QALF hyperkähler spaces

+ rmks about correspondence
(3d Hitchin system / codim-4 ADE sing's G_2-mflds)

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D_k ALF spaces from the Kummer construction

Gibbons–Hawking Ansatz (1978)

$$\begin{cases} \text{flat } (B^3, g_B) \text{ w/ parallel coframe} \\ \triangle_{g_B} h = 0 \text{ w/ } [*dh] \in H^2(B;\mathbb{Z}) \end{cases} \rightsquigarrow \begin{array}{c} \text{1-p family of } S^1\text{–inv. HK}^4 \text{ metrics} \\ g_\epsilon = (1+\epsilon h)\, g_B + \epsilon^2 (1+\epsilon h)^{-1}\theta^2 \end{array}$$

- **cyclic ALF** A_k space: $B = \mathbb{R}^3$, $h = \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|}$

Kummer construction

Gibbons–Pope (1979), Page (1982), Hitchin (1984), Biquard–Minerbe (2011)

- distinct $\pm p_1, \ldots, \pm p_{k-2}$
- **dihedral** $\mathbb{Z}_2 \circlearrowright A_{2k-5}$: 2 fixed points modelled on $\mathbb{C}^2/\mathbb{Z}_2$
- desingularise $A_{2k-5}/\mathbb{Z}_2$ by gluing 2 copies of EH $\rightsquigarrow$
- **dihedral ALF** D_k space $k \geq 2$

Working in fixed complex structure (and 2 EH have same periods up to scale):
holomorphic symplectic manifold $D_k = \mathrm{Hilb}^{\mathbb{Z}_2}(A_{k-3}) = \mathrm{Hilb}^{[2]}(A_{k-3})//_0\mathbb{C}^*$

Review: Hilbert schemes, twistor spaces

Transverse and invariant Hilbert schemes

Beauville (1983), Atiyah–Hitchin (1988), Ito–Nakamura (1996)

holo symplectic $X^2 \rightsquigarrow$ holo sympl $\mathrm{Hilb}^{[k]}(X) \xrightarrow{\phi} \mathrm{Sym}^k(X)$

- holo fibration $\mu\colon X^2 \to C^1 \rightsquigarrow \mathrm{Hilb}^{[k]}_{\mu}(X) \subset \mathrm{Hilb}^{[k]}(X)$
  - $\mathrm{Hilb}^{[2]}_{\mu}(X)$ regular set of $\mathrm{Hilb}^{[2]}(X) \xrightarrow{s^2(\mu)\circ\phi} \mathrm{Sym}^2(C) = \mathrm{Hilb}^{[2]}(C)$
- symplectic involution $\rightsquigarrow \mathrm{Hilb}^{\mathbb{Z}_2}(X) \subset \left(\mathrm{Hilb}^{[2]}(X)\right)^{\mathbb{Z}_2}$

$(M^{4n}, \underline{\omega})$ hyperkähler $\rightsquigarrow$ **twistor space** $Z = M \times S^2$

- complex structure $p\colon Z \to \mathbb{CP}^1$
- antiholomorphic involution $\tau\colon Z \to Z$ lifting antipodal map
- fibrewise symplectic $\omega^c \in H^0(Z, \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2))$

- real holomorphic section s of Z is a **twistor line** if $s^* T_F \simeq \mathcal{O}(1)^{\oplus 2n}$
- $\rightsquigarrow$ components of space of twistor lines carry *pseudo* HK structures

(Hitchin–Karlhede–Lindström–Roček, 1987)

D_k ALF and M-theory/IIA duality

k = 0

Atiyah–Hitchin (1988), Cherkis–Hitchin (2005)

- distinct $\pm p_1, \ldots, \pm p_k \in \mathbb{R}^3$
- A_{2k-1}: holo moment map $\mu \colon X \to \mathbb{C}$ + symplectic involution $\rightsquigarrow \mathrm{Hilb}^{\mathbb{Z}_2}_{\mu}(A_{2k-1})$
- fibrewise on twistor space $Z(A_{2k-1}) \rightsquigarrow$ twistor space $Z(D_k)$

Dihedral ALF spaces from GH + AH Sen (1997), F. (2016), Schroers–Singer (2020)

