# M-theory, 4D gauge theories and New $G_2$-conifolds.

B. Acharya (ICTP+KCL)

Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics Workshop, 8-12 April 2019, KITP Santa Barbara

# Outline

- Review 4d gauge theories from $G_2$ orbifolds
- Bryant-Salamon (and quotients) in M theory
    - "Quantum Bryant-Salamon"
- ALC $G_2$ manifolds and Type IIA limits ($S^1$ collapsed)
- Hyperconifolds and Resolutions
- Foscolo-Haskins-Nordström theorem *
- New $G_2$-manifolds/orbifolds and 4d gauge theory interpretation

Based on work done with

- L. Foscolo
- M. Najjar (PhD student @ KCL, ICTP)
- E. Svanes

paper to appear

# Review: 4d gauge theories from codim-4 orbifold singularities

- M-theory on $X \times \mathbb{R}^{3,1}$

  $\uparrow$ $G_2$ holonomy metric $g_X$

- A codim-4 orbifold singularity supported on $\underline{M^3 \times \mathbb{R}^{3,1}} \subset X \times \mathbb{R}^{3,1}$

$\Updownarrow$

7d gauge theory on $M^3 \times \mathbb{R}^{3,1}$

$\downarrow$ Low energies $E \ll 1/\mathrm{diam}(M^3)$

* 4d gauge theory on $\mathbb{R}^{3,1}$

Codim-4 orbifolds singularities in a $G_2$-holonomy space are always of A-D-E type, modelled locally on

$$\frac{\mathbb{R}^4}{\Gamma_{ADE}} \times \mathbb{R}^3 \cong T(M^3) \subset T(X)$$

A-D-E is the gauge group supported along $M^3 \times \mathbb{R}^{3,1} \subset X \times \mathbb{R}^{3,1}$.

(cf BSA hep-th/9812205)

Note that the strength of gauge interactions, "$g$", is given by

$$\frac{1}{g^2} \sim \mathrm{Vol}(M^3),$$

so we usually want to consider <u>compact</u> $M^3$ in local models.

# Previously Studied LOCAL examples

- Locally flat e.g.'s $X = \dfrac{\left(\mathbb{R}^4/\Gamma_{ADE} \times T^3\right)}{F}$

  here $F$ is a finite, freely acting group

  (BSA, hep-th/9812205)

- Asymptotically Conical $G_2$-orbifolds.

  – Quotients of Bryant-Salamon AC examples

  $$\Rightarrow \quad \frac{S(S^3)}{\Gamma_{ADE}} \cong \mathbb{R}^4/\Gamma_{ADE} \times S^3$$

(BSA, hep-th/0011089; Atiyah-Maldacena-Vafa, Atiyah-Witten)
hep-th/0011256

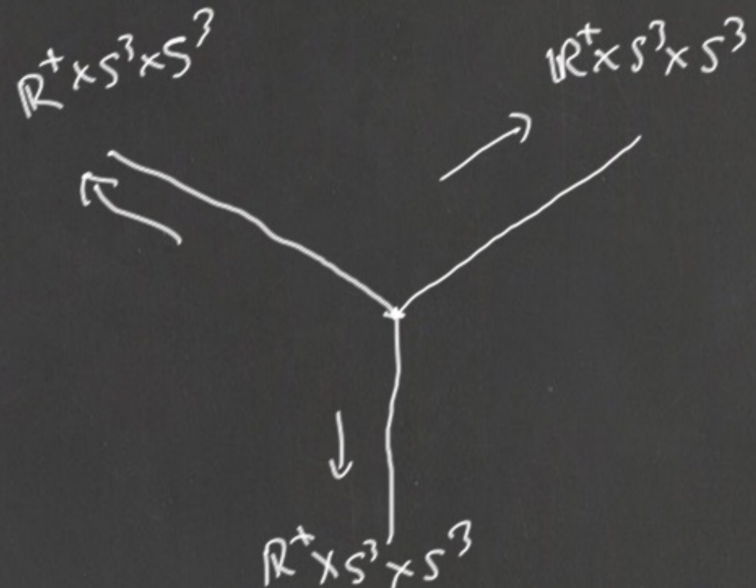
Here: we will study ALC examples

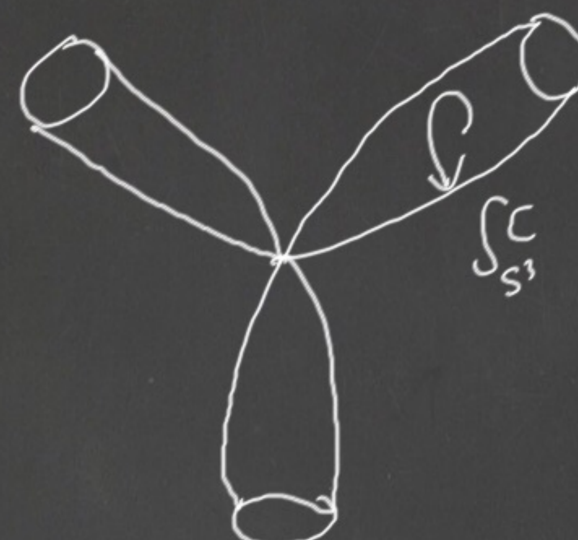
cf Mark Haskins talk.

