M theorists Wishlist for Compact, Singular G_2-holonomy Spaces*

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Outline

1) Review formulation of physics in terms of Lagrangian functionals.
2) M theory in two limits:
   - Type IIA
   - Heterotic (T^3)
3) Adiabatic fibrations of G_2 manifolds:
   - "$S^1 \longrightarrow X \longrightarrow Z/\langle 1,\sigma\rangle$"
   - "$K3 \longrightarrow X \longrightarrow S^3$"

4) Singularities of G_2 hol spaces

Lagrangian Formulation of Physics

(see my lectures on the Standard Model @ Simons)

Defined by a Lagrangian density, $\mathcal{L}$, which is the difference between the kinetic and potential energy densities of a "system of fields" on a Lorentzian manifold $M^{d,1}$ (space time)

Fields: sections of natural bundles over $M^{d,1}$, eg $S(M^{d,1})$, $\Omega^0(M)$, Ω^1, ... often with values in a vector bundle $E \to M$.
E is associated with gauge symmetries, G.

Then $\mathcal{L}$ is the most general function of the fields and their derivatives which is both G-invariant and Lorentz invariant.

Physically the lowest order terms are the most relevant.

Quantum theory (in favourable conditions) then provides rules for predicting the probability for ANY physical process.

- So if you specify gauge group G, the set of fields and how they transform under representations of G, $\mathcal{L}$ is defined, up to coefficients

- Each field corresponds to a particle and polynomials in $\mathcal{L}$ describe interactions.

physics

The next slides describe/define the Standard Model of Particle Physics.

In this lecture we will proceed directly to M theory and string theory.

Each particle in the SM is described by a <u>field</u>. <u>One field for each particle</u>.

A field is a section of some natural bundle $F \to M$ with values in a representation (R) of G_{SM} i.e. in an associated $R_{G_{SM}} \to M$ (associated to E)

F could be $S^+, S^-, \Lambda^0, \Lambda^1, \ldots$

Fermions

The fermions of the SM are classified into QUARKS and LEPTONS.

There are THREE families (generations) of QUARKS and LEPTONS.

Each of the THREE families is described by sections of S^+ or S^- with values in a dimension 15 rep$^{\text{n}}$ of G_{SM}.

FAMILY:	I	II	III
QUARKS	$q = 2/3$ u $m = 2$ MeV	$q = 2/3$ c $m = 1.3$ GeV	$q = 2/3$ t $m = 173$ GeV
	$q = 1/3$ d $m = 3$ MeV	$q = 1/3$ s $m = 95$ MeV	$q = 1/3$ b 4.2 GeV
LEPTONS	$q = -1$ e $m = 0.51$ MeV	$q = -1$ μ $m = 105$ MeV	$q = -1$ τ $m = 1.8$ GeV
	$q = 0$ ν_e $m < 0.23$ eV	$q = 0$ ν_μ $m < 0.23$ eV	$q = 0$ ν_τ $m < 0.23$ eV

Huge range of masses

Gauge bosons

spin 1

$$\left\{\begin{array}{lll} \gamma & \text{photon} & m=0 \\ W^{\pm} & \text{W-bosons} & m=80\,\text{GeV} \\ Z^{0} & \text{Z-boson} & m=90\,\text{GeV} \\ g & \text{gluons} & m=0 \end{array}\right.$$

Higgs $\quad h \quad m=125\,\text{GeV}$

The masses of all of these span <u>many</u> orders of magnitude.

Later I will show that this is <u>natural</u> in <u>M-theory</u> <u>on</u> a <u>G_2-Manifold</u> and that the masses are controlled by the VOLUMES of ASSOCIATIVES!

Each family of fermions is described by a 15-dim rep of G_{SM}, $R_{fermion}$.

$R_{fermion}$ is reducible, the sum of FIVE irreps of G_{SM}:

$$R_{fermion} = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5$$

The fields corresponding to one family are described by sections of:

$$S^+(R_{fermion}) \longrightarrow M^{3,1}$$

and $S^-(\overline{R_{fermion}}) \longrightarrow M^{3,1}$

$$R_{\text{fermion}} = (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1$$

The fermions of <u>each</u> family are described by sections of

$$S^+(R_{\text{fermion}}) \oplus S^-(\overline{R_{\text{fermion}}}).$$

The particles are identified as

$$\begin{array}{ccccccccc} (3,2)_{1/6} & \oplus & (\bar{3},1)_{-2/3} & \oplus & (\bar{3},1)_{1/3} & \oplus & (1,2)_{-1/2} & \oplus & (1,1)_1 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (u_L, d_L) & & \overline{u_R} & & \overline{d_R} & & (\nu_e, e_L) & & \overline{e_R} \end{array}$$

$$(3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1$$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ (u_L, d_L) & \overline{u_R} & \overline{d_R} & (\nu_e, e_L) & \overline{e_R} \end{array}$$

(The bars are here to make all of these left-handed ie sections of S^+)

→ The conjugate, $S^-(\overline{R_{\text{fermion}}})$, gives the "anti-particles" of the above list.

* The SAME is true for the other TWO families with $u \to c$, $e \to \mu$, etc

* The subscript on e.g. $(3,2)_{1/6}$ is the <u>charge</u> under the $U(1)$ factor in G_{SM}

[If you don't like these rational numbers you are free to eg multiply all these factors by <u>6</u>.]

Bosons

Higgs fields : $H \in \Omega^0((1,2)_{-1/2})$

Gauge bosons : $A \in \Omega^1(\mathfrak{g}_{SM})$

adjoint rep of G_{SM}

$$\mathfrak{g}_{SM} = \underbrace{(8,1)_0}_{\text{gluons}} + \underbrace{(1,3)_0}_{\text{W-bosons}} + \underbrace{(1,1)_0}_{\text{Hypercharge gauge boson}}$$

A is a connection on $E \to M$

- There is a covariant derivative D_A

We have now introduced all the fields of the Standard Model.

Now we can define it! ☺

Essentially, it is defined by the set of all "operators" of dimension four which are invariant under

$$G = G_{SM} \times \text{Lorentz}$$

$SO(3,1)$

- An <u>operator</u> is simply a product of any combination of fields plus derivatives acting on them.

- In natural units, H and A have mass dimension ONE. The spinors have dimension 3/2. d, $\partial/\partial x_\mu$ have dimension one.

– This is because the Lagrangian has mass dimension four (it is the difference of kinetic and potential energy densities).

- $\mathcal{L}_{\text{DIRAC}} = i\overline{\psi}\,\not{\partial}\,\psi$, for a massless free fermion described by a section of $S^+ \oplus S^-$

- $\mathcal{L}_{\text{YANG-MILLS}} = -\frac{1}{4}\,\mathrm{tr}(F_A \wedge * F_A)$

$$\Rightarrow \quad [\psi] = [M]^{3/2} \qquad [A] = [M]^1$$

The particles are identified as

$$(3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1$$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ (u_L, d_L) & \overline{u_R} & \overline{d_R} & (\nu_e, e_L) & \overline{e_R} \\ \text{\rotatebox{90}{$\cong$}} & & & \text{\rotatebox{90}{$\cong$}} & \\ Q_L & & & L & \end{array}$$

Introduce an index $i = 1,2,3$ with

$$Q_{L1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad Q_{L2} = \begin{pmatrix} c_L \\ s_L \end{pmatrix} \text{ etc}$$

$$e_{R3} = \tau_R \quad \text{etc}$$

The Lagrangian density for the SM is

$$\mathcal{L} = \mathcal{L}_{\mathrm{DIRAC}} + \mathcal{L}_{\mathrm{Yang\text{-}Mills}} + \mathcal{L}_{\mathrm{Higgs}} + \mathcal{L}_{\mathrm{Yukawa}}.$$

$$\mathcal{L}_{\mathrm{Dirac}} = i\sum_{j=1}^{3}\Big(\overline{Q_{L_j}}\,\not{D}_A\,Q_{L_j} + \overline{u_{R_j}}\,\not{D}_A\,u_{R_j} + \overline{d_{R_j}}\,\not{D}_A\,d_{R_j} + \overline{L_j}\,\not{D}_A\,L_j + \overline{e_{R_j}}\,\not{D}_A\,e_{R_j}\Big)$$

Note: $D_A \equiv d + ig_1 \underset{\uparrow}{B} + ig_2 \underset{\uparrow}{W} + ig_3 \underset{\uparrow}{G}$

corresponding to $U(1) \times SU(2) \times SU(3)$

g_1, g_2, g_3 are dimensionless and define the unit of "charge" for the three forces corresponding to $U(1)$, $SU(2)$, and $SU(3)$.

