More G_2–instantons on twisted connected sums

Thomas Walpuski

Humboldt-Universität zu Berlin

2020-09-16

working paper available at
https://walpu.ski/Research/ArithmeticG2InstantonsTCS.pdf

Twisted connected sums, I

The **twisted connected sum construction** produces a family of G_2–manifolds Y_L ($L \gg 1$) given a matching pair of framed building blocks.

A **building block** (Z, π) consists of a smooth projective 3–fold Z and a projective morphism $\pi\colon Z \to \mathbf{P}^1$ such that:

- $-K_Z \in H^2(Z)$ is primitive, and
- $\Sigma := \pi^*(\infty)$ is a smooth $K3$ surface and $\Sigma \sim -K_Z$.

A **framing** of (Z, π) is hyperkähler structure $\boldsymbol{\omega} = (\omega_I, \omega_J, \omega_K)$ on Σ such that $\omega_J + i\omega_K$ is of type $(2, 0)$ as well as a Kähler class on Z whose restriction to Σ is $[\omega_I]$.

A **matching** of pair of framed building blocks $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm)$ is a diffeomorphism $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$ such that

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}.$$

Twisted connected sums, II

Let W be a 3–fold with $-K_W$ very ample (strong Fano 3–fold). By Bertini, if $|\Sigma_0 : \Sigma_\infty|$ is a general anti-canonical pencil, then its base-locus C is a smooth curve, and $Z = Bl_C W$ together with the map $\pi : Z \longrightarrow \mathbf{P}^1$ defined by the pencil is a building block with $\pi^*(\infty) = \Sigma_\infty$. Moreover, the singular fibers of π have only ODPs.

Matching pairs of framed building blocks (of this type) can be constructed as follows.

Let $N_\pm$ be a pair of lattices. Let $\mathscr{W}_\pm$ be a deformation type of pairs (W, h) consisting of a strong Fano 3–fold W and an isometry $h : N_\pm \cong \mathrm{Pic}(W)$.

Let $\mathrm{Amp}_\pm \subset N_\pm \otimes \mathbf{R}$ be an open cone such that, for every $(W, h) \in \mathscr{W}_\pm$, $h(\mathrm{Amp}_\pm)$ lies in the ample cone of W.

Twisted connected sums, III

Let $N_0 \subset N_\pm$ be a primitive sublattice. Suppose there is an orthogonal pushout N of $N_\pm$ along N_0. ($N_\pm \subset N$, $N_+ + N_- = N$, $N_+ \cap N_- = N_0$, $N_\pm^\perp \subset N_\mp$. This can be checked computationally.)

Suppose N is primitively embedded into the K3 lattice $L = 2E_8 + 3U$ (✓ if $\operatorname{rk} N \leq 11$).

Theorem (Kovalev, Corti–Haskins–Nordström–Pacini)

Given $H_\pm \in \mathrm{Amp}_\pm \cap N_0^\perp$ and $\varepsilon > 0$, there are a pair $(W_\pm, h_\pm) \in \mathscr{W}_\pm$, a smooth $\Sigma_\pm \in |-K_{W_\pm}|$, a hyperkähler structure $\boldsymbol{\omega}_\pm$ on $\Sigma_\pm$, and a hyperkähler rotation $\mathfrak{r}\colon (\Sigma_+, \boldsymbol{\omega}_+) \longrightarrow (\Sigma_-, \boldsymbol{\omega}_-)$ such that $\mathfrak{r}_* \circ h_+(N_0) = h_-(N_0)$. Moreover, $d(\langle H_\pm \rangle, \langle [\omega_{I,\pm}] \rangle) \leq \varepsilon$.

After choosing further general anti-canonical divisors, this yields matching pairs of framed building blocks.

G_2–instantons on twisted connected sums, I

Theorem (Sá Earp–W.)

Let $(Z_\pm, \pi_\pm, \omega_\pm; \mathfrak{r})$ be a matching pair of framed building blocks. Let $\mathscr{E}_\pm \to Z_\pm$ be holomorphic vector bundles of rank r such that:

- *$\mathscr{E}_\pm|_{\Sigma_\pm}$ is μ–stable with respect to $[\omega_{I,\pm}]$. Denote the corresponding ASD instanton by $A_{\infty,\pm}$.*
- *There is an isomorphism of smooth vector bundles $\bar{\mathfrak{r}}\colon E_+|_{\Sigma_+} \to E_-|_{\Sigma_-}$ covering $\mathfrak{r}$ with $\bar{\mathfrak{r}}^* A_{\infty,-} = A_{\infty,+}$.*
- *$\mathscr{E}_\pm$ has no infinitesimal deformations fixing the restriction to $\Sigma_\pm$:*

$$H^1(Z_\pm, \mathscr{E}nd(\mathscr{E}_\pm)(-\Sigma_\pm)) = 0.$$

- *A transverse intersection condition holds.*

Then there is an unobstructed irreducible $\mathbf{P}U(r)$ G_2–instanton on Y_L $(L \gg 1)$ constructed out of $\mathscr{E}_\pm$.