- **Atiyah–Hitchin** metric: asymptotic to $\mathrm{GH}/\mathbb{Z}_2$ with $h = -4\frac{1}{2|x|}$
- incomplete GH for $\epsilon \ll 1$: $B = \mathbb{R}^3$, $h = -4\frac{1}{2|x|} + \sum_{i=1}^{k} \frac{1}{2|x-p_i|} + \frac{1}{2|x+p_i|}$
- **dihedral ALF** D_k space: $\mathrm{GH}^{\epsilon}/\mathbb{Z}_2 \,\#\, \mathrm{AH}$

Classification (G. Chen–X. Chen, 2015)

Any complete HK^4 w/ $\mathrm{Vol}(B_r) = O(r^3)$, $|\mathrm{Rm}| = O(r^{-2})$ is either A_k for some $k \geqslant -1$ or D_k for some $k \geqslant 0$

Hypertoric QALF metrics

Bielawski–Dancer (2000)

- n-torus $T = \mathfrak{h}/\Lambda$ ($\mathfrak{h} \simeq \mathfrak{h}^*$, $\Lambda^* \subset \mathfrak{h}^*$)
- $T \circlearrowright (M^{4n}, \underline{\boldsymbol{\omega}})$ triholomorphically $\rightsquigarrow$ $\underline{\boldsymbol{\mu}}\colon\ M \to \mathfrak{h}^* \otimes \mathbb{R}^3$
  - $u_1, \ldots, u_n \in \Lambda$ + $\underline{\boldsymbol{\lambda}}_1, \ldots, \underline{\boldsymbol{\lambda}}_n \in \mathbb{R}^3$ $\rightsquigarrow$ **flats** $H_{u_i}(\underline{\boldsymbol{\lambda}}_i) = \{\langle u_i, \underline{\boldsymbol{x}} \rangle = \underline{\boldsymbol{\lambda}}_i\}$
  - $G \in \mathrm{Sym}^2_{\geqslant 0}(\mathfrak{h})$: assume $G > 0$ $\rightsquigarrow$
- $(M, \underline{\boldsymbol{\omega}})$ **QALF**
  - large open set of end $\approx (T, G^{-1}) \times (\mathfrak{h}^*, G) \otimes \mathbb{R}^3$
  - local model $(\mathbb{R}^3 \times S^1)^{n-1} \times$ TN at generic points of H
- smoothness: $H_1 \cap \cdots \cap H_k \neq \varnothing \Rightarrow \{u_1, \ldots, u_k\}$ part of a $\mathbb{Z}$-basis of Λ
  - Nagaoka (2019), Eldridge (2020): classification hypertoric manifolds for $n = 2$

Dihedral QALF metrics?

- Weyl group $W \circlearrowright T$ + W-invariant G and $\{H_1, \ldots, H_n\}$ + weight -4 attached to W-fixed flat $\rightsquigarrow$ **approx HK end**
  - large open set of end $\approx (T \times \mathfrak{h}^* \otimes \mathbb{R}^3)/W$
  - local model $(\mathbb{R}^3 \times S^1)^{n-1} \times$ TN at generic points of H_u w/ $\mathrm{Stab}_W(u) = \{1\}$
  - local model $(\mathbb{R}^3 \times S^1)^{n-1} \times$ AH at generic points of H_u w/ $\mathrm{Stab}_W(u) = \mathbb{Z}_2$

Twistor space and holo symplectic structure

Proposition

- Holo sympl X + sympl action finite Coxeter group W
- Decomposition $\bigcup_w X^w$ of set of points with non-triv stab, where X^w codim-2 submfld fixed by reflection w
- $\mathfrak{h}^c = W$–rep w/o triv factors + $\mu\colon\ X \to \mathfrak{h}^c$ holo W-equiv fibration

Then $\mathrm{Hilb}^W_\mu(X)$ is a holo symplectic manifold

- T max torus of cpct ssimple G, Weyl group W, inv metric on $\mathfrak{h} \simeq \mathfrak{h}^*$
- 'coroot-flats' $H_{\alpha^\vee} \subset \mathfrak{h} \otimes \mathbb{R}^3$ fixed by W
- W–invariant collection of further flats H_{u_i}, $u_i \in \Lambda$
  - special case: $\{u_i\}$ = weights of rep V of Langland dual $G^\vee$
  - **Coulomb branch** of 3d N=4 SYM for $G^\vee$ and hypermultiplets in $V \oplus V^*$

Proposal: hypertoric M' from flats H_{u_i} with $\mathbb{C}$-moment map $\mu \rightsquigarrow$