$\mathcal{L}_{\text{Yang Mills}} = -\frac{1}{4} \operatorname{tr}(F_A \wedge * F_A)$ is a sum of three terms for $A = (B, W, G)$

$$\mathcal{L}_{Higgs} = D_A H \wedge * D_A H - m^2 |H|^2 - \lambda |H|^4$$

The last two terms are <u>minus</u> the Higgs potential.

$$\mathcal{L}_{Yukawa} = \lambda^u_{ij}\, \overline{Q_{Li}}\, \tilde{H}^* u_{Rj} + \lambda^d_{ij}\, \overline{Q_{Li}}\, H d_{Rj} + \lambda^L_{ij}\, \overline{L_i}\, H e_{Rj}$$

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

$$\tilde{H}^* \equiv \begin{pmatrix} H_2^* \\ H_1^* \end{pmatrix}$$

Note: apart from Higgs, there are

$$\boxed{\text{NO MASS TERMS in } \mathcal{L}_{SM}.}$$

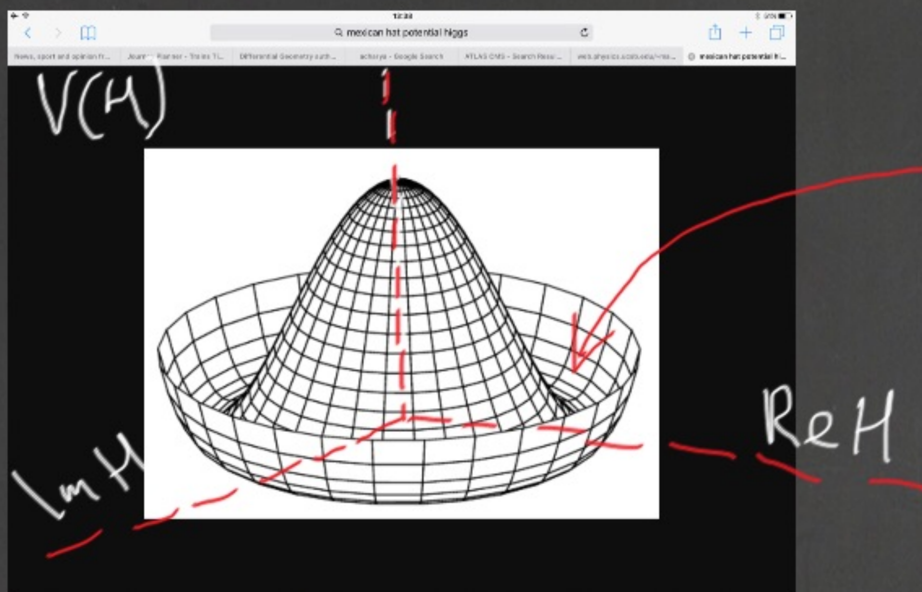
eg a fermion mass requires ψ_L and ψ_R such that $R_L \otimes R_R$ contains a G_{SM} invariant, which it does NOT.

So, if G_{SM} is an exact symmetry, the fermions and gauge bosons would be massless.

The Higgs mechanism solves this problem.

Higgs Mechanism

- $\mathcal{L}_{Higgs}$ contains a potential $V(H)$ for the Higgs field $V(H) = m^2|H|^2 + \lambda|H|^4$
- If the parameter $m^2 < 0$ and $\lambda > 0$



minima

S^1 of vacua at non-zero H.

So, $\mathcal{L}_{SM}$ is G_{SM} invariant,
but its minima (vacua) break
the G_{SM} symmetry.
The choice of vacuum is spontaneous,
without loss of generality we can
write the direction as

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

where $v \sim \sqrt{\frac{-m^2}{2\lambda}}$

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$ has a commutant in G_{SM} which is $SU(3) \times U(1)$. This $U(1)$ is the $U(1)$ of electromagnetism.

Denoting by T_3 and Y the Cartan subalgebra of $\mathcal{L}ie\,(SU(2) \times U(1)_Y)$ (with T_3 for $SU(2)$ and Y for $U(1)_Y$)

$U(1)_{em}$ has a Lie algebra spanned by $Q_{em} \equiv T_3 + Y$ (or $Y/6$ if you multiplied by 6)

$$A_\mu \equiv \frac{g_1}{\sqrt{g_1^2+g_2^2}} W_\mu^3 + \frac{g_2}{\sqrt{g_1^2+g_2^2}} B_\mu$$

$$Z_\mu \equiv \frac{g_2}{\sqrt{g_1^2+g_2^2}} W_\mu^3 - \frac{g_1}{\sqrt{g_1^2+g_2^2}} B_\mu$$

A_μ is the Maxwell gauge field.

Z_μ is the field whose quanta are Z-bosons.

The other two $\mathcal{L}ie(SU(2))$ generators correspond to the W^+ and W^- bosons.

Expanding $\mathcal{L}_{Higgs} \sim D_A H \wedge * (D_A H)^\dagger$

gives $g_1^2 \, B H \wedge * (BH)^\dagger +$

$g_2^2 \, W H \wedge * (WH)^\dagger + \cdots$

expanding around $\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + h \end{pmatrix}$

gives $\frac{g_1^2}{2} B \wedge * B \, v^2 + \frac{g_2^2}{2} W \wedge * W \, v^2 + \cdots$

ie mass terms

Interactions, vertices and Feynman Diagrams

Simple example QED:

Fields (A, e_L, e_R). A connection on $U(1)$ bundle

$$e_L \in S^+, \quad e_R \in S^-$$

e_L, e_R both have charge -1 in units of electric charge e, where $\alpha = e^2/4\pi$ is the fine structure constant (at low energies)

$$\mathcal{L}_{QED} = -\frac{1}{4} dA \wedge {*dA} + i\overline{e_R} \not{D}_A e_R + i\overline{e_L} \not{D}_A e_L - m_e(\overline{e_L} e_R + \overline{e_R} e_L)$$

where $D_A = d + ieA$

$m_e =$ electron mass

All terms in $\mathcal{L}_{QED}$ are quadratic except one: $ei(\overline{e_L} \not{A} e_L + \overline{e_R} \not{A} e_R)$

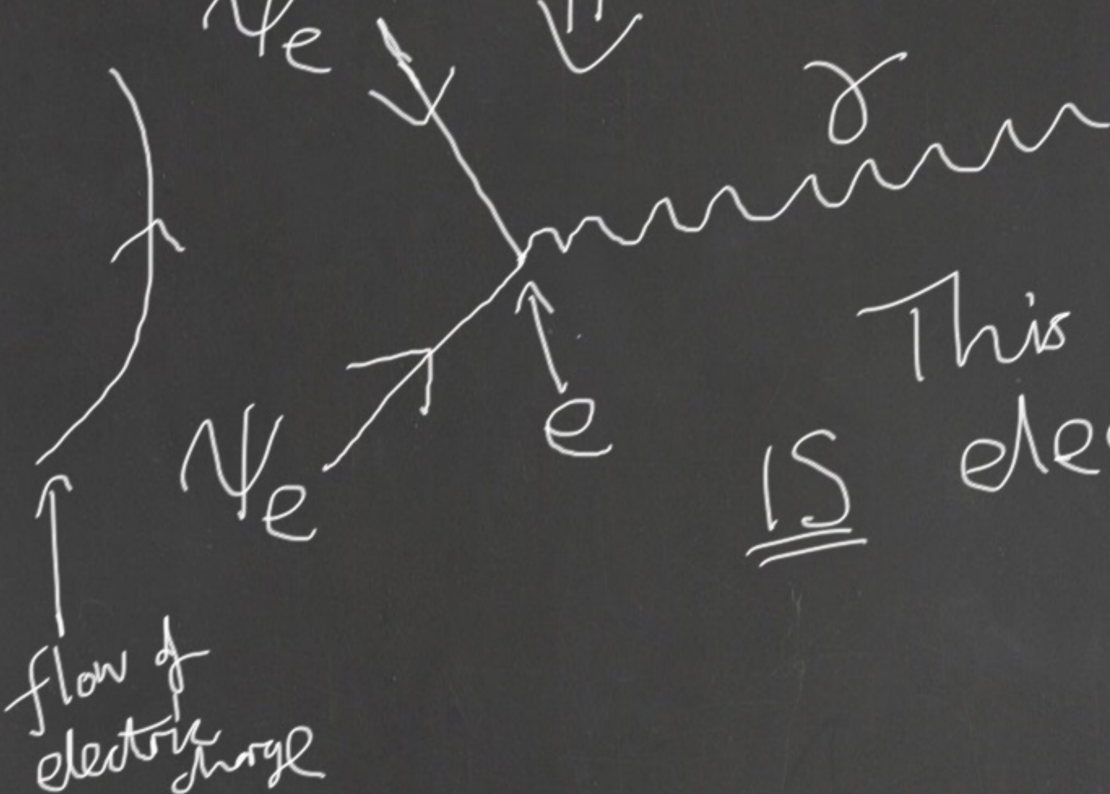
$$= ei(\overline{\psi_e} \not{A} \psi_e)$$

