G_2 instantons and the Seiberg-Witten monopoles

Andriy Haydys

New York

September 14, 2017

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \implies$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\diamond$ $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $\diamond$ $d\varphi = 0$ and $d(*\varphi) = 0$.

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\diamond$ $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $\diamond$ $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

Examples:

- $\diamond$ Bryant'87: local examples;

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

Examples:

- Bryant'87: local examples;
- Bryant–Salamon'89: 3 complete examples;

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\diamond$ $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $\diamond$ $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

Examples:

- $\diamond$ Bryant'87: local examples;
- $\diamond$ Bryant–Salamon'89: 3 complete examples;
- $\diamond$ Joyce'96: $\sim$100 compact examples;

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$
there is $\varphi \in \Omega^3(Y)$ s.t.

- $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

Examples:

- Bryant'87: local examples;
- Bryant–Salamon'89: 3 complete examples;
- Joyce'96: $\sim$100 compact examples;
- Kovalev'02, Corti–Haskins–Nordström–Pacini'15: $\sim$500 000 000 compact examples ?

G_2 manifolds

(Y^7, g) is a G_2 manifold, if $\mathrm{Hol}(g) \subset \mathrm{G}_2 \quad \Longrightarrow$
there is $\varphi \in \Omega^3(Y)$ s.t.

- $\mathrm{Stab}(\varphi_y) = \mathrm{G}_2 \subset \mathrm{GL}(T_y Y)$;
- $d\varphi = 0$ and $d(*\varphi) = 0$.

Remark

φ determines g and is called the associative 3-form.

Examples:

- Bryant'87: local examples;
- Bryant–Salamon'89: 3 complete examples;
- Joyce'96: $\sim$100 compact examples;
- Kovalev'02, Corti–Haskins–Nordström–Pacini'15: $\sim$500 000 000 compact examples ?

This motivates the search for invariants of compact G_2 manifolds.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t.
$\nabla s = 0 \implies s$ vanishes nowhere.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- $\diamond$ Represent $Y = \partial W$;
- $\diamond$ Extend s to a transverse section $\hat{s}$ of $\$_W^+$ (NB. $\mathrm{rk}\, \$_W^+ = 8$);

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\not{S}_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- ◇ Represent $Y = \partial W$;
- ◇ Extend s to a transverse section $\hat{s}$ of $\not{S}_W^+$ (NB. $\operatorname{rk} \not{S}_W^+ = 8$);
- ◇ Define $\nu(Y, g) = -2\,\#\Big(\hat{s}^{-1}(0)\Big) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- $\diamond$ Represent $Y = \partial W$;
- $\diamond$ Extend s to a transverse section $\hat{s}$ of $\$_W^+$ (NB. $\operatorname{rk} \$_W^+ = 8$);
- $\diamond$ Define $\nu(Y, g) = -2\,\#\Big(\hat{s}^{-1}(0)\Big) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\not{S}_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- $\diamond$ Represent $Y = \partial W$;
- $\diamond$ Extend s to a transverse section $\hat{s}$ of $\not{S}_W^+$ (NB. $\operatorname{rk} \not{S}_W^+ = 8$);
- $\diamond$ Define $\nu(Y, g) = -2\,\#\Big(\hat{s}^{-1}(0)\Big) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Important property: If g_t is a 1-parameter family of G_2 metrics on Y, then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend s to a transverse section $\hat{s}$ of $\$_W^+$ (NB. $\operatorname{rk} \$_W^+ = 8$);
- Define $\nu(Y, g) = -2\,\#\big(\hat{s}^{-1}(0)\big) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Important property: If g_t is a 1-parameter family of G_2 metrics on Y, then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

Theorem

There are closed 7-manifolds admitting at least two inequivalent metrics with holonomy G_2.