$S^1$- bundles over AC CY 3-folds

# Classical Bryant-Salamon Moduli Space

Real $G_2$-moduli space $\longrightarrow$ Complexified in M-theory

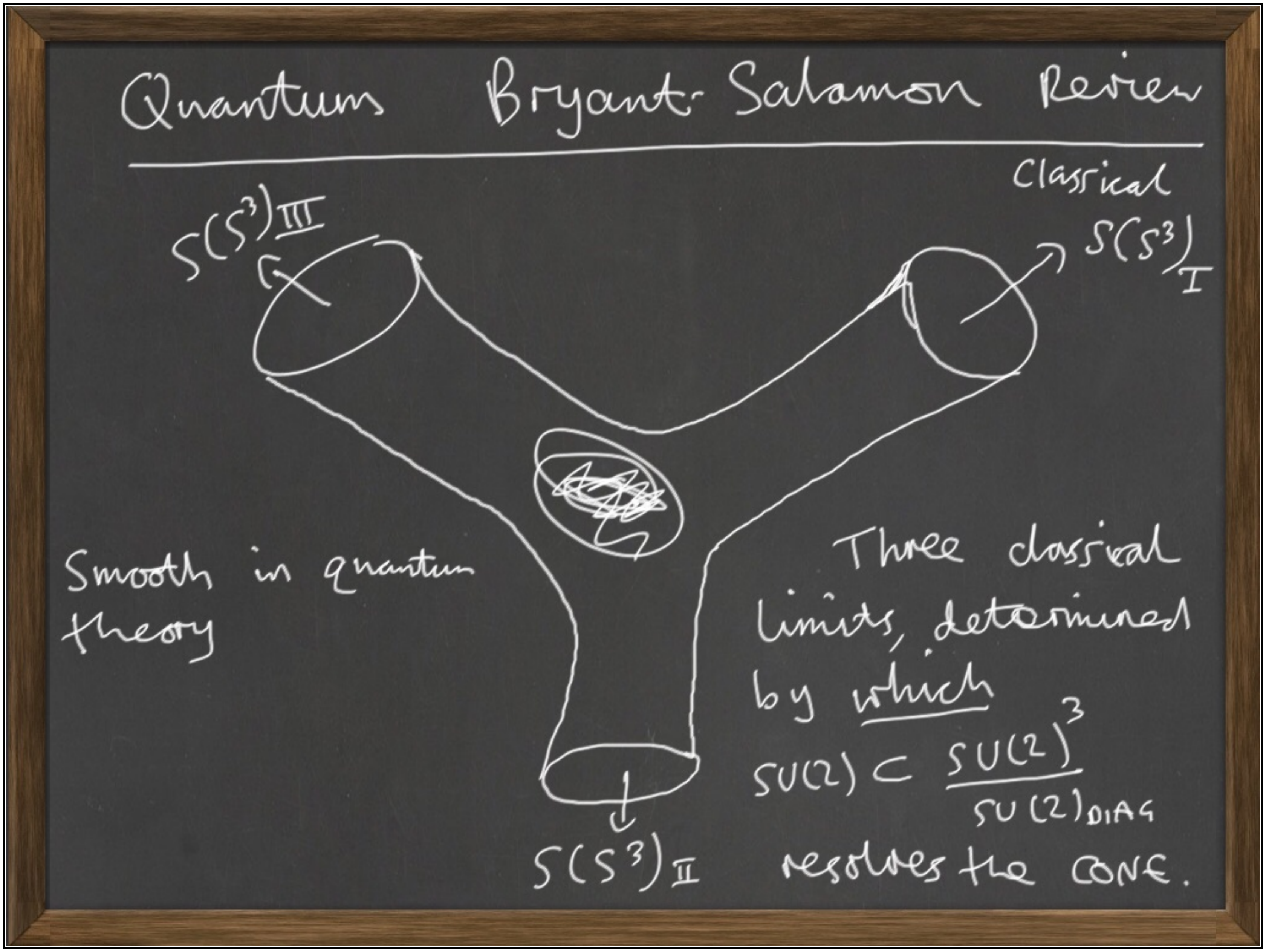


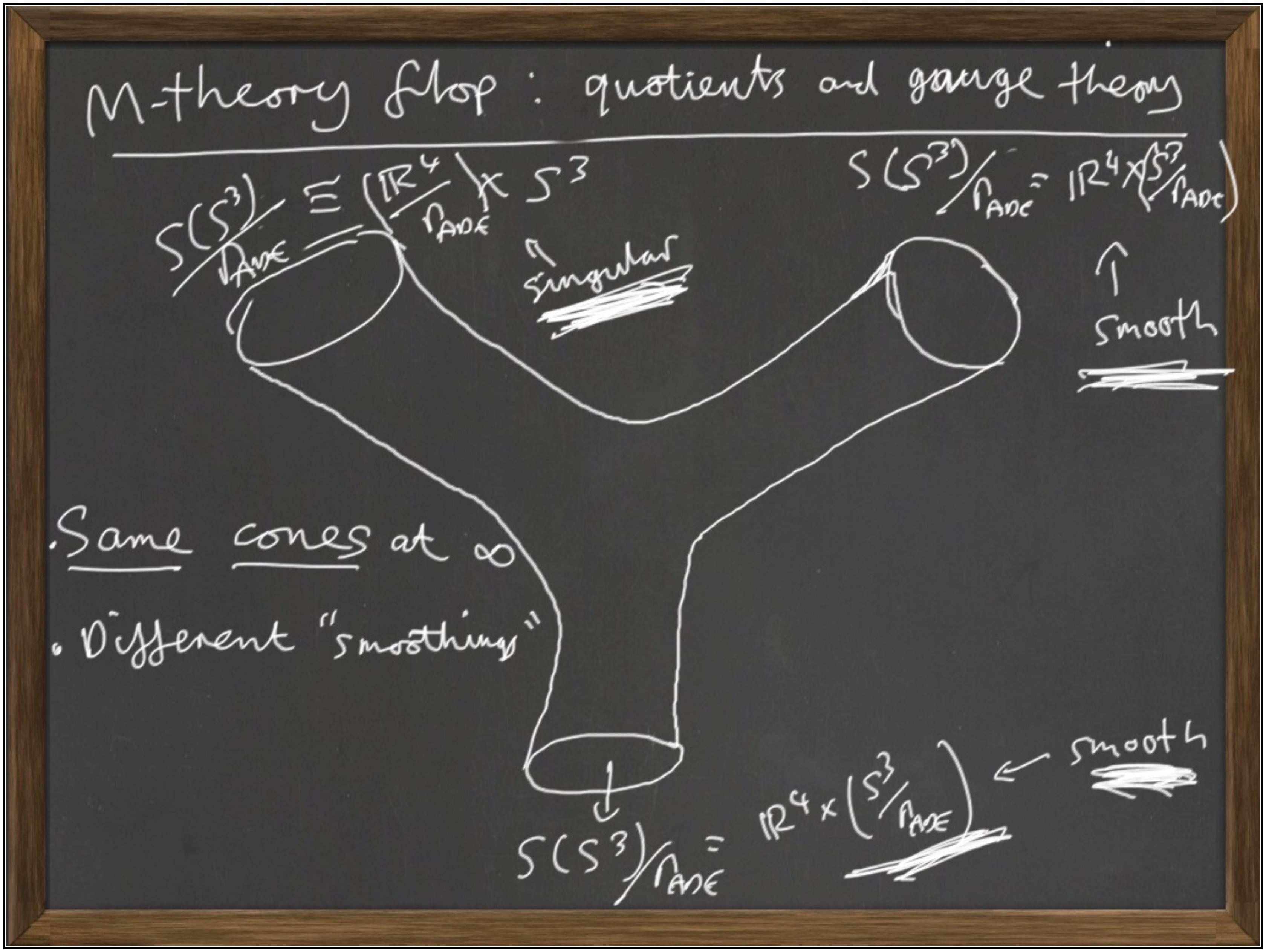




Three AC manifolds, filled in by a choice of $SU(2) \subset SU(2)^3$ at $\infty$

Complex moduli space

$$Z \sim \int_{S^3} (\varphi + iC)$$

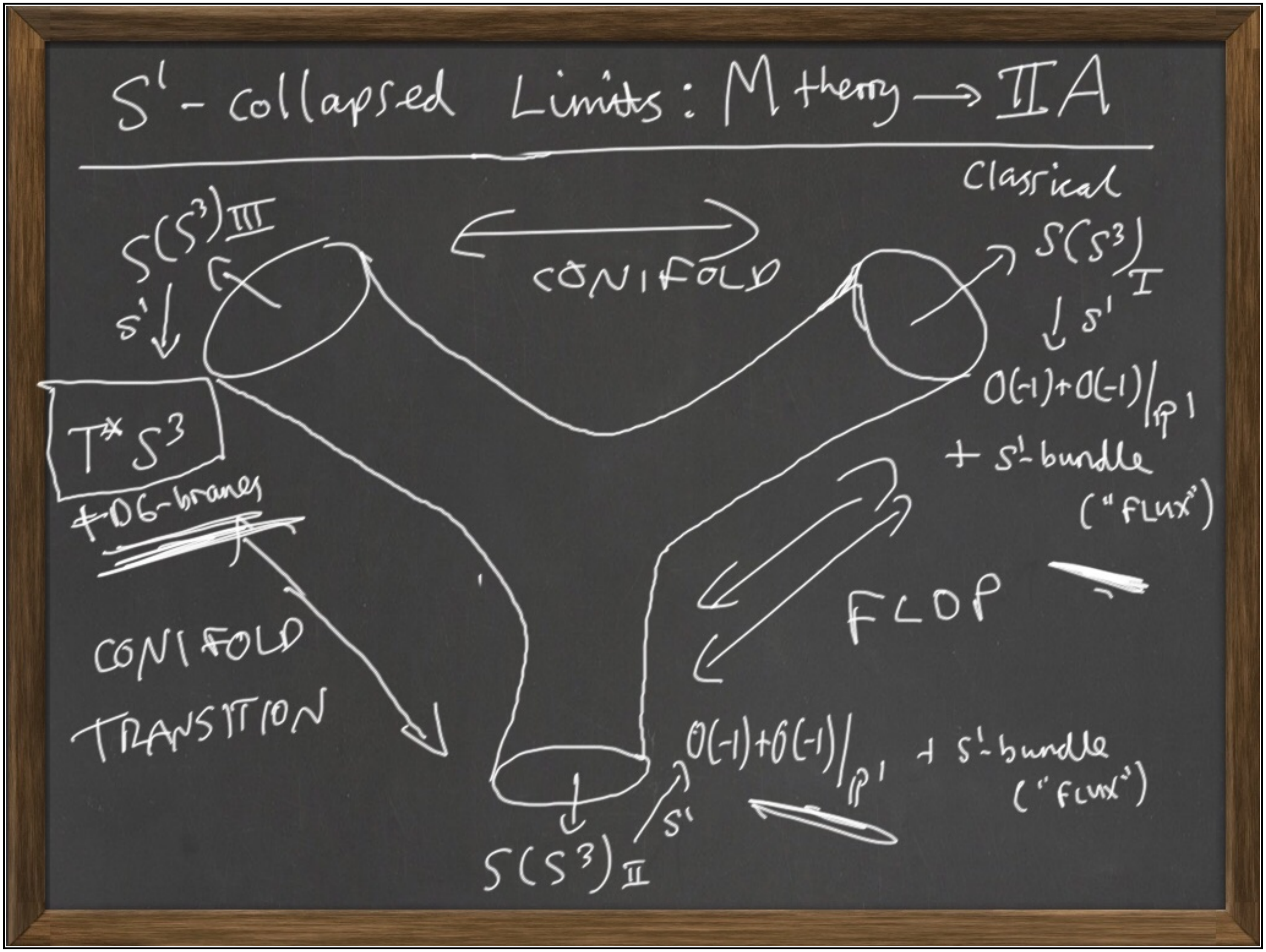

Quantum Bryant-Salamon Review
Classical
S(S³)_III
S(S³)_I
S(S³)_II
Smooth in quantum theory
Three classical limits, determined by which
SU(2) ⊂ SU(2)³ / SU(2)_DIAG
resolves the CONE.


M-theory flop : quotients and gauge theory
$S(S^3)/\Gamma_{ADE} \equiv (\mathbb{R}^4/\Gamma_{ADE}) \times S^3$
singular
$S(S^3)/\Gamma_{ADE} = \mathbb{R}^4 \times (S^3/\Gamma_{ADE})$
smooth
· Same cones at $\infty$
· Different "smoothings"
$S(S^3)/\Gamma_{ADE} = \mathbb{R}^4 \times (S^3/\Gamma_{ADE})$
smooth

## Type IIA limits

ie Examples which collapse to 6-mflds

- These generalise Bryant-Salamon, but instead of AC, they are ALC
  - At $\infty$, $S^1$-bundles over Calabi Yau Cone