$[\psi = e_L + e_R]$

$$\mathcal{L}_{int} = e\, i\, \bar{\Psi}_e \not{A} \Psi_e$$

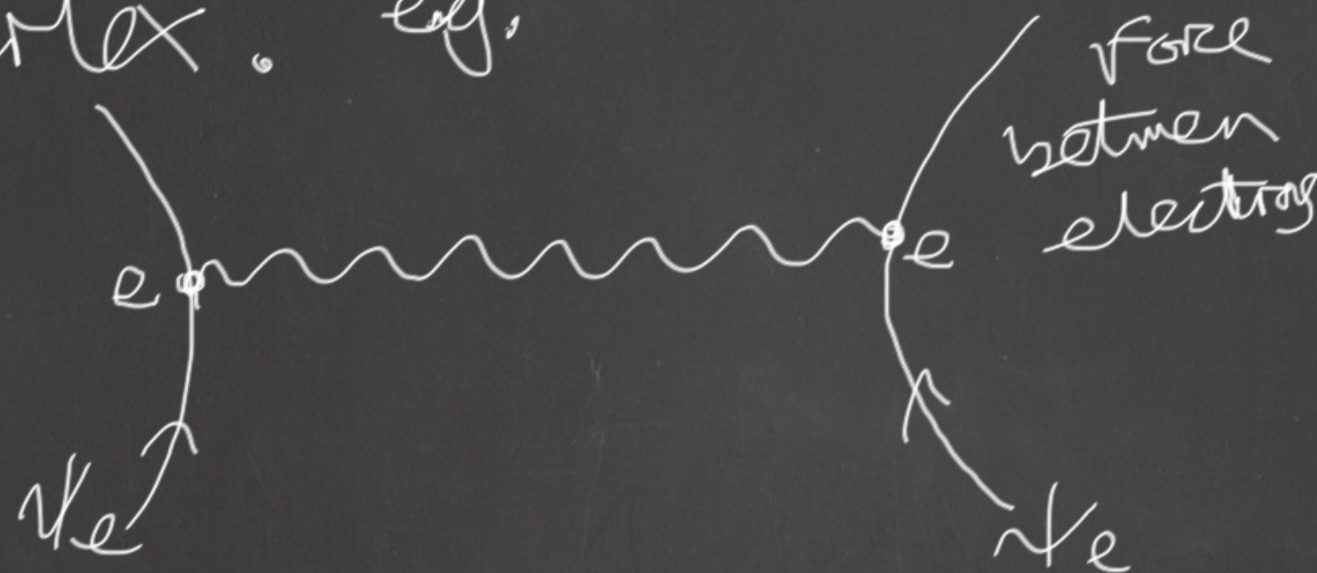
$\Updownarrow$

γ = photon



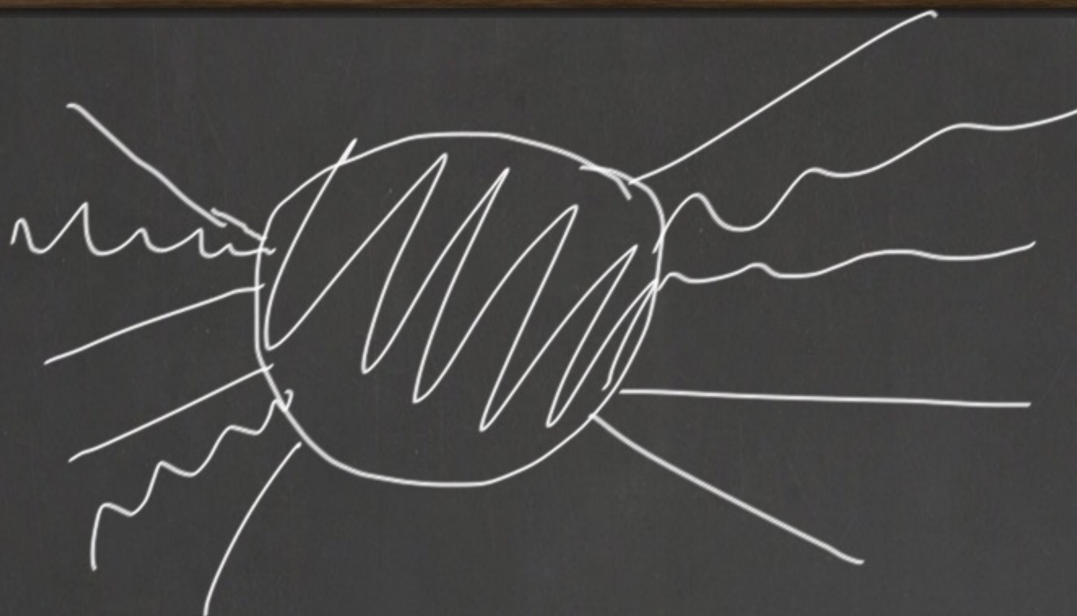
This vertex <u>IS</u> electromagnetism!

Any physical process involving electrons and photons only can be described by building Feynman diagrams using this vertex. e.g.



Feynman rules assign a function, called the AMPLITUDE, $\mathcal{M}$, to any Feynman diagram.

$\mathcal{M}$ is a $\mathbb{C}$-function of the energies and momenta of the initial and final state particles as well as couplings and masses.
(charges)



Probability for any process is

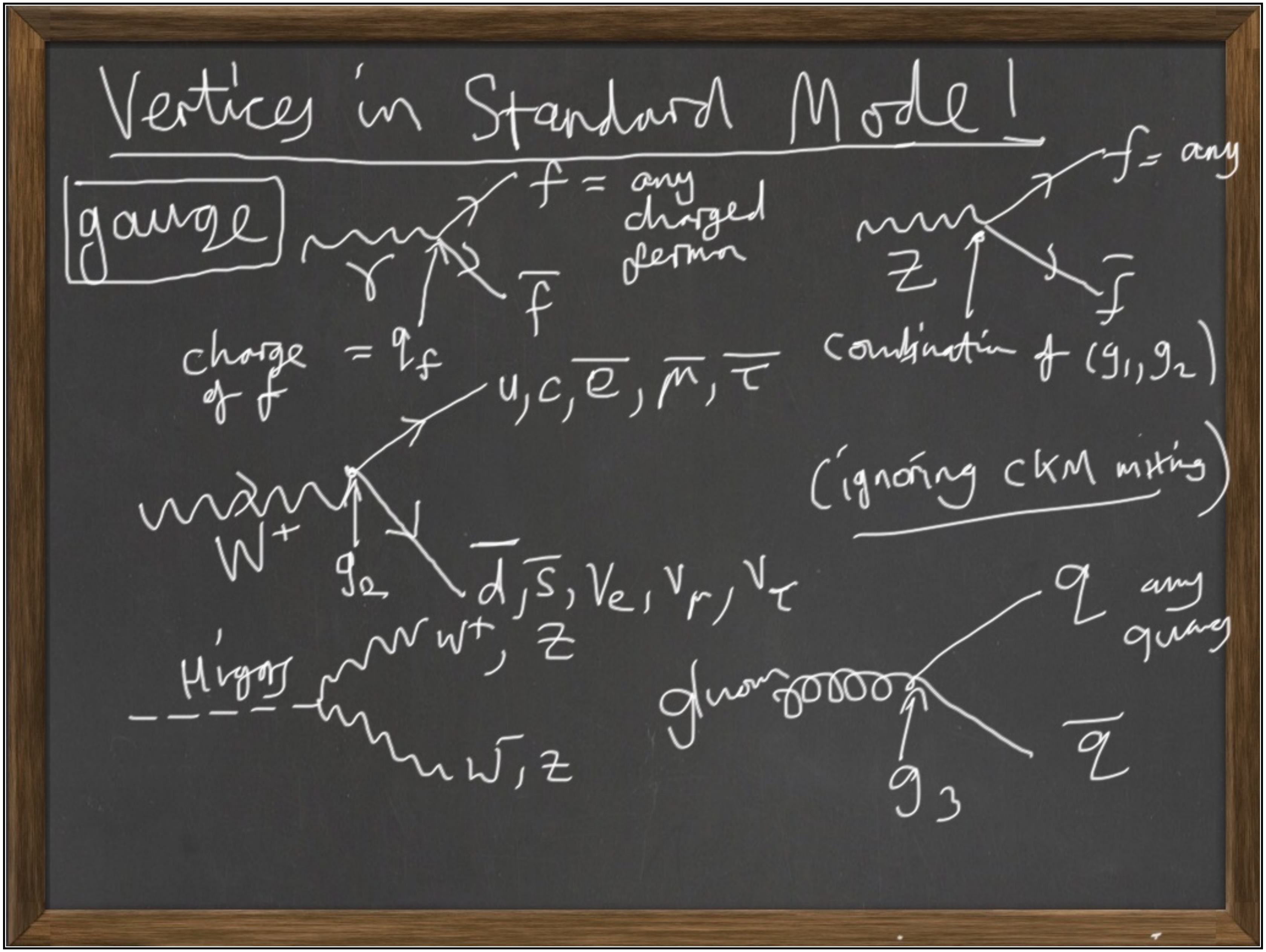
$$P = \int |M|^2 \rho(E,P)$$

where $\rho(E,P)$ is the density of states.

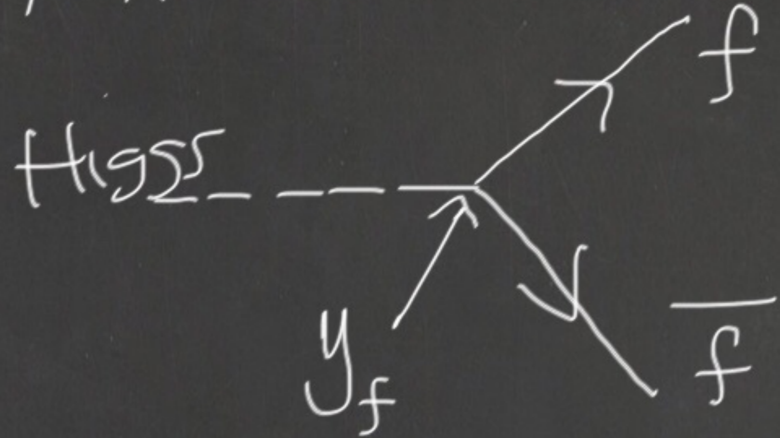
$$P = \int |M|^2 \rho$$

Integral is over final state particles subject to E and p conservation. "Phase space" integral.