G_2–instantons on twisted connected sums, II

Combining the theorem with results due to Donaldson and Mukai yields:

Corollary ("rigid case")

Let $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ be a matching pair of framed building blocks. Let $\mathscr{E}_\pm \longrightarrow Z_\pm$ be holomorphic vector bundles of rank r such that:

- $\mathscr{E}_\pm|_{\Sigma_\pm}$ *is μ–stable with respect to $[\omega_{I,\pm}]$.*
- $\mathfrak{r}^* c_1(\mathscr{E}_-|_{\Sigma_-}) = c_1(\mathscr{E}_+|_{\Sigma_+})$ *and* $\mathfrak{r}^* c_2(\mathscr{E}_-|_{\Sigma_-}) = c_2(\mathscr{E}_+|_{\Sigma_+})$.
- $\mathscr{E}_\pm$ *and* $\mathscr{E}_\pm|_{\Sigma_\pm}$ *both are infinitesimally rigid:*

$$H^1(Z_\pm, \mathscr{E}nd(\mathscr{E}_\pm)) = 0 \quad \text{and} \quad H^1(\Sigma_\pm, \mathscr{E}nd(\mathscr{E}_\pm|_{\Sigma_\pm})) = 0.$$

Then there is an unobstructed irreducible $\mathbf{P}U(r)$ *G_2–instanton on Y_L ($L \gg 1$) constructed out of $\mathscr{E}_\pm$.*

There are only two examples of G_2–instantons constructed using the gluing theorem in the literature (W., Menet–Nordström–Sá Earp).

Finding examples, I

Let $v = (r, B, s) \in \{2, 3, \dots\} \times N_0 \times \mathbf{Z}$ and let $H_\pm$ be $\pi_\pm$–ample. Consider the relative moduli problem of sheaves over $Z_\pm \longrightarrow \mathbf{P}^1$ with Mukai vector v ($\mathrm{rk} = r$, $c_1 = B$, $\chi = r + s$) and subject to some condition (here: $H_\pm$–Gieseker stability). Suppose that a corresponding fine moduli space $\rho_\pm : M_\pm \longrightarrow \mathbf{P}^1$ exists (✓ r, s coprime).

In this situation, the conditions of the gluing theorem translate to conditions on sections $s_\pm$ of $\rho_\pm : M_\pm \longrightarrow \mathbf{P}^1$:

- $s_\pm(\infty)$ are μ–stable ($\rightsquigarrow$ ASD instanton $A_{\infty,\pm}$).
- $\mathfrak{r}^* A_{\infty,-}$ is gauge-equivalent to $A_{\infty,+}$.
- $s_\pm(b)$ is a vector bundle for every $b \in \mathbf{P}^1$.
- $s_\pm$ admits no infinitesimal deformations fixing $s_\pm(\infty)$.
- A transverse intersection condition at ∞ holds.

The strategy is to find cases where $M_\pm$ can be understood very well.

Finding examples, II

The relative dimension of $M_\pm$ is $2 + v^2 = 2 + B^2 - 2rs$.

Theorem (Kuleshov, Mukai, Thomas, Bridgeland–Maciocia, …)

Suppose that the fibers of $Z_\pm$ have at worst RDP singularities.

- *If $v^2 = -2$ and r, s coprime, then $M_\pm = \mathbf{P}^1$ and it parametrizes vector bundles.*
- *If $v^2 = 0$ and r, s coprime, then $M_\pm$ is a building block, and smooth (singular) points of the fibers parametrize vector bundles (reflexive sheaves).*

Proposition

Let $\mathscr{E}$ be an H–μ–semistable sheaf on Σ with Mukai vector $v = (r, B, s)$. If r and the divisibility of B are coprime and, for every $x \in H^{1,1}(\Sigma, \mathbf{Z})$ perpendicular to H,

$$x^2 \leq -\frac{1}{4} r^2 (2r^2 + v^2),$$

then $\mathscr{E}$ is H–μ–stable.

