PLURIPOTENTIAL THEORY ON CALIBRATED MANIFOLDS

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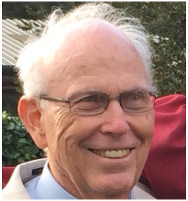
with Reese Harvey



General Comments

A Calibration

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$\swarrow$

ϕ-submanfiolds

rectifiable ϕ-currents

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Misha Verbitsky, Semyon Alesker

Elementary Concepts and Notation

1. **Definition:** A **Calibrated manifold** is a pair (X, ϕ) where
   - X is a riemannian manifold, and
   - $\phi \in \mathcal{E}^p(X)$ is a parallel p-form ($\nabla\phi \equiv 0$) with

$$\text{comass}(\Phi) \stackrel{\text{def}}{=} \sup\{\phi(\xi) : \xi \text{ is a unit simple p-vector}\} = 1.$$

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Tangent p-planes P corresponding to $\xi \in G(\phi)$ are called **ϕ-planes**.

ϕ-Submanifolds and Currents

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where $\|T\|$ is a Radon measure on X and $\overrightarrow{T}$ is a field of unit simple p-vectors

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Definition: T is a **positive ϕ-current** if $\vec{T}_x \in$ *Convex Hull of* $G(\phi)$, $\|T\|$-a.a x.

Fundamental Results:

THEOREM 1. *A positive ϕ-current T is homologically mass-minimizing.*

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THEOREM 3. (Almgren Regularity). *Any d-closed ϕ-recitfiable current is a smooth submanifold (with integer multiplicities) outside a closed subset of Hausdorff codimension-2.*

Some Questions:

When (X, ω) is Kähler and $\phi = \frac{1}{p!}\omega^p$, the positive ϕ-currents are the classical positive (p, p)-currents of Lelong and the French school.

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• Tangents: **Tangents always exist** and are again positive ϕ-currents.

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(The Bombieri-Siu Theorem)

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Suppose T is a **rectifiable ϕ-current** with $dT = 0$.

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- **Stratification:** Does $\text{supp}(T)$ have a Whitney stratification?

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• **Analyticity:** Is supp(T) a real analytic variety?

• **Stratification:** Does supp(T) have a Whitney stratification? Can the stratification results of Cheeger-Naber be made stronger in the setting of classical calibrations?

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Define the d^{ϕ}**-operator**

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Proposition 1.

$$\boxed{dd^\phi u = \mathcal{L}_{\nabla u}\phi = D_{\mathrm{Hess}\, u}\phi}$$

where for $A \in \mathrm{Sym}^2(\mathbb{R}^n)$, considered as a map $A : \mathbb{R}^n \to \mathbb{R}^n$,

$$D_A : \Lambda^p\mathbb{R}^n \to \Lambda^p\mathbb{R}^n$$

is its extension as a derivation.

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Note: The RHS does **not** depend on an orientation for P.

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$$u \text{ is quasi } \phi\text{-psh} \qquad \iff \qquad \mathrm{tr}\left\{ \left(\tfrac{1}{p}\mathrm{Id} + \mathrm{Hess}\, u \right)\Big|_P \right\} \ \geq \ 0 \quad \text{for all } \phi\text{-planes } P.$$

Another View of ϕ-Plurisubharmonicity

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Proposition 2 says that for $u \in C^2(X)$

$$\mathbf{u} \text{ is } \phi\text{-plurisubharmonic} \quad \Longleftrightarrow \quad \mathrm{Hess}_{\mathbf{x}}\mathbf{u} \in \mathcal{P}(\phi) \quad \forall\, \mathbf{x} \in \mathbf{X}.$$

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$$\mathrm{PSH}(X, \phi) \equiv \text{the set of these functions.}$$

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(2) $\mathrm{PSH}(X, \phi)$ is closed under uniform limits and decreasing limits.

(3) For any family $\mathcal{F} \subset \mathrm{PSH}(X, \phi)$, locally bounded above,

$$U \equiv \left\{ \sup_{u \in \mathcal{F}} u \right\}^* \in \mathrm{PSH}(X, \phi)$$

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So the ϕ-harmonic functions are direct generalizations of (smooth) solutions to Monge-Ampère.

Basic Example.

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For any calibration $\phi \in \Lambda^p \mathbb{R}^n$ on euclidean space,

$$u(x) = -\frac{1}{|x|^{p-2}} \quad \text{is } \phi\text{-harmonic when} \quad p \geq 3$$

and

$$u(x) = \log |x| \quad \text{is } \phi\text{-harmonic if} \quad p = 2$$

(outside the origin).

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Tangents always exist.

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Theorem. Suppose $p \geq 3$ and $G(\phi)$ satisfies a certain transitivity condition (which holds for all the classical calibrations). Then

$$\lim_{r \downarrow 0} u_r(x) \;=\; -\frac{\Theta}{|x|^{p-2}}$$

for some constant $\Theta \geq 0$.

This fails completely in Kähler case (Kiselman).

Question.

Does strong uniqueness of tangents continue to hold for calibrations on special holonomy manifolds?