- For metrics on $S(S^3)$ we have the $B_7$ and $D_7$ metrics (Brandhuber et al) (Cvetic et al) 2002 (See Mark Haskins talk)
- Have $SU(2)^2 \times U(1)$ symmetry instead of $SU(2)^3$


S¹ - collapsed Limits : M theory → IIA
Classical
S(S³) III
s¹
T* S³
+ D6-branes
CONIFOLD
CONIFOLD
TRANSITION
S(S³) I
s¹
O(-1)+O(-1)/P¹
+ S¹-bundle
("FLUX")
FLOP
O(-1)+O(-1)/P¹ + S¹-bundle
("FLUX")
S¹
S(S³) II

# New Examples Studied Here:

- $S^1$-bundles over resolutions of CY-cones
- CY-cones we consider: finite quotients of the conifold
- Conifold, $\mathcal{C} :\equiv \{ z_1^2+z_2^2+z_3^2+z_4^2=0 \mid z_i \in \mathbb{C}^4 \}$.
- $\dfrac{\text{Conifold}}{\Gamma}$ called Hyperconifolds (Rhys-Davies).
- Toric examples admit crepant resolutions with AC CY metrics ($\widetilde{\mathcal{C}/\Gamma}$)

* Foscolo-Haskins-Nordstrom Theorem: shows "$S^1$-collapsed $G_2$-mflds exist if $c_1(\mathcal{L}) \cup \omega = 0$"

To motivate our examples, consider the resolved conifold, $\mathcal{O}(-1)\oplus\mathcal{O}(-1)|_{\mathbb{P}^1} \equiv \tilde{\mathcal{C}}$

- As a manifold $\simeq \mathbb{R}^4 \times S^2$.