Crowley–Nordström's ν-invariant and its refinement

Observation: Any G_2 manifold is equipped with a spinor $s \in \Gamma(\$_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend s to a transverse section $\hat{s}$ of $\$_W^+$ (NB. $\operatorname{rk} \$_W^+ = 8$);
- Define $\nu(Y, g) = -2\,\#\Big(\hat{s}^{-1}(0)\Big) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Important property: If g_t is a 1-parameter family of G_2 metrics on Y, then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

Theorem

There are closed 7-manifolds admitting at least two inequivalent metrics with holonomy G_2.

Theorem

For all G_2 manifolds constructed previously by CHNP we have $\bar{\nu} = 0$.

A gauge-theoretic approach

Pick a G-bundle $P \to Y$ (can assume $\mathrm{G} = \mathrm{SU}(2)$).

Definition

A connection A on P is called a G_2-instanton, if

$$F_A \wedge *\varphi = 0. \qquad (*)$$

A gauge-theoretic approach

Pick a G-bundle $P \to Y$ (can assume $\mathrm{G} = \mathrm{SU}(2)$).

Definition

A connection A on P is called a G_2-instanton, if

$$F_A \wedge *\varphi = 0. \tag{$*$}$$

Remark

- $\Lambda^2 T^*M = \Lambda^2_{14} T^*M \oplus \Lambda^2_7 T^*M; \qquad (*) \iff \pi_7(F_A) = 0;$

A gauge-theoretic approach

Pick a G-bundle $P \to Y$ (can assume $\mathrm{G} = \mathrm{SU}(2)$).

Definition

A connection A on P is called a G_2-instanton, if

$$F_A \wedge *\varphi = 0. \qquad (*)$$

Remark

- $\Lambda^2 T^*M = \Lambda^2_{14} T^*M \oplus \Lambda^2_7 T^*M$; $\quad (*) \iff \pi_7(F_A) = 0$;
- $(*)$ can be viewed as a first order non-linear elliptic PDE.

A gauge-theoretic approach

Pick a G-bundle $P \to Y$ (can assume $\mathrm{G} = \mathrm{SU}(2)$).

Definition

A connection A on P is called a G_2-instanton, if

$$F_A \wedge *\varphi = 0. \qquad (*)$$

Remark

- $\Lambda^2 T^*M = \Lambda^2_{14} T^*M \oplus \Lambda^2_7 T^*M; \qquad (*) \iff \pi_7(F_A) = 0;$
- $(*)$ can be viewed as a first order non-linear elliptic PDE.

Fact

The moduli space of G_2 instantons

$$\mathcal{M} = \{A \mid F_A \wedge \psi = 0\}/\mathcal{G}(P)$$

is finite dimensional; Moreover, $\mathrm{v}\text{-dim}\, \mathcal{M} = 0$.

Question (Donaldson–Thomas'98)

Can we define a G_2 Casson invariant $\lambda(Y, g)$ by "counting" G_2-instantons on Y such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \qquad (**)$$

provided there is a smooth path g_t of G_2-metrics connecting g_0 and g_1.

Question (Donaldson–Thomas'98)

Can we define a G_2 Casson invariant $\lambda(Y, g)$ by "counting" G_2-instantons on Y such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \qquad (**)$$

provided there is a smooth path g_t of G_2-metrics connecting g_0 and g_1.

Problem: $\mathcal{M}$ may be non-compact, i.e., G_2 instantons can degenerate.

Question (Donaldson–Thomas'98)

Can we define a G_2 Casson invariant $\lambda(Y, g)$ by "counting" G_2-instantons on Y such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \qquad (**)$$

provided there is a smooth path g_t of G_2-metrics connecting g_0 and g_1.

Problem: $\mathcal{M}$ may be non-compact, i.e., G_2 instantons can degenerate.

Theorem (Tian'00; (Nakajima, Price, Uhlenbeck))

Let A_i be any sequence of G_2 instantons. Then there is a closed subset $S \subset Y$, $\dim S \leq 3$, s.t. A_i converges in $C^\infty_{loc}(M \setminus S)$ to a G_2 instanton.

Question (Donaldson–Thomas'98)

Can we define a G_2 Casson invariant $\lambda(Y, g)$ by "counting" G_2-instantons on Y such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \qquad (**)$$

provided there is a smooth path g_t of G_2-metrics connecting g_0 and g_1.