Finding examples, III

Corollary

Let $N_0, N_\pm, N, \mathscr{W}_\pm, \mathrm{Amp}_\pm, ..$ be as before. Let

$$H_\pm \in \mathrm{Amp}_\pm \cap N_0^\perp \quad \text{and} \quad v = (r, B, s) \in \{2, 3, \ldots\} \times N_0 \times \mathbf{Z}$$

such that the following hold:

- $B^2 = 2(rs - 1)$ *(that is: $v^2 = -2$),*
- *r and the divisibility of B are coprime, r and s are coprime; and*
- *for every non-zero $x \in N_\pm$ perpendicular to $H_\pm$,*

$$(\star) \qquad x^2 < -\frac{1}{2} r^2 (r^2 - 1).$$

Then there is a matched pair of framed building blocks constructed as before and an irreducible and unobstructed $\mathbf{P}U(r)$ G_2–instanton on Y_L ($L \gg 1$).

Finding examples, IV

The input required by the corollary can be found by brute-force computer search given Pic and a reasonably large choice of Amp for all strong Fano 3–folds $(b_2 \leq 3)$:

1. Choose suitable $v = (r, B, s)$ with $v^2 = -2$, r and the divisibility of B coprime, and r and s coprime.
2. Check whether orthogonal pushout of $N_\pm$ along $N_0 = \langle B \rangle$ exists.
3. Try to find $H_\pm \in \mathrm{Amp}_\pm \cap N_0^\perp$ such that condition $(\star)$ holds.

After about 3 days of computation on my laptop, I found 298 new examples. Here are some statistics:

r	2	3	4	5	6	7	8	13	17	19
#	23	43	6	16	1	181	3	2	22	1

More statistics and the Python/Sage code are in `walpu.ski/Research/ArithmeticG2InstantonsTCS.pdf`.

Further directions: $v^2 = 0, 1$

Suppose that v and $H_\pm$ such that $M_\pm \cong Z_\pm$ and such that the fiber over ∞ parametrizes μ–stable bundles.

Since $Z_\pm = Bl_{C_\pm} W_\pm$, it contains $\mathbf{P}^1 \times C_\pm$ and every point $p \in C_\pm$ defines a section of $\pi_\pm : Z_\pm \longrightarrow \mathbf{P}^1$. Every transverse intersection point $\mathfrak{r}(C_+) \cap C_-$ gives rise to a pair of sections satisfying the matching and transverse intersection condition; hence, it induces a G_2–instanton over Y_L.

The same pair of sections also yield an associative in Y_L (cf. work in progress by NORDSTRÖM–RODRÍGUEZ DÍAZ–SÁ EARP).

NIKULIN gives arithmetic conditions on v for the smooth **fibers** of $M_\pm$ to be isomorphic to smooth **fibers** of $Z_\pm$. In some cases, TYURIN explains these isomorphism geometrically via the Serre correspondence and certain constructions on holomorphic vector bundles.

Its remains to work out in which cases NIKULIN's isomorphism can be made global, and to find examples satisfying the required arithmetic conditions.

Further directions: $v^2 = 0$, II

Suppose, more generally, that v and $H_\pm$ such that the fiber over ∞ parametrizes μ–stable bundles.

Jun Li and Le Potier construct two natural line bundles $\mathscr{L}_0$ and $\mathscr{L}_1$ on (the smooth fibers of) $M_\pm$. For $k \gg 1$, $\mathscr{L}_0 \otimes \mathscr{L}_1^k$ is relatively ample. If the fiber over $b \in \mathbf{P}^1$ is smooth and parametrizes μ–stable bundles, then the restriction of $\mathscr{L}_1$ to that fiber is ample and the induced metric agrees with the Weil–Petersson metric g_{WP} on the ASD moduli space (up to scale).

A matching of $\mathfrak{r}$ of $\Sigma_\pm$ induces a matching of the corresponding moduli spaces of ASD instantons with the Weil–Petersson hyperkähler structure. **If $\mathscr{L}_1$ is relatively ample**, then we can form the dual twisted connected sum $\widehat{Y}_L$ with building blocks $M_\pm$.

There is a correspondence between (certain) associatives in $\widehat{Y}_L$ and (certain) G_2–instantons on Y_L (and vice versa).

Further directions: $v^2 = 0$, III

It is not hard to find cases where $\mathscr{L}_1$ is ample on the generic fiber, but it might still fail to be relatively ample.

One can always contract $M_\pm = M_\pm^{Gi}$ to $M_\pm^{Uh}$ so that $\mathscr{L}_1$ becomes relatively ample, but $M_\pm^{Uh}$ might have singularities and not be a building block anymore.

The work of HEIN–SUN, GAO CHEN, DONALDSON, YANG LI, ... **suggests** that there might be a singular G_2–space $\widehat{Y}_L$ dual to Y_L.