- Candelas-de la Ossa metric has $\mathbb{Z}_K$ symmetry with quotient $\left.\dfrac{\mathcal{O}(-1)\oplus\mathcal{O}(-1)}{\mathbb{Z}_K}\right|_{\mathbb{P}^1}$. $\mathbb{Z}_K$ acts only on fibers fixing $\{0\}\times\mathbb{P}^1$
- This is a collapsed limit of a $G_2$-holonomy metric on $\mathbb{R}^4/\mathbb{Z}_K \times S^3$ (as considered previously), plus a $c_1(\mathcal{L})=1$ line bundle on $S^2$.

$D_7$

- Replacing $c_1(\mathcal{L})|_{\mathbb{P}^1} = 1 \Rightarrow N$, we get:

$$\mathbb{R}^4/\mathbb{Z}_K \times S^3/\mathbb{Z}_N$$

# Comments on $X = \frac{\mathcal{O}(-1)\oplus\mathcal{O}(-1)|_{\mathbb{P}^1}}{\mathbb{Z}_K} \times S^1$

- Special case when $X$ has $SU(3)\times 1$ holonomy
- 4d gauge theory has $N=2$ supersymmetry (as opposed to $G_2$ hol $\Leftrightarrow N=1$ ")
- Physics described by "pure" $SU(K)$ $N=2$ supersymmetric gauge theory (solved by Seiberg-Witten theory).
- Classical moduli space $\cong$ Kähler cone of resolutions of $\frac{\mathcal{O}(-1)+\mathcal{O}(-1)|_{\mathbb{P}^1}}{\mathbb{Z}_K}$
- Quantum moduli space $\cong$ cplx str of mirror CY.