Problem: $\mathcal{M}$ may be non-compact, i.e., G_2 instantons can degenerate.

Theorem (Tian'00; (Nakajima, Price, Uhlenbeck))

Let A_i be any sequence of G_2 instantons. Then there is a closed subset $S \subset Y$, $\dim S \leq 3$, s.t. A_i converges in $C^\infty_{loc}(M \setminus S)$ to a G_2 instanton. Moreover, $S = S_0 \cup M$ and

- *(Regularity):* *M is an integer multiplicity rectifiable current;*
- *(PDE):* *M is associative, i.e. $\imath_M^* \varphi = \mathrm{vol}_M$;*
- (S_0 *is small*) : $\mathcal{H}^3(S_0) = 0$.

Degenerations of G_2 instantons

Conjecture (Tian)

$\dim S_0 \leq 1$.

Degenerations of G_2 instantons

Conjecture (Tian)

$\dim S_0 \leq 1$.

A sequence of G_2 instantons can develop:

- ◇ a "bubble" along an associative submanifold;
- ◇ a non-removable singularity along a 1-dim. submanifold of M.

Degenerations of G_2 instantons

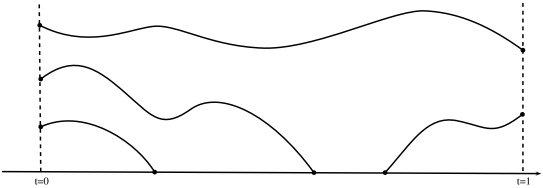
Conjecture (Tian)

$\dim S_0 \leq 1$.

A sequence of G_2 instantons can develop:

- ◇ a "bubble" along an associative submanifold;
- ◇ a non-removable singularity along a 1-dim. submanifold of M.

Expected behavior of G_2 instantons in 1-parameter families:



Degenerations of G_2 instantons

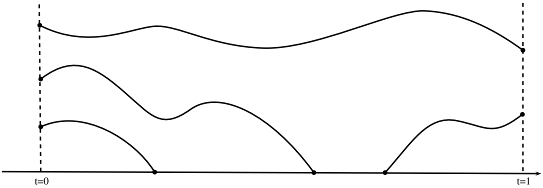
Conjecture (Tian)

$\dim S_0 \leq 1$.

A sequence of G_2 instantons can develop:

- ⋄ a "bubble" along an associative submanifold;
- ⋄ a non-removable singularity along a 1-dim. submanifold of M.

Expected behavior of G_2 instantons in 1-parameter families:



Question (Donaldson–Segal)

Is there a way to "compensate" for jumps of the number of G_2 instantons?

Bubbles of G_2 instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $\mathrm{SU}(n)$-instantons on $\mathbb{R}^4$;
hyperKähler manifold, (I_1, I_2, I_3) cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

Bubbles of G_2 instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $\mathrm{SU}(n)$-instantons on $\mathbb{R}^4$;
hyperKähler manifold, (I_1, I_2, I_3) cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

Definition

A map $u\colon \mathbb{R}^3 \to \mathring{\mathcal{M}}_{k,n}$ is called Fueter, if $I_1\partial_1 u + I_2\partial_2 u + I_3\partial_3 u = 0$.

Bubbles of G_2 instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $\mathrm{SU}(n)$-instantons on $\mathbb{R}^4$;
hyperKähler manifold, (I_1, I_2, I_3) cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

Definition

A map $u\colon \mathbb{R}^3 \to \mathring{\mathcal{M}}_{k,n}$ is called Fueter, if $I_1\partial_1 u + I_2\partial_2 u + I_3\partial_3 u = 0$.

Remark

Fueter sections can also be defined.
Ex. If $\not{S} \to M^3$ is a spinor bundle, $\not{S}/\pm 1$ can be thought of as a bundle with fibers $\mathring{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.