- holomorphic symplectic manifold $M = \mathrm{Hilb}^W_\mu(M')$
- $Z(M)$ by fibrewise construction on fibres of $Z(M')$

From the twistor space to Nahm's equations

Simplest case $M' = T \times (\mathfrak{h} \otimes \mathbb{R}^3)$, $\mathfrak{h}$ W-irrep (pure SYM for simple $G^\vee$)
$Z(M') =$ T^c–bundle $\mathcal{L}^{1/\epsilon} \xrightarrow{\mu} \mathfrak{h}^c \otimes \mathcal{O}(2)$ with trans funct $\exp\left(-\frac{1}{\epsilon}\zeta^{-1} v\right)$

- twistor lines in $Z(M)$: W–inv real curves in $Z(M')$ of deg $|W|$ over $\mathbb{CP}^1$ and with normal bundle a sum of $\mathcal{O}(1)$'s (Bielawski, 2014)
  - W–inv curves $S \subset \mathfrak{h}^c \otimes \mathcal{O}(2)$ from $q_S \in H^0(\mathbb{CP}^1, \bigoplus_{i=1}^r \mathcal{O}(2d_i))$
  - $\mathcal{L}^{1/\epsilon}|_S \simeq S \times T^c$
- $(S, \mathcal{L})$ 'cameral' data of **coHiggs G^c–bundle** $(\mathcal{E}, \Phi)$ on $\mathbb{CP}^1$
  Hurtubise (1997), Scognamillo (1998), Donagi–Gaitsgory (2002)
  - $(S, S \times T^c)$ cameral data of $(\mathcal{E}_{max}, \Phi)$ determined by $\sum_{\alpha \in R+} \alpha^\vee$ and q_S
- **Nahm's flow**: $(\mathcal{E}, \Phi) \rightsquigarrow (\bar{\partial}_{\mathcal{E}} + s\, \iota_\Phi \omega_{\mathrm{FS}}, \Phi)$
  - $\mathcal{E} \simeq S \times G^c \rightsquigarrow \mathcal{E}_s \xrightarrow{u_s} \mathcal{E}$
  - $\mathrm{Ad}_{u_s}(\Phi) = (\Phi_1 + i\Phi_2) - 2i\Phi_3\zeta + (\Phi_1 - i\Phi_2)\zeta^2$ solution of Nahm's eqns

$$\dot{\Phi}_i + [\Phi_j, \Phi_k] = 0 \qquad \text{(Hitchin, 2011/2018)}$$

  - $\mathcal{E}_s \to \mathcal{E}_{max}$ as $s \to 0 \rightsquigarrow \underline{\mathbf{\Phi}}$ simple pole w/ residue = regular $\mathfrak{su}_2$–triple

$-\rho_i + [\rho_j, \rho_k] = 0$ $\qquad$ $\Phi_i = \frac{1}{s}\rho_i + \cdots \quad s \to 0$

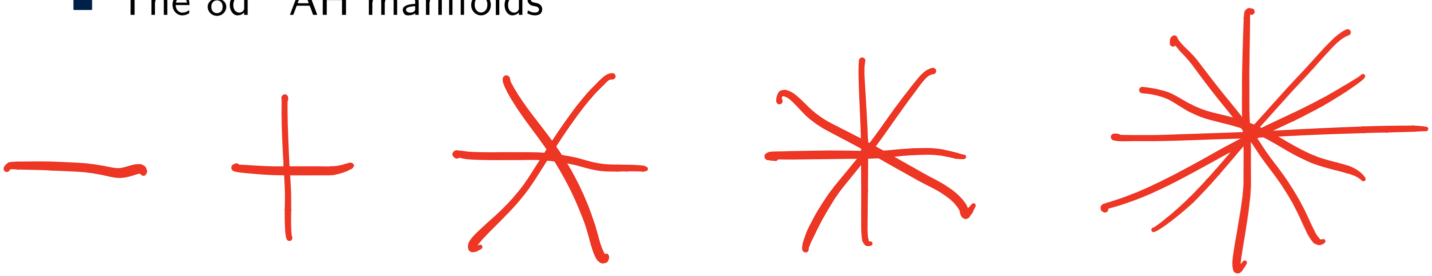
The AH manifold of a cpct ssimple Lie group

Claim: $M = \mathrm{Hilb}^W_\mu(T^*T^c)$ can be identified with the the moduli space of solutions to Nahm's eqs on $(0, 1/\epsilon)$ with structure gp G and regular poles at the endpoints. $Z(M)$ is the twistor space of the HK L^2-metric on this moduli space. This metric is QALF.