(Klem, Lerche, Mayr, Vafa)

## Comments continued

When $X$ is a non-trivial $S^1$-bundle over $\left( \dfrac{\mathcal{O}(-1) \oplus \mathcal{O}(-1)|_{\mathbb{P}^1}}{\mathbb{Z}_K}, \omega \right)$ ie has genuine hol $= G_2$ the $N=2$ moduli space gets lifted to a set of critical points of a superpotential,

- $W = \int c_1(\mathcal{L}) \wedge \omega \wedge \omega$ + Instantons

  ↑ Gromov-Witten Invariants

(Hori-Vafa
Aganagic, Marino, Vafa: presumably $W$ is computed by the "Topological Vertex".)

## $S^1$ bundles over Hyperconifolds

- At $\infty$ we consider $S^1$ bundles over $\mathbb{Z}_k$ quotients of the conifold.
- So, $S^1$ bundles over $\boxed{S^3/\mathbb{Z}_k \times S^2}$.
- Classified by pairs $(q, N)$
  - $q$ is a flat $S^1$ connection on $S^3/\mathbb{Z}_k$ ie if $\eta^k = 1$, $\eta^q$ is the connectin.
  - $N$ is $c_1(\mathcal{L})|_{S^2}$
- This speifies Type IIA vacuum at $\infty$.

# M-theory on $\mathbb{R}^4/\mathbb{Z}_K \times S^3/\mathbb{Z}_N$

- Considered by Tamar Friedmann (hep-th/0203256) for AC case (quotient of B-S)
- SU(k) gauge theory on $S^3/\mathbb{Z}_N$

* Here, ALC, pick a flat SU(k) connection on $S^3/\mathbb{Z}_N$ PLUS the N choices of M-theory circle at $\infty$ ($\equiv$ flat $S^1$-bundle)

# Vacua $= N \times$ # flat SU(k) connections on $S^3/\mathbb{Z}_N$.

# Examples

## * $(K=2, N)$ (Hosomichi-Page)

- $SU(N)$ flat connections on $S^3/\mathbb{Z}_2$ &
- Let $N = n_1 + n_2$
- Flat connection breaks $SU(N) \to SU(n_1) \times SU(n_2) \times U(1)$

## * Foscolo-Haskins-Nordström:

$M_{P,Q}$ are $S^1$ bundles over $\widetilde{\mathcal{O}(-1)+\mathcal{O}(-1)|_{\mathbb{P}^1}}/\mathbb{Z}_2 = \tilde{\mathcal{C}}/\mathbb{Z}_2$

- $H_2(\tilde{\mathcal{C}}/\mathbb{Z}_2) \cong \mathbb{Z}^2$ generated by two $S^2$'s. One base of conifold, one fiber
- $P, Q$ are degrees of $S^1$ bundle.
- Solving $c_1(\mathcal{L}) \cup \omega = 0$ gives ALC $G_2$ metrics on $M_{P,Q}$

- $P + Q = N$

eg if $N = 5$

| | |
|---|---|
| 5 | 0 |
| 4 | 1 |
| 3 | 2 |
| 2 | 3 |
| 1 | 4 |
| 0 | 5 |

- $SU(5)$ flat connections on $S^3/\mathbb{Z}_2$:

$(+++++)$ $SU(5)$

$(--+++)$ $SU(3)\times SU(2)\times U(1)$

$(----+)$ $SU(4)\times U(1)$

- $\times 2$ due to 2 flat $S^1$-bundles at $\infty$

$\Rightarrow$ 6 vacua arranged

So, $n_1 = P$ $n_2 = Q$.

# (4, 3) example

$S^1$-bundles over $\dfrac{\widetilde{\mathcal{O}(-1)+\mathcal{O}(-1)}\Big|_{\mathbb{P}^1}}{\mathbb{Z}_3} \cong M_{p_1 p_2 p_3}$

- Fix $\sum p_i = 4 \Rightarrow 15$ $S^1$-bundles (fluxes)

$c_1(\mathcal{L})$

| | | |
|---|---|---|
| 4 | 0 | 0 |
| 3 | 1 | 0 |
| 3 | 0 | 1 |
| 2 | 2 | 0 |
| 2 | 1 | 1 |
| 0 | 4 | 0 |
| 0 | 3 | 1 |
| 1 | 3 | 0 |
| 0 | 2 | 2 |
| 1 | 2 | 1 |

| | | |
|---|---|---|
| 0 | 0 | 4 |
| 1 | 0 | 3 |
| 0 | 1 | 3 |
| 2 | 0 | 2 |
| 1 | 1 | 2 |

# (4,3) example

- The 15 manifolds $M_{p_1 p_2 p_3}$, $\Sigma p_i = 4$ are in natural correspondence with Type IIA theory on $T^* S^3/\mathbb{Z}_3$ + 4 D6-branes on $\{0\} \times S^3/\mathbb{Z}_3$

- Five flat SU(4) connections on $S^3/\mathbb{Z}_3$:

$$\underset{SU(4)}{(1,1,1,1)}, \quad \underset{SU(3)\times U(1)}{(\lambda\lambda\lambda 1)}, \quad \underset{SU(3)\times U(1)}{(\bar\lambda,\bar\lambda,\bar\lambda,1)}, \quad \underset{U(1)^2\times SU(2)}{(\lambda\bar\lambda 1 1)}, \quad \underset{SU(2)\times SU(2)}{(\lambda\lambda,\bar\lambda\bar\lambda)}$$