Fermi's Golden Rule


Vertices in Standard Model!
gauge
f = any charged fermion
γ
f̄
charge of f = q_f
Z
f = any
f̄
combination of (g_1, g_2)
W^+
g_2
u, c, ē, μ̄, τ̄
d̄, s̄, ν_e, ν_μ, ν_τ
(ignoring CKM mixing)
Higgs
W^+, Z
W^-, Z
gluon
q any quark
q̄
g_3

Yukawa Interactions



$$y_f = \frac{m_f}{\sqrt{2}\, V}$$

There are also quartic vertices but I won't discuss these.

Predictions from $\mathcal{L}_{SM}$:

Any Feynman diagram with initial and final states consistent with energy, momentum, spin, charge, conservation gives a prediction of the Standard Model.

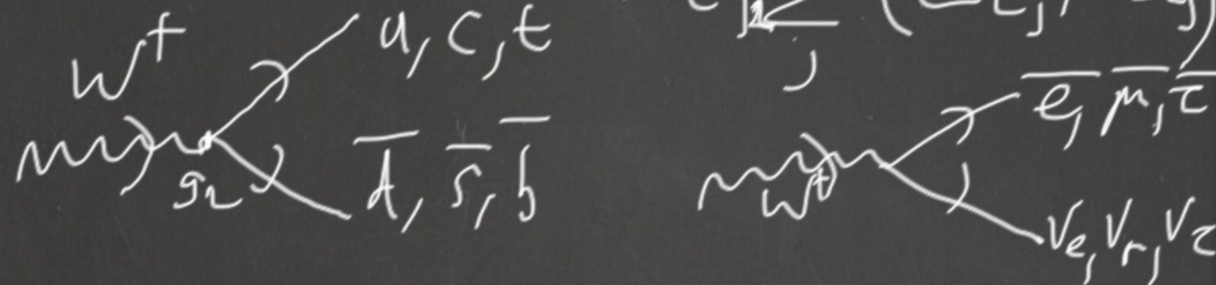
Many thousands of such predictions have been tested successfully, some to rather high precision (>10 sig figs)

An example: Decay probabilities of W-bosons or Q: if you have 100 W-bosons, what do they decay into?

The gauge interactions of W-bosons come from

$$\mathcal{L}_{DIRAC} = \cdots + i g_2 \sum_j \left(\bar{Q}_{L_j} \not{W} Q_{L_j}\right) + i g_2 \sum_j \left(\bar{L}_{L_j} \not{W} L_{L_j}\right)$$

Vertices



The vertices are actually Feynman diagrams for the processes:

$$\left.\begin{array}{l} W^+ \to u\bar{d} \\ W^+ \to c\bar{s} \end{array}\right\}$$ called hadronic as the quarks "hadronise" into jets of hadrons

$$\left.\begin{array}{l} W^+ \to \bar{e}\,\nu_e \\ W^+ \to \bar{\mu}\,\nu_\mu \\ W^+ \to \bar{\tau}\,\nu_\tau \end{array}\right\}$$ called Leptonic channels

(But not $W^+ \to t\bar{b}$ as $M_t \gg M_W$)

The vertex factors for all of these are the same ($= g_2$) and so the $SU(2)$ interactions are universal.

Moreover $M_W \gg M_b, M_c, M_s, M_u, M_d, M_e, \ldots$

So the fermions can be taken as massless to a good approximation

So, expect $\Gamma(W \to \bar{e}\nu) = \Gamma(W \to \bar{\mu}\nu)$
$= \Gamma(W \to \bar{\tau}\nu)$

One might conclude that $P(W \to u\bar{d}) = P(W \to \bar{e}\nu)$, but we have to remember that there are 3 u's, 3 d's, 3 c's and 3 s's.

$$\therefore\ P(W \to u\bar{d}) = 3\,P(W \to \bar{e}\nu)$$

Hence, there are $2 \times 3 = 6$ hadronic modes and 3 leptonic decay modes.

So we predict that

$$p\,(W \to \text{hadrons}) = \frac{6}{9} = 2/3$$

$$p\,(W \to \text{leptons}) = 1/3$$

$$p\,(W \to \bar{e}\,\nu_e) = 1/9$$

$$p\,(W \to \bar{\mu}\,\nu_\mu) = 1/9$$

$$p\,(W \to \bar{\tau}\,\nu_\tau) = 1/9$$

Lets compare this to real data:

pdg.lbl.gov

Cut-off date for this update was January 15, 2015.

Particle Listings

Search Listings

Gauge & Higgs Bosons (gamma, g, W, Z, ...)

Leptons (e, mu, tau, neutrinos, heavy leptons ...)

Quarks (u, d, s, c, b, t, ...)

Mesons (pi, K, D, B, psi, Upsilon, ...)

Baryons (p, n, Lambda_b, Xi, ...)

Other Searches (SUSY, Compositeness, ...)

Open the section on bosons:

Cut-off date for this update was January 15, 2015.

Particle Listings

Search Listings

Gauge & Higgs Bosons (gamma, g, W, Z, ...)

- gamma
- g (gluon)
- graviton
- W boson
- Z boson
- H^0
- Neutral Higgs Bosons, Searches for

Scroll down to W-decay modes:

15:07
pdg.lbl.gov

W^+ DECAY MODES

W^- modes are charge conjugates of the modes below.

	Mode		Fraction (Γ_i/Γ)	Confidence level
Γ_1	$\ell^+\nu$	[a]	(10.86 ± 0.09) %	
Γ_2	$e^+\nu$		(10.71 ± 0.16) %	
Γ_3	$\mu^+\nu$		(10.63 ± 0.15) %	
Γ_4	$\tau^+\nu$		(11.38 ± 0.21) %	
Γ_5	hadrons		(67.41 ± 0.27) %	
Γ_6	$\pi^+\gamma$		$< 7 \times 10^{-6}$	95%
Γ_7	$D_s^+\gamma$		$< 1.3 \times 10^{-3}$	95%
Γ_8	cX		(33.3 ± 2.6) %	
Γ_9	$c\bar{s}$		$(31 {}^{+13}_{-11})$ %	
Γ_{10}	invisible	[b]	(1.4 ± 2.9) %	

These are pretty close to our predictions!

[a] ℓ indicates each type of lepton (e, μ, and τ), not sum over them.

[b] This represents the width for the decay of the W boson into a charged particle with momentum below detectability, p< 200 MeV.

These predictions test the full symmetry group $G = G_{SM} \times \text{Lorentz}$.

eg on the fact that there are 3 of each quark, that fermions obey SU(2) invariant interactions, also that $m_t > M_W$.

Also that there are 3 lighter lepton families.

In our example, we didn't calculate the overall <u>decay rate</u> of the W's, only the ratios of decay rates between different channels.
The overall "rate" is usually required for most other predictions, but I hope that our example illustrates the general idea for predictions in <u>general</u> particle physics models.

Remarks on R_{fermion}.

- Why this particular 15-dim rep?
- The rep explains why the charge of a proton is exactly opposite to the electron charge. But why?
- A possible explanation: The smallest simple group containing G_{SM} (actually mod $\mathbb{Z}_6$) is $SU(5)$.

Viewed as a rep$^{\underline{n}}$ of $SU(5)$

$$R_{\text{fermions}} = \overline{5} \oplus 10$$

i.e. $\overline{\Lambda^1(\mathbb{C}^5)} \oplus \Lambda^2(\mathbb{C}^5)$

So, two of the low dim$^{\underline{l}}$ irreps of $SU(5)$ "explain" R_{fermion}!

This is part of the idea behind Grand Unified Theories.

Other gauge theories can be obtained by replacing G_{SM} by another compact group and choosing $S^+(R)$, $\Omega^0(R')$, etc ...

[There are restrictions called <u>anomalies</u>, but we will not get into that]

Supersymmetric Standard Model

Supersymmetry pairs bosons and fermions.