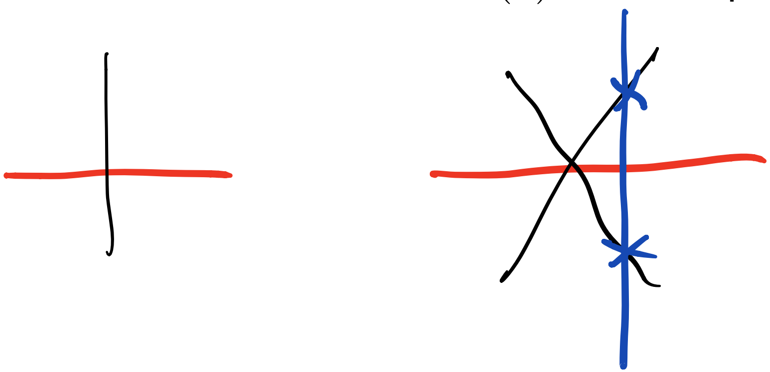
- **regular** poles $\Rightarrow \underline{\boldsymbol{\Phi}}_k$ diverges in M iff $\lambda_k^{-1} := |\underline{\boldsymbol{\Phi}}_k(\text{mid})| \to \infty$
- rescale: $\lambda_k \underline{\boldsymbol{\Phi}}_k(\lambda_k \cdot\,) \xrightarrow{k}$ pairs of solns $\underline{\boldsymbol{\Phi}}^{\pm}$ on $(0, +\infty)$ and $(-\infty, 0)$
- $\underline{\boldsymbol{\Phi}}^{+} \approx \underline{\boldsymbol{\tau}} + \frac{1}{s}\underline{\boldsymbol{\rho}}^{\infty} + O(s^{-1-\delta})$ or $O(e^{-\delta s})$
  - $[\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\tau}}] = 0 \Rightarrow \underline{\boldsymbol{\tau}} \in \mathfrak{h} \otimes \mathbb{R}^3$
  - $\underline{\boldsymbol{\rho}}^{\infty}$ regular $\mathfrak{su}_2$-triple in $\ker[\underline{\boldsymbol{\tau}}, \cdot\,]$
- if $\underline{\boldsymbol{\tau}}$ regular ACyl gluing $\rightsquigarrow g_{L^2} \approx$ hypertoric $+ O(e^{-\delta s})$
- $n = 2$: if $\underline{\boldsymbol{\tau}}$ not regular gluing 'bridge' given by sol for $G = \mathrm{SU}(2)$ $\rightsquigarrow$ bubbling-off $\mathbb{R}^3 \times S^1 \times AH$ along coroot-flats

8d examples

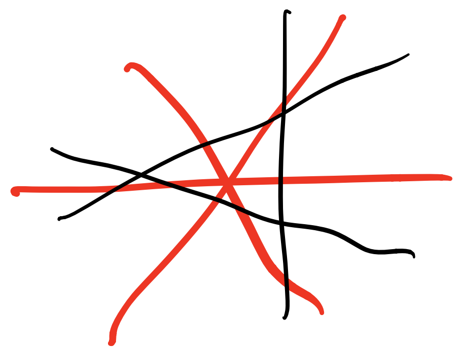
- The 8d "AH manifolds"



- Dancer's manifold: SU(2) Nahm eqn's on $(0, 1/\epsilon]$ with regular pole at 0



- More complicated examples



Coda: G_2 Hitchin-type system / codim-4 ADE singularities in G_2-mflds & deformations of HK cones

$M^3, g, S \to M$ locus of ADE singularities in a G_2-mfld

$\mathbb{C}^2/\Gamma$ $\quad \Gamma \leftrightsquigarrow G$ cpct ssimple Lie gp

$\mathfrak{g}, \mathfrak{h}, W$

$\mathfrak{h} \cong H^2(\widetilde{\mathbb{C}^2/\Gamma})$

G_2 Hitchin-type system:

$P \to M$ princ. G-bdle

A conn. on P

$\phi \in \Omega^1(M; \mathrm{ad}\,P)$

s.t.
$$\begin{cases} F_A - [\phi \wedge \phi] = 0 \\ d_A \phi = 0 = d_A^* \phi \end{cases}$$