- Three choices of flat $U(1)$ connection at $\infty$

Three $\times$ Five $=$ 15

# (4, 3) example

$S^1$-bundles over $\frac{\mathcal{O}(-1)+\mathcal{O}(-1)}{\mathbb{Z}_3}\Big|_{\mathbb{P}^1} \cong M_{p_1 p_2 p_3}$

- Fix $\sum p_i = 4 \Rightarrow 15$ $S^1$-bundles (fluxes)

1)

| $c_1(\mathcal{L})$ | Flat SU(4) |
|---|---|
| 4 0 0 | $SU(4)$ |
| 3 1 0 | $SU(3)\times U(1)$ |
| 3 0 1 | $SU(3)\times U(1)$ |
| 2 2 0 | $SU(2)\times SU(2)$ |
| 2 1 1 | $SU(2)\times U(1)^2$ |

2)

| $c_1(\mathcal{L})$ | Flat SU(4) |
|---|---|
| 0 4 0 | $SU(4)$ |
| 0 3 1 | $SU(3)\times U(1)$ |
| 1 3 0 | $SU(3)\times U(1)$ |
| 0 2 2 | $SU(2)\times SU(2)$ |
| 1 2 1 | $SU(2)\times U(1)^2$ |

3)

| $c_1(\mathcal{L})$ | Flat SU(4) |
|---|---|
| 0 0 4 | $SU(4)$ |
| 1 0 3 | $SU(3)\times U(1)$ |
| 0 1 3 | $SU(3)\times U(1)$ |
| 2 0 2 | $SU(2)\times SU(2)$ |
| 1 1 2 | $SU(2)\times U(1)^2$ |

# Overall picture

$\frac{\mathcal{O}(-1)+\mathcal{O}(-1)|_{\mathbb{P}^1}}{\mathbb{Z}_k} \cong \mathbb{R}^4/\mathbb{Z}_k \times S^2$ and resolutions

$\Updownarrow$

$\mathcal{N}=2$ susy $SU(k)$ gauge theories

$\exists$ $\underline{\mathcal{N}=1}$ susy deformations of these with superpotentials described by "$\int_Z c_1(J) \wedge \omega^2$"

$\Updownarrow$

ALC $G_2$-manifolds (and orbifolds)

Conjecture: quantum moduli space described by Gromov-Witten prepotential of $\frac{\mathcal{O}(-1)+\mathcal{O}(-1)|_{\mathbb{P}^1}}{\mathbb{Z}_k}$

IIA on $\left.\widetilde{\frac{\mathcal{O}(-1)\oplus\mathcal{O}(-1)}{\mathbb{Z}_K}}\right|_{\mathbb{P}^1} + \sum_{1}^{K} p_i = N$

- $\infty = S^3/\mathbb{Z}_K \times S^2 \Rightarrow$ K flat M-theory $S^1$'s.

  $\Updownarrow$ Conifold transition

- IIA on $T^* S^3/\mathbb{Z}_K$ + N D6-branes

  SU(N) gauge theory on $S^3/\mathbb{Z}_K$.

So, the $S^1 \longrightarrow M_{p_1 \dots p_K} \longrightarrow \left.\widetilde{\frac{\mathcal{O}(-1)+\mathcal{O}(-1)}{\mathbb{Z}_K}}\right|_{\mathbb{P}^1}$

$G_2$ manifolds are M-theory "gravity duals" of IIA on $T^* S^3/\mathbb{Z}_K$ + N D6-branes

(7,3) example

| | | |
|---|---|---|
| 5 | 0 | 0 |
| 4 | 1 | 0 |
| 4 | 0 | 1 |
| 3 | 2 | 0 |
| 3 | 0 | 2 |
| 3 | 1 | 1 |
| 2 | 2 | 1 |
| 2 | 1 | 2 |
| 2 | 3 | 0 |
| 2 | 0 | 3 |
| 1 | 4 | 0 |
| 1 | 0 | 4 |
| 1 | 3 | 1 |
| 1 | 1 | 3 |
| 1 | 2 | 2 |
| 0 | 5 | 0 |
| 0 | 0 | 5 |