Take $S^+(R_{fermion})$ of the SM

Add $\Omega^0(R_{fermion})$ (scalar partners of quarks and leptons)

Take $A \in \Omega^1(\mathcal{L}ie(G_{SM}))$

Add $\lambda \in S^+(\mathcal{L}ie(G_{SM}))$ (fermionic partners of gauge bosons)

Take $H \in \Omega^0((1,2)_{+1/2})$ add $H' \in \Omega^0((1,2)_{-1/2})$

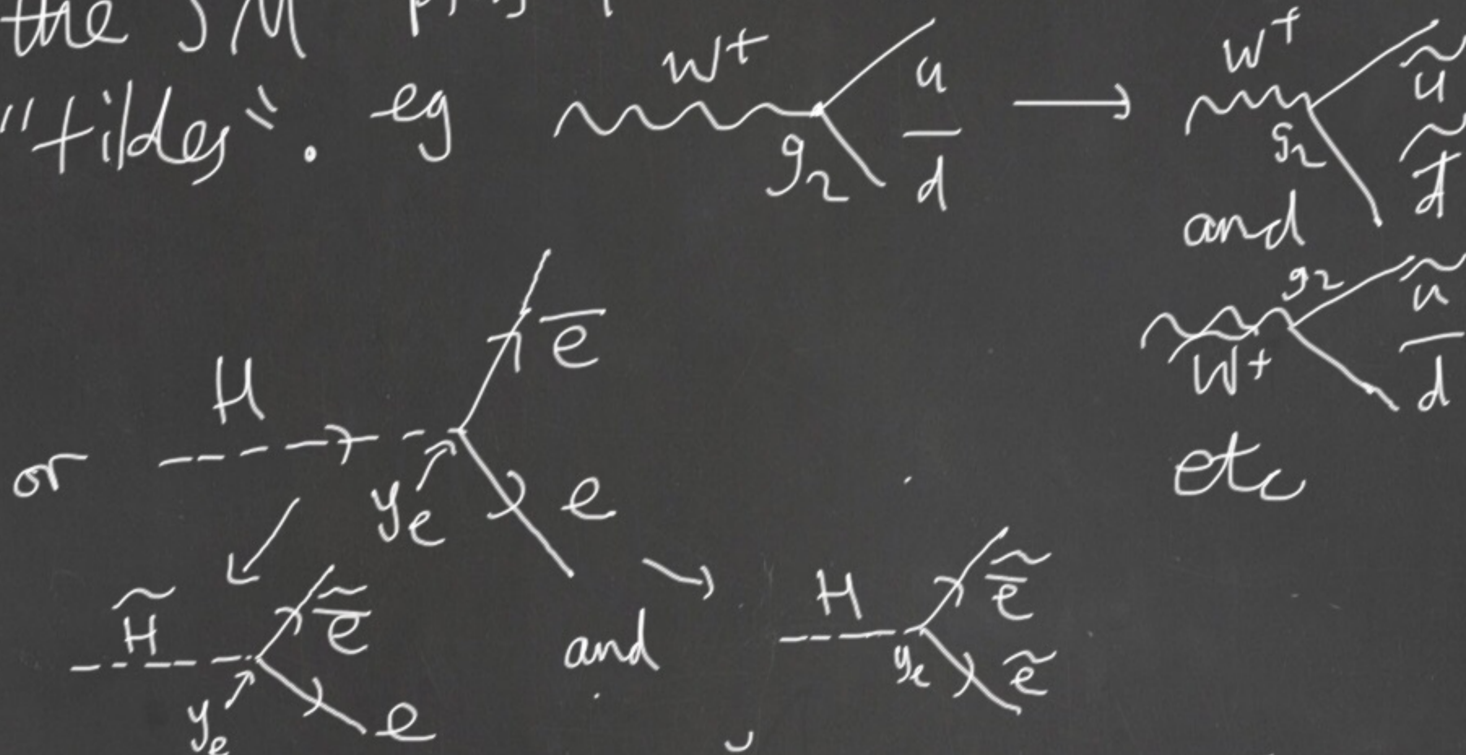
Add $\tilde{H} \in S^+((1,2)_{+1/2}) \leftarrow$ (Higgsinos: fermionic partners of Higgs)

The superpartners of $Q_L, \overline{u_R}, \overline{d_R}, L, \overline{e_R}$ are denoted $\widetilde{Q_L}, \widetilde{\overline{u_R}}, \widetilde{\overline{d_R}}, \widetilde{L}, \widetilde{\overline{e_R}}$ and are sections of $\Omega^0(R_{\text{fermion}})$.

Redenote $H \in \Omega^0(R=(1,2)_{-1/2})$ as H_d and add $H_u \in \Omega^0(R=(1,2)_{1/2})$

The superpartners of H_u and H_d are described by fields $\widetilde{H_u} \in S^+((1,2)_{1/2})$ and $\widetilde{H_d} \in S^+((1,2)_{-1/2})$

Many vertices of the Minimal Supersymmetric Standard Model consist of those of the SM plus the same vertices with <u>Two</u> "tildes". eg



There are also some additional quadratic terms which give masses

eg $$M^2_{AB}\overline{\tilde{\Psi}_A}\tilde{\Psi}_B + \mu H_u H_d + B\mu H_u H_d$$

$$+ iM_1\overline{\tilde{B}}B + iM_2\overline{\tilde{W}}W + iM_3\overline{\tilde{g}}g$$

and some cubic ones

eg $$A_{ij}H_u\tilde{Q}_{L_i}\tilde{u}_{R_j} + \cdots$$

Particle Physics from G_2-manifolds

Consider M-theory on $X_7 \times \mathbb{R}^{3,1}$ with metric $g_{10,1} = g(X) + \eta(\mathbb{R}^{3,1})$ where $Hol(g(X)) = G_2$ and η is flat.

If X is smooth (and large), we can use 11d supergravity to describe the low energy physics.

M theory and its limits

- M theory is a physical theory in (10+1) Lorentzian dimensions
- It has a Lagrangian which is known in various limits
- fields are a metric g_{11}, a 3-form "gauge field" C_3 and a gravitino, $\psi_{3/2} \in S(TM)$.

M theory difficult to work with explicitly as a quantum theory, but it has limits which <u>are</u> well understood.

Two important ones are:

Type $\underline{II}A$ limit : $S^1 \to M^{10,1} \longrightarrow M^{9,1}$

Heterotic limit : $K3 \to M^{10,1} \longrightarrow M^{6,1}$

We are interested in quarks and leptons in these limits, which tell us something about singular G_2 manifolds.

<u>11d Sugra fields</u> $\left(\underset{\Omega^3}{C_3},\ \underset{\text{metric}}{g_{11}},\ \underset{\substack{S(TM) \\ \text{gravitino}}}{\Psi_{3/2}} \right)$

1. We fix a backround solution of the E-L equations of the form

$$(C_3, g_{11}, \Psi_{3/2}) = \left(0,\ g^0(x) + \eta(\mathbb{R}^{3,1}),\ 0\right)$$

2. We perturb
3. We extract the ~~zero~~ modes
4. We integrate over $\int_X \sqrt{g^0}\, \mathcal{L}^{11d}$
5. We get left with $\int d^4x\, \mathcal{L}^{4d}$

The zero modes:

– From C_3 $\left\{ \begin{array}{l} b^2(X) \;\; U(1) \text{ gauge fields } A_I \in \Omega^1(M^{3,1}) \\ b^3(X) \;\; S^1\text{-valued scalars } a_i \in \Omega^0(M^{3,1}) \end{array} \right.$

axions

– From g_{11} $\left\{ \begin{array}{l} 1 \;\; \text{4d metric } g^{3+1}_{\mu\nu} \in S^2(M^{3,1}) \\ b^3(X) \;\; \text{scalars } \quad s_i \in \Omega^0(M^{3,1}) \end{array} \right.$

G_2-moduli

– from $\Psi_{3/2}$ $\left\{ \begin{array}{l} 1 \;\; \text{4d gravitino } \psi^4_{3/2} \in S^+(TM) \\ b^2(x) \;\; \text{spinors } \quad \lambda_I \in S^+(M) \\ b^3(x) \;\; \text{spinors } \quad \chi_i \in S^+(M) \end{array} \right.$

$(g^4$ and $\psi^4_{3/2})$ give the $N=1$ supergravity multiplet

(A_I, λ_I) give $b^2(X)$ vector multiplets

the a_i and s_i's become $\mathbb{R}$ and $\mathbb{I}$m parts of $b^3(X)$ complex scalars $\boxed{z_j = a_j + \sqrt{-1}\, s_j}$

(z_j, χ_j) give $b^3(X)$ "chiral" multiplets

The z_j's should be local coordinates on the complexified moduli space of G_2 manifolds.

$\mathcal{M}_{G_2}(X, \varphi, C_3)$

$$Z_j \sim \int_{\Sigma_j} C_3 + i\,\Phi_3$$

G_2 form

are periods of a complexified G_2 3-form over a basis $\{\Sigma_j\}$ for $H_3(X)$.

The $\mathcal{L}^4$ that we get from this has a metric on the moduli space, which is a Kähler metric with potential;

$$\boxed{K(z_j, \overline{z_j}) = -3 \ln \int_X \Phi \wedge * \Phi}$$

- We would very much like to know the properties of this Kähler metric!
- Can we compute it approximately for the TCS G_2-manifolds?

- The components of the moduli space metric are homogeneous of degree minus two, so it looks like the metric has "–ve curvature" in some sense that would be good to make precise.

In general, $N=1$ $d=4$ supergravity theories also depend on a superpotential $W(z_i)$, $\partial_i W = 0$, locally holomorphic. Witten/Bagger: W is a section of a line bundle $L \to \mathcal{M}_{G_2}(X, \ldots)$.

Because the $a_i = \mathrm{Re}\, z_i$ is periodic,

$$W = \text{zero},$$ up to instanton effects.

– Instantons are associative submanifolds (suitably rigid)

There is also a third function, also holomorphic, called the gauge coupling function $f(z_i)$.