Bubbles of G_2 instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $\mathrm{SU}(n)$-instantons on $\mathbb{R}^4$;
hyperKähler manifold, (I_1, I_2, I_3) cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

Definition

A map $u\colon \mathbb{R}^3 \to \mathring{\mathcal{M}}_{k,n}$ is called Fueter, if $I_1\partial_1 u + I_2\partial_2 u + I_3\partial_3 u = 0$.

Remark

Fueter sections can also be defined.
Ex. If $\$ \to M^3$ is a spinor bundle, $\$/\pm 1$ can be thought of as a bundle with fibers $\mathring{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.

Observation: If $M \subset Y$ associative, then $\$ \to M$ is the normal bundle.

Bubbles of G_2 instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $\mathrm{SU}(n)$-instantons on $\mathbb{R}^4$;
hyperKähler manifold, (I_1, I_2, I_3) cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

Definition

A map $u\colon \mathbb{R}^3 \to \mathring{\mathcal{M}}_{k,n}$ is called Fueter, if $I_1\partial_1 u + I_2\partial_2 u + I_3\partial_3 u = 0$.

Remark

Fueter sections can also be defined.
Ex. If $\$ \to M^3$ is a spinor bundle, $\$/\pm 1$ can be thought of as a bundle with fibers $\mathring{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.

Observation: If $M \subset Y$ associative, then $\$ \to M$ is the normal bundle.

Conjecture

Let A_i be any seq. of G_2 instantons such that $|F_{A_i}|^2$ "concentrates" near an associative M. Then there is a seq. $\varepsilon_i \to 0$ s.t. $\rho_\varepsilon^* A_i$ converges to a Fueter section with values in $\mathring{\mathcal{M}}_{k,n}$. Here $\rho_\varepsilon\colon \$ \to \$,\ s \mapsto \varepsilon^{-1} s$.

The Seiberg–Witten equations with multiple spinors

M closed oriented Riemannian three-manifold;
$L \to M$ Hermitian line bundle; $A \in \mathcal{A}(L)$;
$E \to M$ Hermitian vector bundle, $\operatorname{rk} E = n$, $\Lambda^n E \cong \underline{\mathbb{C}}$; $B \in \mathcal{A}(E)$;

The Seiberg–Witten equations with multiple spinors

M closed oriented Riemannian three-manifold;
$L \to M$ Hermitian line bundle; $A \in \mathcal{A}(L)$;
$E \to M$ Hermitian vector bundle, $\operatorname{rk} E = n$, $\Lambda^n E \cong \underline{\mathbb{C}}$; $B \in \mathcal{A}(E)$;

$\Psi \in \Gamma\big(\mathrm{Hom}(E; \$ \otimes L)\big)$;
$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2$ Hermitian endom. of $\$$, i.e., $\mu(\Psi) \in \Omega^1(M; i\mathbb{R})$;

The Seiberg–Witten equations with multiple spinors

M closed oriented Riemannian three-manifold;
$L \to M$ Hermitian line bundle; $A \in \mathcal{A}(L)$;
$E \to M$ Hermitian vector bundle, $\operatorname{rk} E = n$, $\Lambda^n E \cong \underline{\mathbb{C}}$; $B \in \mathcal{A}(E)$;

$\Psi \in \Gamma\big(\mathrm{Hom}(E; \not{S} \otimes L)\big)$;
$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2$ Hermitian endom. of $\not{S}$, i.e., $\mu(\Psi) \in \Omega^1(M; i\mathbb{R})$;

The Seiberg–Witten equations with n spinors:

$$\not{D}_{A,B}\Psi = 0, \quad *F_A = \mu(\Psi). \tag{SWn}$$

The Seiberg–Witten equations with multiple spinors

M closed oriented Riemannian three-manifold;
$L \to M$ Hermitian line bundle; $A \in \mathcal{A}(L)$;
$E \to M$ Hermitian vector bundle, $\operatorname{rk} E = n$, $\Lambda^n E \cong \underline{\mathbb{C}}$; $B \in \mathcal{A}(E)$;

$\Psi \in \Gamma\big(\mathrm{Hom}(E; \not{S} \otimes L)\big)$;
$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2$ Hermitian endom. of $\not{S}$, i.e., $\mu(\Psi) \in \Omega^1(M; i\mathbb{R})$;

The Seiberg–Witten equations with n spinors:

$$\not{D}_{A,B}\Psi = 0, \quad *F_A = \mu(\Psi). \tag{SWn}$$

Remark

A is a variable in (SWn), whereas B is a parameter.