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(2) $-u$ satisfies the *dual* subequation

$$\widetilde{\mathcal{P}}(\phi) \equiv \sim \{-\mathrm{Int}\mathcal{P}(\phi)\}$$

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Theorem. (The Dirichlet Problem for ϕ-Harmonic Functions). Suppose that Ω has a strictly ϕ-plurisubharmonic defining function. Then for every $\varphi \in C(\partial\Omega)$, there exists and unique $u \in C(\overline{\Omega})$ such that

(1) u is ϕ-harmonic on Ω, and

(2) $u\big|_{\partial\Omega} = \varphi$.

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(2) $u\big|_{\partial\Omega} = \varphi$,

(3) $\exists\ C > 0$ such that for $j = 1, ..., k$

$$-\frac{\Theta_j}{|x - x_j|^{p-2}} - C \quad \leq \quad u(x) \quad \leq \quad -\frac{\Theta_j}{|x - x_j|^{p-2}} + C$$

Problem

In the Special Lagrangian, Associative, Co-Associative and Cayley Cases

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Do the smooth ϕ-harmonic functions satisfy a natural polynomial differential equation?

The First Relation of PSH(X, ϕ) to ϕ-Submanifolds

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Restricition Theorem. For any $u \in \mathrm{PSH}(X, \phi)$ and any ϕ-submanifold $M \subset X$, the restriction

$$u\big|_M \text{ is } \Delta_M\text{-subharmonic on } M$$

(or $\equiv -\infty$).

ϕ-Convexity

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Definition. For $K \subset\subset X$, the **ϕ-convex hull** of K is the set $\widehat{K}$ of points $x \in X$ such that

$$\mathbf{u}(\mathbf{x}) \leq \sup_{\mathbf{K}} \mathbf{u}$$

for all (smooth) $u \in \mathrm{PSH}(X, \phi)$.

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Definition. The calibrated manifold X is **ϕ-convex** if

$$K \subset\subset X \qquad \Rightarrow \qquad \widehat{K} \subset\subset X.$$

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Theorem.

$$X \text{ is } \phi\text{-convex} \quad \iff$$

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If, in addition, X admits any **strictly** ϕ-psh function, the exhaustion can be taken to be **strictly** ϕ-psh, and X is called **strictly ϕ-convex**.

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These are the **ϕ-analogues of Stein manifolds**

The Free Dimension of ϕ

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Theorem. If X is strictly ϕ-convex, then

X has the homotopy-type of a complex of dimension $\leq \mathrm{fd}(\phi)$.

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Theorem If $Y \subset X$ is ϕ-free, every tubular neighborhood

$$\Omega_\epsilon \equiv \{x \in X : \mathrm{dist}(x, Y) < \epsilon\}$$

for $\epsilon > 0$ sufficiently small, is strictly ϕ-convex.

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Characterizing Boundaries of ϕ-submanifolds.

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Theorem 1. Suppose (X, ϕ) is a non-compact, strictly ϕ-convex manifold, and that ϕ is exact. Then given a $(p-1)$-dimensional current $S \in \mathcal{E}'_{p-1}(X)$ with compact support,

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$\int_S \alpha \ \geq \ 0$ for all smooth p-forms $\alpha \in \mathcal{E}^p(X)$ such that $d\alpha(\xi) \geq 0 \quad \forall\, \xi \in G(\phi)$.

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Necessary and sufficient conditions involve local geometry (maximal complexity) of the boundary together with moment conditions.

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If S is the smooth boundary $S = \partial T$ of a Special Lagrangian submanifold T in $\mathbb{C}^n$, then

$$\int_S \psi \;=\; 0 \qquad \forall\, \psi \in \mathcal{E}^n \;\; \textbf{s.t.} \;\; d\psi \in \mathcal{I}.$$

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Theorem. (Lei Fu). For $n \geq 3$ the Moment Condition is not sufficient to characterize boundaries of Special Lagrangian submanifolds. There exist (explicitly stated) additional necessary conditions.

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General Question: Which rational classes in $\mathbb{H}_p(\phi)$

are represented by rational combinations of rectifiable ϕ-cycles?

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Refined Question: Suppose $\alpha \in \mathbb{H}_p(\phi) \cap \widetilde{H}_p(X;\mathbb{Z})$

Is it true that for some large integer N,

the class $\alpha + N[*\phi]$ (or some multiple)

is represented by a rectifiable ϕ-cycle?

An Idle Question

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On any calibrated manifold (X, ϕ), there is a natural second-order equation given by :

$$dd^{\phi}u \wedge dd^{*\phi}u / *1 \;\equiv\; L(u)$$

and a “quasi” analogue

$$(\phi + dd^{\phi}u) \wedge (*\phi + dd^{*\phi}u) / *1 \;\equiv\; \widetilde{L}(u)$$

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Of particular interest is the Cayley calibration Φ since $\Phi = *\Phi$, and so

$$(\Phi + dd^{\Phi}u) \wedge (\Phi + dd^{*\Phi}u) \ = \ f(*1)$$

is something like a Cayley Calabi-Yau equation.