| | | |
|---|---|---|
| 0 | 1 | 4 |
| 0 | 4 | 1 |
| 0 | 2 | 3 |
| 0 | 3 | 2 |

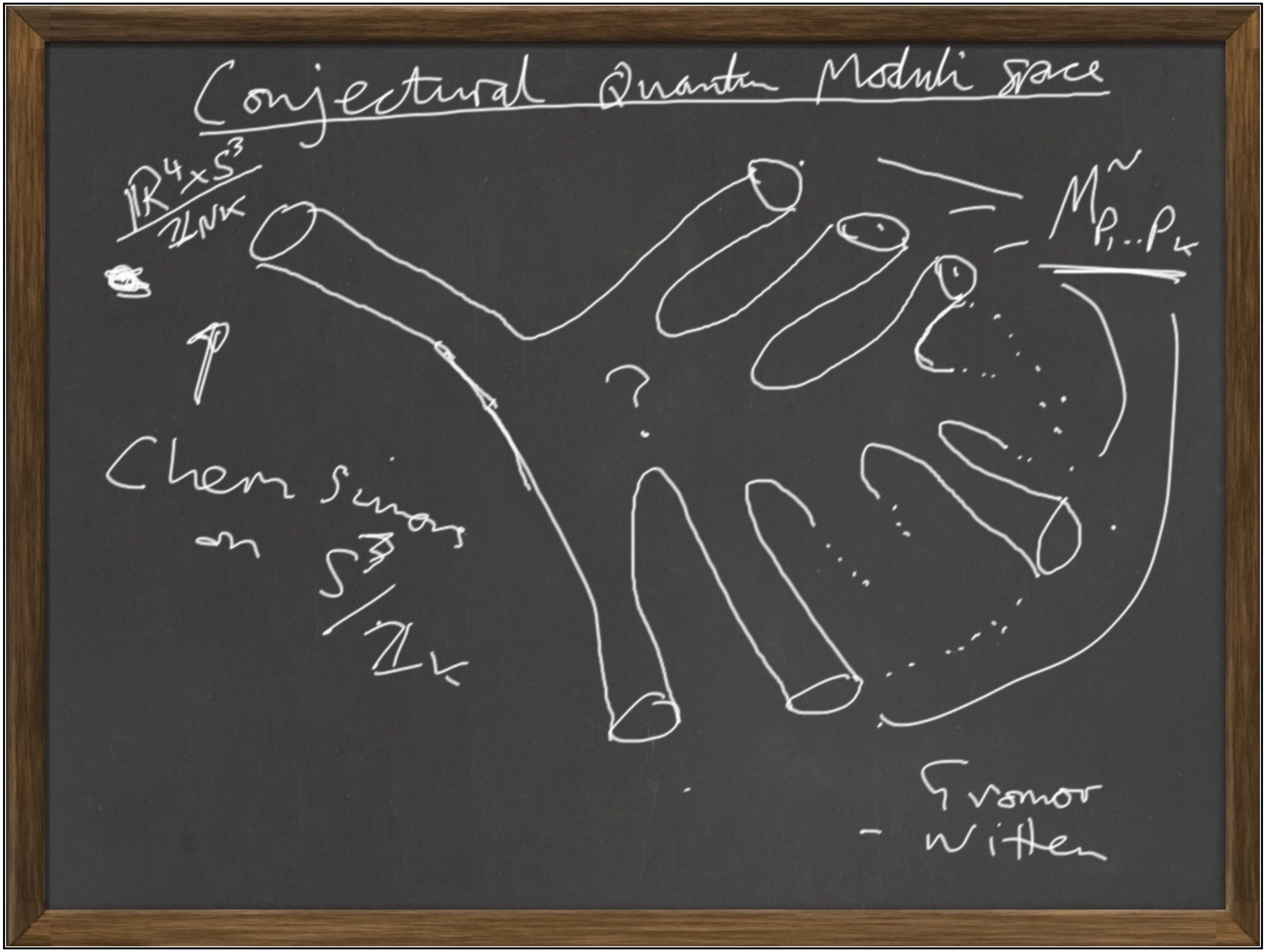
$$1\ 1\ 1\ 1\ 1$$
$$\lambda\ \lambda\ \lambda\ 1\ 1$$
$$\bar\lambda\ \bar\lambda\ \bar\lambda\ 1\ 1$$
$$\lambda\ \lambda\ \lambda\ \lambda\ \bar\lambda$$
$$\bar\lambda\ \bar\lambda\ \bar\lambda\ \bar\lambda\ \lambda$$
$$\lambda\ \bar\lambda\ 1\ 1\ 1$$
$$\lambda\ \lambda\ \bar\lambda\ \bar\lambda\ 1$$

7 flat $SU(5)$ on $S^3/\mathbb{Z}_3$

$\times 3 = 21$ vacua

$= 21$ bundles

$M_{p_1 p_2 p_3}$, $\Sigma p_i = 7$.


Conjectural Quantum Moduli space
$\mathbb{R}^4 \times S^3 / \overline{\mathbb{Z}_{NK}}$
?
$\tilde{M}_{P_1 \ldots P_K}$
Chern Simons on $S^3/\mathbb{Z}_K$
Gromov – Witten

# Unbroken $U(1)$'s and Harmonic 2-forms

- When flat connection is $\neq \mathbb{1}$,

$$SU(N) \to SU(P_1) \times SU(P_2) \ldots U(1)^d$$

So low energy theory has $d$ massless photons

**Predict:**

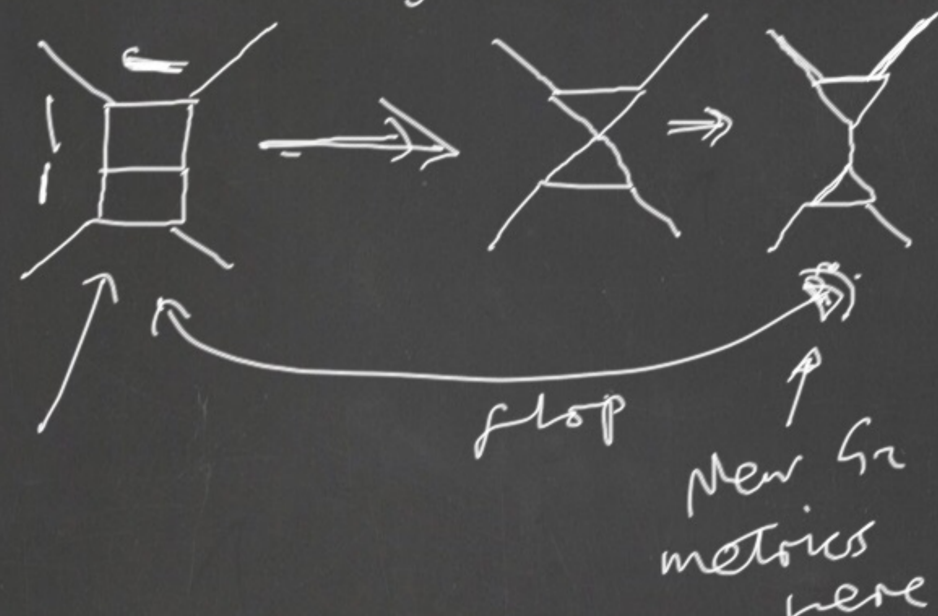
- If the picture is correct, the corresponding $G_2$-manifold/orbifold $M_{P_1 \ldots P_K}$ has $d$ $L^2$-normalisable harmonic 2-forms.