For the $b^2(U(1))$ gauge fields their contribution to $\mathcal{L}^4$ is calculated to be:

$$d^{jIK}\,\mathrm{Im}\,z_j\,F_I \wedge * F_K + d^{jIK}\,\mathrm{Re}\,z_j\,F_I \wedge F_K$$

where $d^{jIK} : H^3(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$

$$\int \alpha^j \wedge \beta^I \wedge \beta^K$$

So $$f(Z) = \int_X (C_3 + i\phi) \wedge \beta^I \wedge \beta^K$$

$\{\beta^I\}$ basis for $H^2(X,\mathbb{Z})$

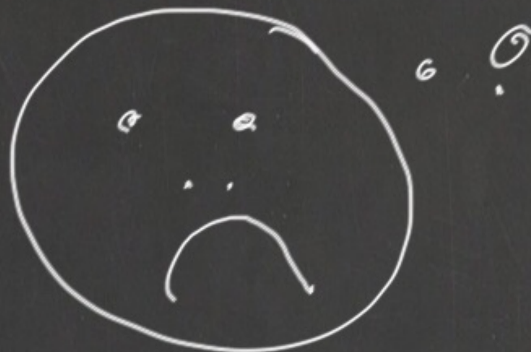
Can ~~wr~~ K

The low energy physics is Gravity + $b^2(X)$ $U(1)$'s and

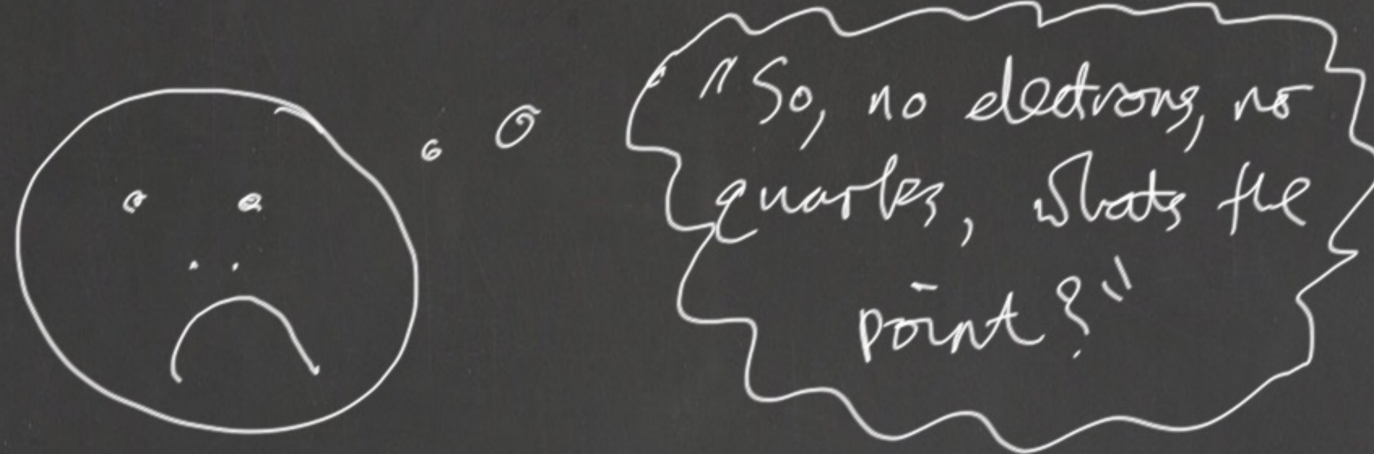
- $b^3(X)$ neutral moduli multiplets

- There are no light charged particles!

- No non-Abelian gauge symmetries



"So, no electrons, no quarks, what's the point?"



We will try and show that particles like quarks and leptons and non-Abelian gauge bosons emerge from special kinds of singularities of G_2-manifolds.

Yang-Mills Fields from codim 4 singularities

Consider a special case when

$X = K3 \times \mathbb{R}^3$ w/ product metric

Then $M^{10,1} = X \times \mathbb{R}^{3,1} = K3 \times \mathbb{R}^{6,1}$, leading to a 6+1 d Lagrangian.

In this case the moduli space = space of Einstein metrics on

$$K3 = \mathbb{R}^+ \times SO(3,19;\mathbb{Z}) \backslash \frac{SO(3,19)}{SO(3)\times SO(19)}$$

There are also 22 $U(1)$ gauge fields

$C_3 \}$ $b^2(K3)$ $U(1)$ gauge fields

$g_n \}$ 58 scalars $= \left(\mathrm{Vol}(K3), \int_{\Sigma_\alpha} \omega_I \equiv \phi_{I\alpha} \right)$

"SD cycles"

$\psi_{3/2} \}$ fermions which make everything supersymmetric

This looks v. similar to the theory obtained by considering $E_8 \times E_8$ Heterotic superstring theory on

$M^{9,1} = T^3 \times \mathbb{R}^{6,1}$.

Heterotic fields on $M^{9,1}$:

$\Omega^0, \Omega^2, S^2, \Omega^1 \otimes \mathrm{ad}\,E$

$\phi \quad B \quad g^{10}, \quad A$

E ← $E_8 \times E_8$ bundle

Massless Bose fields in Het string on $T^3 \times \mathbb{R}^{6,1}$

10d	$\int_{T^3}$	6+1d massless fields
dilaton $\phi \in \Omega^0(M^{10})$	INTEGRATE	$\to$ 1 scalar $\in \mathbb{R}^+$
metric $g^{10} \in S^2 TM^{10}$		$\to$ 6 scalars in $SL(3,\mathbb{R})/SO(3)$
B-field $B \in \Omega^2(M)$		$\to$ 3 scalars in $H^2(T^3; U(1))$
$E_8\times E_8$ gauge field $A \in \Omega^1(\mathrm{Lie}(E_8\times E_8))$ $E \to M$ $E_8\times E_8$ v bundle	OVER T^3	Flat $E_8\times E_8$ connections on T^3. The Identity connected component is 48-dim$^{\underline{l}}$ and is $\dfrac{(\hat{T}^3)^{16}}{W(E_8\times E_8)}$ $W(E_8\times E_8)$ is Weyl group

So the moduli space is 58-dim$^{\text{l}}$

$$58 = \underset{\underset{\phi}{\uparrow}}{1} + \underset{\underset{SL(3)/SO(3)}{\uparrow}}{6} + \underset{\underset{H^2(T^3)}{\uparrow}}{3} + \underset{\underset{\left(H_1(T^3)\right)^{rk(E_8\times E_8)}}{\uparrow}}{48}$$

What about gauge bosons?

$g_{11} \longrightarrow \quad U(1)^3$ from the 3 Killing vectors on T^3.

$B \longrightarrow \quad U(1)^3$ from the 3 harmonic 1 forms on T^3.

$A \longrightarrow \quad U(1)^{16}$ at generic points in space of flat $E_8 \times E_8$ connections (Id comp)

So $U(1)^{22}$, as in M-theory on K3.

- In fact, the Heterotic moduli space on T^3 is also, locally $\mathbb{R}^+ \times \frac{SO(3,19)}{SO(3)\times SO(19)}$
- String dualities assert that the heterotic string on T^3 is "dual" to M theory on K3.
- Non-Abelian gauge symmetry is present in Het string from the start

Non-Abelian symmetries in Het on T^3

The $U(1)^{16}$ gauge group is identified as the commutant of a generic flat connection on T^3 inside $E_8 \times E_8$.

But: at special codim 3 and 3n subspaces, the commutant enhances to non-Abelian groups

E.g. consider flat $SU(2)$ connections on T^3. These are globally

$$\mathcal{M}_{flat}(SU(2),T^3) = \mathrm{Hom}(\pi_1(T^3), SU(2)) = \frac{T^3}{\mathbb{Z}_2}$$

At a general pt $\in \mathcal{M}$, the commutant of the flat connection is $T(SU(2)) = U(1)$

At the origin $(0,0,0)$, the commutant is the full $SU(2)$.

In general for $\mathcal{M}_{flat}(E_8 \times E_8; T^3)$, one requires

choosing at least one $U(1) \subset T(E_8 \times E_8)$

and the 3 holonomies of this connection

must vanish in order to ENHANCE

the GAUGE SYMMETRY.

ie codim 3 singularities

McKay Correspondence for M-theory on K3

$$H_2(K3,\mathbb{Z}) \cong \Gamma(E_8) \oplus \Gamma(E_8) \oplus \underbrace{3 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{related to } (g, B, \varphi \text{ on } T^3)}$$

sig $(19,3)$ lattice.

* According to Het/M duality, there is a codim^{n} 10 subspace of $\mathcal{M}(K3)_{\text{einstein}}$ which one is tempted to identify with $\mathcal{M}(E_8 \times E_8, T^3)_{\text{flat}}$. Coords $\Phi_{I\alpha} = \int_{\Sigma_\alpha} \omega_I$ $\quad \Sigma_\alpha \in \Gamma(E_8) \oplus \Gamma(E_8)$

M - branes

$dC_3 = G_4$ In the absence of branes and $[G_4] = 0$ coh

$d G_4 = 0$

$d *_{11} G_4 = 0$

M-branes are sources of currents

M5: $d G_4 = q_5 \delta_5 (M^{5,1} \subset M^{10,1})$

M2: $d * G_4 = q_2 \delta_8 (M^{2,1} \subset M^{10,1})$

Near singularities of $\mathcal{M}^{(K3)}_{\text{EINSTEIN}}$,

The K3 becomes singular and has codim 4 orbifold singularities. We can model these on orbifold singularities of the form $\mathbb{R}^4/\Gamma_{ADE}$ where Γ_{ADE} is a finite subgroup of $SU(2) \subset SO(4)$ acting on $\mathbb{C}^2 \cong \mathbb{R}^4$ in the fundamental rep$^{\underline{n}}$.