Theorem (H–Walpuski'15)

Let (A_k, Ψ_k) *be any sequence of the SW monopoles with n spinors.*

(a) *If* $\|\Psi_k\|_{L^2} \leq C$, *then a subsequence of* (A_k, Ψ_k) *converges to a solution of* (SWn);

Theorem (H–Walpuski'15)

Let (A_k, Ψ_k) be any sequence of the SW monopoles with n spinors.

(a) If $\|\Psi_k\|_{L^2} \leq C$, then a subsequence of (A_k, Ψ_k) converges to a solution of (SWn)*;*

(b) If $\lim_k \|\Psi_k\|_{L^2} = \infty$, then there is a closed nowhere dense $Z \subset M$ and a subsequence of $\big(A_k, \|\Psi_k\|_{L^2}^{-1}\Psi_k\big)$, which converges to some (A, Ψ) over $M \setminus Z$; Moreover,

- *(A, Ψ) solves* $\quad \not{D}_{A,B}\Psi = 0, \quad \mu(\Psi) = 0 \quad$ *on $M \setminus Z$;*
- *$|\Psi|$ extends as a C^0-function on M and $Z = |\Psi|^{-1}(0)$.*

Theorem (H–Walpuski'15)

Let (A_k, Ψ_k) be any sequence of the SW monopoles with n spinors.

(a) *If $\|\Psi_k\|_{L^2} \le C$, then a subsequence of (A_k, Ψ_k) converges to a solution of* (SWn)*;*

(b) *If $\lim_k \|\Psi_k\|_{L^2} = \infty$, then there is a closed nowhere dense $Z \subset M$ and a subsequence of $\big(A_k, \|\Psi_k\|^{-1}_{L^2}\Psi_k\big)$, which converges to some (A, Ψ) over $M \setminus Z$; Moreover,*

- *(A, Ψ) solves* $\quad \not{D}_{A,B}\Psi = 0, \quad \mu(\Psi) = 0 \quad$ *on $M \setminus Z$;*
- *$|\Psi|$ extends as a C^0-function on M and $Z = |\Psi|^{-1}(0)$.*

Remark

$$(\text{SWn}) \quad \Longleftrightarrow \quad \not{D}_{A,B}\hat{\Psi} = 0, \quad \varepsilon^2 * F_A = \mu(\hat{\Psi}), \quad \|\hat{\Psi}\|_{L^2} = 1.$$

Theorem (H–Walpuski'15)

Let (A_k, Ψ_k) be any sequence of the SW monopoles with n spinors.

(a) *If $\|\Psi_k\|_{L^2} \leq C$, then a subsequence of (A_k, Ψ_k) converges to a solution of* (SWn);

(b) *If $\lim_k \|\Psi_k\|_{L^2} = \infty$, then there is a closed nowhere dense $Z \subset M$ and a subsequence of $\big(A_k, \|\Psi_k\|^{-1}_{L^2}\Psi_k\big)$, which converges to some (A, Ψ) over $M \setminus Z$; Moreover,*
- *(A, Ψ) solves* $\quad \not{D}_{A,B}\Psi = 0, \quad \mu(\Psi) = 0 \quad$ *on $M \setminus Z$;*
- *$|\Psi|$ extends as a C^0-function on M and $Z = |\Psi|^{-1}(0)$.*

Remark

$$\text{(SWn)} \quad \Longleftrightarrow \quad \not{D}_{A,B}\hat{\Psi} = 0, \quad \varepsilon^2 * F_A = \mu(\hat{\Psi}), \quad \|\hat{\Psi}\|_{L^2} = 1.$$

Theorem (Taubes)

$\dim Z \leq 1$.