$A + i\phi$ flat

Slogan: (A, ϕ) encode information to construct a fibration

$$F^4 \hookrightarrow (X^7, \varphi)$$
$$\downarrow$$
$$(M^3, g)$$

closed φ w/ "small" torsion
(Donaldson: adiabatic limit coassoc. fibr.)
Joyce - Karigiannis

F^4: HK at leading order, "deforming" $\mathbb{C}^2/\Gamma$

Problem clear interpretation only if $[\phi \wedge \phi] = 0$ $\quad (\phi_1, \phi_2, \phi_3) \in (\mathfrak{h} \otimes \mathbb{R}^3)/W$

§1. An analogue of the Hitchin map

$\phi \in \Omega^1(M; \mathrm{ad}\,P) \longleftrightarrow \phi : S \times_M P \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}$ $\quad SU(2) \times G$-equiv.

[or loc. $\phi = \phi_i dx^i$ $\quad (\phi_1, \phi_2, \phi_3) \in C^\infty(U; \mathbb{R}^3 \otimes \mathfrak{g})$]

Lemma Ad-inv. poly on $\mathfrak{g}$ induce $SU(2)$-equiv., G-inv. map

$$\mathbb{R}^3 \otimes \mathfrak{g} \longrightarrow \bigoplus_{i=1}^{r} \mathrm{Sym}^{2d_i}(\mathbb{C}^2)_{\mathbb{R}}$$

proof. $\mathbb{R}^3 \otimes \mathbb{C} \simeq H^0(\mathbb{P}^1; \mathcal{O}(2))$ $\quad \mathrm{Sym}^{2d_i}(\mathbb{C}^2) \simeq H^0(\mathbb{P}^1; \mathcal{O}(2d_i))$ ☒

Def Hitchin map: $\phi \longmapsto s_\phi \in C^\infty(M; \bigoplus_i \mathrm{Sym}^{2d_i}(\mathcal{S}))$

$\mathcal{D} s_\phi = 0$

§2. Deformations of HK cones

Prop. (BF) $\bigoplus_{i=1}^{r} H^0(\mathbb{P}^1; \mathcal{O}(2d_i))_{\mathbb{R}}$ parametrises HK metrics (gen. incomplete) "deforming" the flat cone metric on $\mathbb{C}^2/\Gamma$.

proof.
uses twistor methods & deformation theory:

Versal def. of $\mathbb{C}^2/\Gamma$:

$$\mathfrak{X} \xrightarrow{\ \pi\ } \mathfrak{h}^c/W \simeq \mathbb{C}^r \text{ via inv. poly}$$

holo. sympl. fibres, $\mathbb{C}^*$ acts, + antiholo. involution

$$\rightsquigarrow \mathfrak{X}(2) \xrightarrow{\ \pi\ } \bigoplus_i \mathcal{O}(2d_i) \quad \text{w/ real structure}$$

$$\mathfrak{X}(2) \searrow \mathbb{P}^1 \swarrow \bigoplus_i \mathcal{O}(2d_i)$$

* $s \in \bigoplus_i H^0(\mathbb{P}^1; \mathcal{O}(2d_i))_{\mathbb{R}} \rightsquigarrow Z_s = \pi^{-1}(s)$ twistor space (Hitchin, 1979)
* ℓ twistor line in Z_s is smooth pt of space of sections of $\mathfrak{X}(2)$ & $H^0(\pi)$ submersion at ℓ

$\rightsquigarrow$ deform to ℓ_t twistor line in Z_{ts} ☒

Example: $\mathbb{C}^2/\mathbb{Z}_2$ $\quad d_1 = 2 \quad \mathrm{Sym}^4(\mathbb{C}^2)_{\mathbb{R}} \simeq \mathrm{Sym}^2_0(\mathbb{R}^3)$

Hitchin map: $\mathbb{R}^3 \otimes \mathfrak{su}_2 \ni (\phi_1, \phi_2, \phi_3) \longmapsto \begin{pmatrix} |\phi_2|^2 - |\phi_3|^2 & \langle \phi_1, \phi_2 \rangle & \langle \phi_1, \phi_3 \rangle \\ & \ddots & \end{pmatrix}$

HK metric g_s from Z_s, $s \in \mathrm{Sym}^2_0(\mathbb{R}^3)$ explicit: (Belinskiĭ et al. 1978)

HK "end" asymptotic to $\mathbb{C}^2/\mathbb{Z}_2$

complete only when s comes from $\phi_i = -\lambda_i \xi$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$