The flat metric on $\mathbb{R}^4/\Gamma_{ADE}$ admits a desingularisation with a smooth hyperkähler metric

π : Smooth $\widetilde{\mathbb{R}^4/\Gamma_{ADE}} \longrightarrow$ Singular, flat $\mathbb{R}^4/\Gamma_{ADE}$

$\pi^{-1}(0)$ is a set of $rk(A\text{-}D\text{-}E)$ S^2's which intersect according to the Dynkin diagram of A-D-E.

$H_2(\widetilde{\mathbb{R}^4/\Gamma_{ADE}}, \mathbb{Z})$ = Root lattice of ADE

Intersection form = $-$ Cartan matrix

$$A_\alpha = \int_{\Sigma_\alpha \subset H_2\left(\widetilde{\mathbb{R}^4/\Gamma_{ADE}}\right)} C_3$$ are a set of

$rk(ADE)$ $U(1)$ gauge fields. The gauge group $U(1)^{rk(ADE)}$ is naturally identified with $T(ADE\text{ group})$.

max torus

Moreover, since $d \star G_4 = q_2 \delta_8(M^{2+1})$,

M2-branes wrapped on the $\Sigma_\alpha \subset H_2$

are like charged particles in $\mathbb{R}^{6,1}$:

$$d\star G_4 = \tilde{\Sigma}_\alpha \wedge d \star_{6+1} dA_\alpha = q_2 \delta_6(P^+) \wedge [\Sigma_\alpha]$$

$\tilde{\Sigma}_\alpha$ = Poincaré dual of Σ_α.

Because $H_2\left(\widetilde{\frac{\mathbb{R}^4}{\Gamma_{ADE}}}\right)$ = root lattice ADE,

These M2-branes, plus the rk(ADE) "photons"

have charges of the adjoint rep^n of ADE.

We also get $3 \times \mathrm{rk}(ADE)$ moduli fields from the moduli space of Einstein metrics.

$$\int_{\Sigma_\alpha} \omega_I \sim \phi_{I\alpha} \qquad I = 1,2,3$$

The wrapped M2-branes are "BPS" states whose masses are <u>exactly</u> given by the volumes of the exceptional S^2's

ie $\quad \mathrm{Mass}_2 = |\vec{\phi}_{I\alpha}|$

$\therefore$ At the origin of moduli space, we have a copy of the adjoint rep^n of ADE which is MASSLESS.

- The whole system is also supersymmetric
- How do we describe this?

☆ At the two derivative level, $\exists$ a unique supersymmetric Lagrangian theory in $\mathbb{R}^{6,1}$ with non-Abelian gauge symmetry: 6+1 d Super Yang-Mills.

∴ The physics of M-theory on

$$\left(\frac{\widetilde{\mathbb{R}^4}}{\Gamma_{ADE}} \times \mathbb{R}^{6,1}, \; \omega_I\right) \text{ is described}$$

H-K

by (6+1)d Super-Yang-Mills theory with $G =$ A-D-E gauge symmetry:

$$\mathcal{L}^{6+1d} = -\frac{1}{4}\,\mathrm{tr}\, F_A * F_A + i\bar{\lambda}\,\not{D}_A \lambda + \sum_I D_A \Phi_I * D_A \Phi_I - \mathrm{tr}\,[\phi_I, \phi_J]^2$$

$$A \in \Omega^1(\mathrm{Lie}(ADE))$$

$$\Phi_I \in \Omega^0(\mathrm{Lie}(ADE) \otimes \mathbb{R}^3)$$

$$\lambda \in \underset{\substack{\uparrow \\ \text{Spin}}}{S}(\mathrm{Lie}(ADE) \otimes \mathbb{C}^2)$$

$\exists$ $SU(2)$ global symmetry $(SU(2)_R)$

The $SO(3) \subset SU(2)$ rotates the HK ω_I and the 3 Φ_I's

The λ's take values in the fundamental of $SU(2)$

The vacua are described by the set of

$$\Phi_I \in \Omega^0(\mathfrak{Lie}(ADE) \otimes \mathbb{R}^3)$$

with $[\phi_I, \Phi_J] = 0$

modulo gauge transforms.

ie a <u>local</u> description of a flat <u>ADE</u> connection on T^3.

[commuting triple]

This is exactly the moduli space of hyperkähler metrics on $\widetilde{\mathbb{R}^4 / \Gamma_{ADE}}$. (Kronheimer)

To obtain an identity connected flat A-D-E connection on T^3, one simply exponentiates the 3 ϕ_I's

$$g_I = e^{i\Phi_I} \qquad \left(e^{i\int_\Sigma \omega_I}\right)$$

Non-Abelian symmetry in M-theory on G_2-manifolds, X.

One can consider codim 4 singularities in X.
These are supported on 3d subspaces $Q \subset X$.
Nice local model: Bryant-Salamon metric on $S(S^3) \cong \mathbb{R}^4 \times S^3$

This $Hol(G_2)$ metric has $SU(2)^3$ symmetry.

There is an associative

$\{0\} \times S^3 \subset \mathbb{R}^4 \times S^3$.

One can take a quotient by Γ_{ADE}, $S(S^3)/\Gamma_{ADE}$, which fixes the associative S^3 ie X modelled by $\frac{\mathbb{R}^4}{\Gamma_{ADE}} \times S^3$.

The volume of the associative S^3 is a free parameter.

In the large volume limit, the manifold gets flatter and flatter and looks more and more like $\mathbb{R}^4/\Gamma_{ADE} \times \mathbb{R}^3$ locally.

$$\mathrm{Vol}(S^3) \sim \frac{1}{g^2}$$

charge or gauge coupling $\longrightarrow g$

ie weak coupling limit in gauge theory!

So we can describe the physics of the singularity by taking the 6+1 d Super Yang-Mills on $\{0\} \times S^3$, and any suitably large associative Q supported at a codim 4 singularity.

Expanding in "harmonics",
keeping only zero modes and integrating over Q, we find that:

low energy physics of a codim 4 ADE singularity on Q
is given by
3+1 d Super Yang-Mills theory
plus $b_1(Q)$ "chiral" multiplets
$(\Omega^0(\mathrm{Lie}(ADE)),\ S^+(\mathrm{Lie}(ADE)))$

Actually, precise statement is chiral multiplets are the moduli space of complex flat $ADE_{\mathbb{C}}$ connections on Q.

Essentially, when we consider the 6+1 d Super Yang-Mills theory on $Q \times \mathbb{R}^{3,1} \subset X_7 \times \mathbb{R}^{3,1}$, the <u>three</u> scalars of the 6+1 d theory in flat spacetime, Φ_I, <u>become</u> <u>components</u> of a 1-form, Φ on Q. $\Phi \in \Omega^1(Q, \mathcal{L}ie(ADE))$

The components of the gauge field A on Q then combine with Φ to give a <u>complex</u> ADE connection

$$\mathcal{A}_s = A + i\,\Phi$$

The "BPS" or supersymmetry conditions then assert that $F_{\mathcal{A}_s}$ is flat and $d_A {*} \Phi = 0$

i.e., stable, complex, flat connections

One expects to find situations in which there is a coassociative fibration over Q, where the fibers are diffeomorphic to the $\widetilde{\mathbb{R}^4/\Gamma_{ADE}}$.
(cf S. Donaldson)

If we assume the generic fiber is smooth, then we have a $U(1)^{rk(ADE)}$ gauge theory on Q

There are rk(ADE) complex gauge fields A^α_4 which are essentially the integrals of the complexified 3-form $C_3 + i\varphi$ over the 2-cycles in $\widetilde{\mathbb{R}^4/\Gamma_{ADE}}$

$$A^\alpha_4 \sim \int_{\Sigma_\alpha} (C_3 + i\varphi)$$

Note that for the Cartan directions in Lie(ADE), the one form Φ_α is both closed and co-closed and is presumably related to the 1-form discussed by Donaldson recently in "Adiabatic Co-associative Fibrations".

Chiral Fermions from Codim 7 Singularities

- We have seen that a key ingredient of the SM of particles, ie non-Abelian gauge fields, reside at codim 4 orbifold singularities.
- The other key ingredient are chiral fermions: fermions whose left and right handed components transform in different complex representations of G_{SM}.
- These have to arise from higher codimension singularities.

- One expects this to be codim 7, because, lower codimension singularities will effect-ively give L and R spinors in the same rep$^{\underline{n}}$ of G_{SM}.

- We will motivate this study via duality with the heterotic string on T^3

M theory $\approx$ Het String $E_8 \times E_8$

$$K3 \to X \to S^3 \qquad \mathrm{Hol}(G_2)$$

$$T^3 \to Z \to S^3 \qquad (\text{Calabi Yau})$$

Adiabatic fibration to give a duality between M theory on X and het string on Z.

