\]

Applied to critical loci in the TCS

\[
C^1 = \Omega^0(S^1)^k, \quad C^2 = \Omega^1(S^1)^k
\]

\[
H^1(M_3, \Sigma_-) \cong \mathbb{R}^k, \quad H^2(M_3, \Sigma_-) \cong \mathbb{R}^k.
\]
Singular Transitions in TCS $G_2$-manifolds

Can TCS be deformed to yield chiral 4d theories?

Deformation of concentric circular charge configurations to e.g. ellipses: gives 4 critical points with equal chiral and conjugate-chiral matter:
Singular Transitions in TCS $G_2$-manifolds

To change chirality, recall:

$n_\pm = \# \text{components with charge } \pm$

$\ell_\pm = \# \text{loops with charge } \pm$

$r = \# \text{- charged loops that are independent in homology in } S^3 \setminus \Gamma_+$

Then the zero-mode spectrum is

$$b^1(M_3, \Sigma_-) = \ell_+ + n_- - r - 1, \quad b^2(M_3, \Sigma_-) = \ell_- + n_+ - r - 1,$$

and the chiral index is simply

$$\chi = (n_+ - \ell_+) - (n_- - \ell_-)$$

Singular transitions in the local model that will generate chirality:
Recent resurgence of insights in 3d $\mathcal{N} = 1$ theories and dualities.
Geometric engineering of these in M-theory: $Spin(7)$ 8-manifold.

[Alternatively: M5-branes on associative three-cycles in $G_2$ [Eckhard, SSN, Wong]]

Compact $Spin(7)$ manifolds are equally sparse:

- [Joyce (2000)] orbifold $T^8/\Gamma$
- Calabi-Yau four-fold orientifold [Kovalev (2018?)]
- Inspired by TCS for $G_2$ we developed a Generalized Connected Sum construction. [Braun, SSN (2018)]
Generalized Connected Sum $Spin(7)$-manifolds

Generalized Connected Sum (GCS):  

Field theoretic construction: $Z_\pm$ preserves 3d $\mathcal{N} = 2$. Central region preserves 3d $\mathcal{N} = 4$, but gluing retains only common 3d $\mathcal{N} = 1$. Examples of new compact $Spin(7)$ manifolds [Braun, SSN].

Higgs bundle for Spin(7): [Heckman, Lawrie, Lin, Zoccarato]
Summary and Outlook

• $G_2$ manifolds provide a purely geometric way of engineering gauge theories in 4d with minimal susy.

• Local Higgs bundle model gives insights into the structure of the gauge sector

• Future: using insights into deformations of TCS form local model, try to construct compact $G_2$ with codim 7 singularities