# Other examples

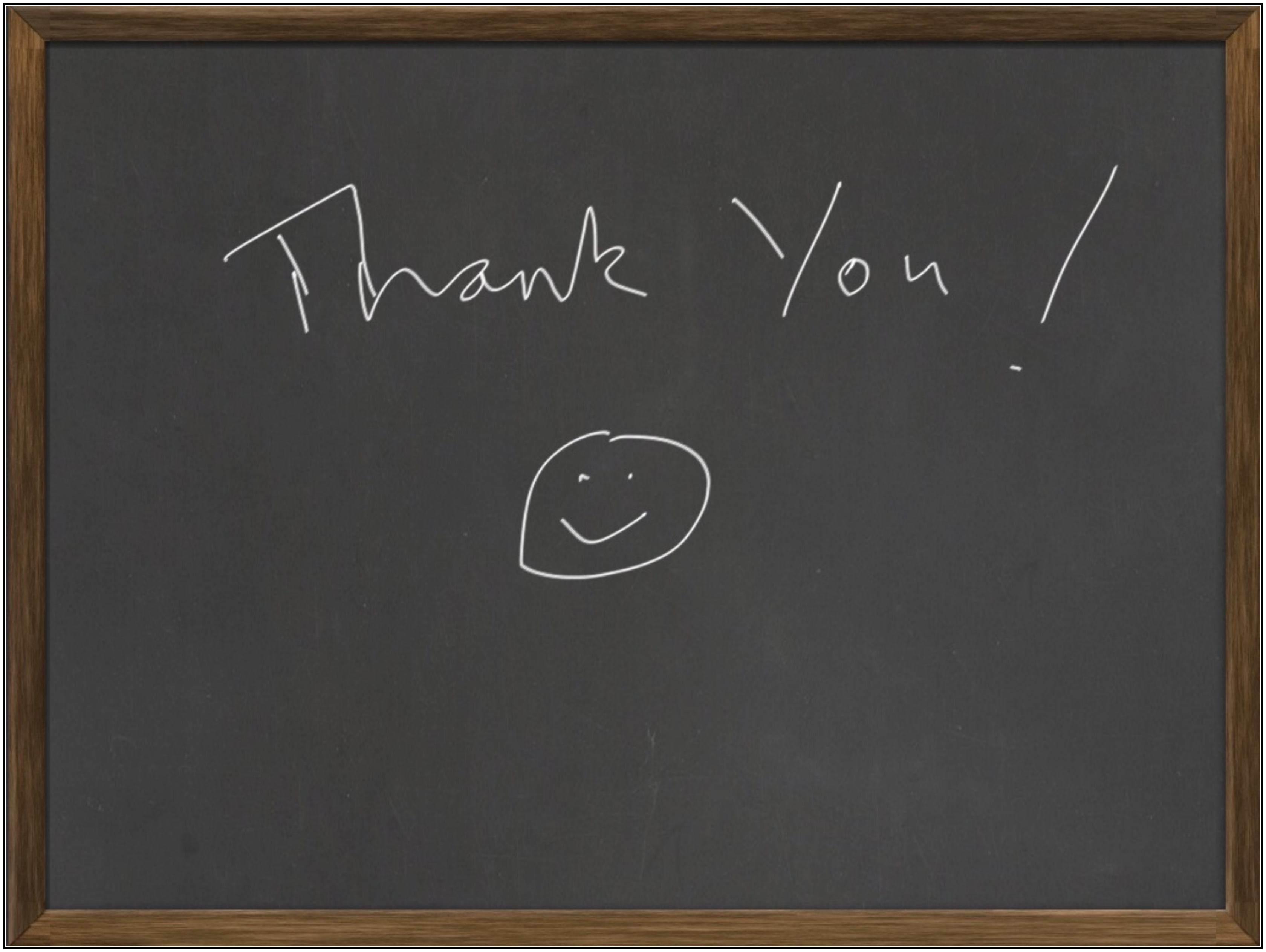
- $S^1 \longrightarrow M_7 \longrightarrow \widetilde{\mathcal{O}(-1)\oplus\mathcal{O}(-1)/\mathbb{P}^1} \big/ \Gamma_{ADE}$

- Flops of $\mathbb{Z}_k$ examples eg

$\widetilde{\mathcal{C}}/\mathbb{Z}_3 =$



- These new examples are under investigation
- Can use "Topological Vertex" (Aganagic/Marino/Vafa) to compute quantum superpotential (in progress)


Thank You!