Hol vector bundle required here

Chiral Fermions in Het String on Z

- Data for the background:

(Z, g) – Calabi-Yau ; $(E \to Z)$ $E_8 \times E_8$ hol vector bundle

A – a Hermitian Yang-Mills connection on E with $c_2(E) = c_2(TZ)$.

- Chiral fermions arise from zero modes of the Dirac operator on Z, coupled to the connection A.

$$\not{D}_A \psi = 0.$$

- We suppose Z is T^3-fibered over a large S^3, with small fibers, of length L.
- The Dirac equation on Z will have a contribution from $\left.\not{D}_A\Psi\right|_{T^3}$ ie the Dirac operator on the fibers, coupled to the connection restricted to the fibers.
- We do not have to worry about singular fibers in what follows!

- The restriction of the HYM eq^{ns} to the fibers, require the connection A to be flat on each fiber.
- This is expected from our discussion of M on $K3$ vs flat on T^3.
- If $\tilde{A}$ is a generic flat connection on T^3, $\not{D}_{\tilde{A}}\psi$ has no zero modes, hence will have eigenvalues of order $1/L$

- But $1/L \gg$ <u>any</u> of the other terms in $\not{D}_A \psi$ on Z, hence we require zero modes from the $\not{D}_A \psi|_{T^3}$ contribution.
- Zero modes of $\not{D}_A$ on T^3 arise precisely when A has <u>non-Abelian</u> commutant in $E_8 \times E_8$.
- These naturally occur at <u>isolated</u> points on the base S^3.

So chiral fermions are localised at pts on S^3 above which the T^3 has a connection with "enhanced symmetry".

- In M theory, these are precisely the K3 fibers with ADE singularities.

- Lets describe the local models for these codim 7 singularities.

M theory dual of a 3d family of flat $SU(2)$ connections on T^3, trivial at the origin is a 3d family of "Eguchi-Hanson" spaces in which the S^2 collapses at the origin to give an $A_1 = SU(2)$ singularity.

This 7d total space $\widetilde{\mathbb{R}^4/\mathbb{Z}_2} \to \underline{\underline{X_7 \to \mathbb{R}^3}}$ $\underline{\underline{\text{has}}}$ to

- have a holonomy G_2 metric
- have a singular point over $\{0\} \subset \mathbb{R}^3$

Ex 1: Bryant-Salamon Cone: $\mathbb{R}^+ \times \mathbb{CP}^3$

$$\widetilde{\mathbb{R}^4/\mathbb{Z}_2} \hookrightarrow \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^3$$

$\mathbb{R}^+ \times \mathbb{CP}^3$ is a 3d family of "Eguchi-Hanson" 4-mflds. At the origin of $\mathbb{R}^3$, the S^2 in $\widetilde{\mathbb{R}^4/\mathbb{Z}_2}$ collapses so the fiber over this point is singular, $\mathbb{R}^4/\mathbb{Z}_2$, and the total space has a conical singularity.

Exactly what was required!

Generalisation: a 3d family of flat $SU(N+1)$ connections whose generic commutant is $SU(N)\times U(1)$, enhanced to $SU(N+1)$ at $\{0\}\in\mathbb{R}^3$.

In M theory would correspond to

$$\underbrace{\mathbb{R}^4/\mathbb{Z}_{N+1}}\hookrightarrow X_7 \longrightarrow \mathbb{R}^3$$

where generic fiber has A_{N-1} singularity, enhanced to A_N at $\{0\}\in\mathbb{R}^3$.

(BSA, Witten)

Ex2 Conjectural Hol(G_2) metric on

$$\mathbb{R}^+ \times \mathbb{WCP}^3_{NN11} :$$

$$\mathbb{R}^4/\mathbb{Z}_{N+1} \longrightarrow \mathbb{R}^+ \times \mathbb{WCP}^3_{N,N,1,1} \longrightarrow \mathbb{R}^3$$

↑ fibers are a partial resolution of $\mathbb{R}^4/\mathbb{Z}_{N+1}$ along the "N-th node"

ie each fiber has an $\mathbb{R}^4/\mathbb{Z}_N$ orbifold singularity and a non-zero 2-sphere. At the origin in $\mathbb{R}^3$ the fiber is $\dfrac{\mathbb{R}^4}{\mathbb{Z}_{N+1}}$.

In general, expect Hol(G_2) metrics on:

1) Start with any flat orbifold

$$\mathbb{R}^4/\Gamma_{ADE} \quad \text{with } rk(ADE) = k+1$$

2) The nodes on the "boundary" of the ADE Dynkin diagram correspond to 2-spheres in partial resolutions of $\mathbb{R}^4/\Gamma_{ADE}$. Each of these nodes breaks ADE to a rk(k) subgroup

- Each S^2 has three associated moment maps $(\underline{\mu})$ in Kronheimer's construction
- So we get a 3d family of partially resolved ALE spaces

$$\widetilde{\mathbb{R}^4/\Gamma_{ADE}} \longrightarrow \left(\widetilde{\mathbb{R}^4/\Gamma_{ADE}}, \underline{\mu}\right) \longrightarrow \mathbb{R}^3$$

where the fibers each have an $\mathbb{R}^4/\Gamma_G$ orbifold singularity.

" At the origin, the fiber becomes

$$\frac{\mathbb{R}^4}{\Gamma_{ADE}} \quad \text{of} \quad r K(K+1).$$

These 7d spaces are conjectured to have holonomy G_2 metrics.

(BSA, Witten)

Type IIA limit:

$$S^1 \to M^{10,1} \longrightarrow M^{9,1}$$

If $M^{10,1}$ collapses to $M^{9,1}$ (in a reasonable way), we get Type IIA superstring theory on $M^{9,1}$!

Let $\underset{g}{M^{9,1}} \equiv \underset{g_Z}{Z} \underset{+}{\times} \underset{flat}{\mathbb{R}^{3,1}}$

If we make g_Z a Calabi-Yau metric,

Then $X_7 = S^1 \times Z$.

In this case the $\mathcal{L}^{i\!d}$ we get ~~cannot~~ describe quarks and leptons, even if Z is singular.

If Z admits an anti-holomorphic involution, things are more interesting,
σ
because the S^1 must now vary over

$Z/(1,\sigma)$.

If L is fixed pt set of σ, $L \equiv Z_\sigma$
then L is "magnetically charged"
−ve
wrt the $U(1) = S^1$ local symmetry.
ie a monopole localised on L.

In fact the magnetic charge is "negative" $-4[L]$ in $H_3(Z,\mathbb{Z})$.

Gauss' Law requires additional +ve sources of charge to cancel this.

These are D6-branes on sLag 3-cycles $[L_i]$ whose $\sum_i n_i[L_i] = 4[L]$. All of these are sLag with <u>same</u> phase.

There are <u>exact</u> Type IIA solutions of this kind which are known.

These must correspond to singular holonomy G_2 spaces.

Near one of the L_i, the model is something like a fibration of Taub-NUT spaces over L_i with very small S^1's

Near L it is a family of Atiyah-Hitchin manifolds over L.

For the very special case of

$$Z = T^3 \times T^3, \quad X_7 = K3 \times T^3$$

$$\sigma = (-1, 1)$$

The collapsed limits were worked out by Lorenzo Foscolo (2016).

- Another special case:

$$X_7 = \frac{Z \times S^1}{\mathbb{Z}_2} \qquad \mathbb{Z}_2 \quad \underset{(\sigma, -1)}{Z \times S^1 \to Z \times S^1}$$

But more generally...

General picture / conjecture:

If $(Z, \omega, \Omega, \sigma)$ is a compact Calabi Yau 3-fold with anti-hol involution σ and L_i are a set of sLagrangians $\subset Z$ all with the same phase as $L \equiv Z|_\sigma$ and $\exists$ +ve integers s.t. $\sum_i n_i [L_i] = 4[L]$ in $H_3(\cancel{Z}, \mathbb{Z})$, then $\exists$ a G_2 holonomy 'metric' on

$$S^1 \longrightarrow X_7 \longrightarrow \frac{Z - \{L, L_i\}}{\mathbb{Z}_2}$$

with degenerations of the S^1 along the $\{L, L_i\}$ modelled on Taub-NUT and Atiyah-Hitchin

Note: in general the L_i's intersect in codimension six in Z.

At these points, X_7 will have codimension seven singularities of the kind discussed earlier.

Ex 3 Bryant-Salamon Cone $\mathbb{R}^+ \times \mathbb{CP}^3$

(Atiyah-Witten) The G_2-holonomy cone metric on $\mathbb{R}^+ \times \mathbb{CP}^3$ admits a $U(1)$ action (isometric) whose quotient is $\mathbb{R}^6$

$$U(1) \longrightarrow \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^6$$

The fixed pt set of $U(1)$ on $Z = \mathbb{R}^6$ is $\mathbb{R}^3 \cup \mathbb{R}^3 \subset \mathbb{R}^6$ which intersect at the origin.

Summary:

The IIA and Heterotic limits of M-theory imply the existence of singular, compact G_2-holonomy spaces with S^1 and K3 fibrations. Codim 4 and 7 singularities.

The string picture suggests a construction of such G_2 manifolds.