The limit (A, Ψ)

Fix $m \in M \setminus Z$.

$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/\mathrm{U}(1)$.

The limit (A, Ψ)

Fix $m \in M \setminus Z$.

$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/\mathrm{U}(1)$.

Fact (ADHM construction of instantons on $\mathbb{R}^4$)

$\mu^{-1}(0)/\mathrm{U}(1)$ *is isometric to* $\mathring{\mathcal{M}}_{1,n}$.

The limit (A, Ψ)

Fix $m \in M \setminus Z$.

$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/\mathrm{U}(1)$.

Fact (ADHM construction of instantons on $\mathbb{R}^4$)

$\mu^{-1}(0)/\mathrm{U}(1)$ *is isometric to* $\mathring{\mathcal{M}}_{1,n}$.

Theorem (H'12)

For any solution of

$$\not{D}_{A,B}\Psi = 0, \;\; \mu(\Psi) = 0 \qquad \text{on } M \setminus Z$$

the projection of Ψ *is a Fueter section.*

The limit (A, Ψ)

Fix $m \in M \setminus Z$.

$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/\mathrm{U}(1)$.

Fact (ADHM construction of instantons on $\mathbb{R}^4$)

$\mu^{-1}(0)/\mathrm{U}(1)$ *is isometric to* $\mathring{\mathcal{M}}_{1,n}$.

Theorem (H'12)

For any solution of

$$\not{D}_{A,B}\Psi = 0,\ \ \mu(\Psi) = 0 \qquad \text{on } M \setminus Z$$

the projection of Ψ *is a Fueter section.*

Proposal (H–Walpuski'15)

Count G_2 *instantons together with the Seiberg–Witten monopoles on associative submanifolds* $M \subset Y$.

What we need to understand better

- ◇ Unremovable singularities of G_2 instantons;

What we need to understand better

- ⋄ Unremovable singularities of G_2 instantons;
- ⋄ $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of Z;

What we need to understand better

- $\diamond$ Unremovable singularities of G_2 instantons;
- $\diamond$ $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of Z;
- $\diamond$ The rôle of reducible Seiberg–Witten monopoles;

What we need to understand better

- $\diamond$ Unremovable singularities of G_2 instantons;
- $\diamond$ $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of Z;
- $\diamond$ The rôle of reducible Seiberg–Witten monopoles;
- $\diamond$ Singularities of associative submanifolds.

Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

◇ $Z \neq \varnothing$ provided $c_1(L^2) \neq 0$.

Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

◇ $Z \neq \varnothing$ provided $c_1(L^2) \neq 0$.

Assume $Z = \varnothing$ and $c_1(L^2) \neq 0$. Then

$$\Psi\Psi^* = \tfrac{1}{2}|\Psi|^2 \neq 0 \implies \ker\Psi^* = \{0\} \implies \Psi \text{ epi } \implies \Psi \text{ iso;}$$

Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

◇ $Z \neq \varnothing$ provided $c_1(L^2) \neq 0$.

Assume $Z = \varnothing$ and $c_1(L^2) \neq 0$. Then

$$\Psi\Psi^* = \tfrac{1}{2}|\Psi|^2 \neq 0 \implies \ker \Psi^* = \{0\} \implies \Psi \text{ epi } \implies \Psi \text{ iso;}$$
$$\Psi \colon E \xrightarrow{\cong} \$ \otimes L \implies \Lambda^2 E \cong \Lambda^2(\$ \otimes L) \cong L^2,$$

which is a contradiction.

Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

- $Z \neq \varnothing$ provided $c_1(L^2) \neq 0$.

Assume $Z = \varnothing$ and $c_1(L^2) \neq 0$. Then

$$\Psi\Psi^* = \tfrac{1}{2}|\Psi|^2 \neq 0 \implies \ker \Psi^* = \{0\} \implies \Psi \text{ epi} \implies \Psi \text{ iso};$$
$$\Psi\colon E \xrightarrow{\cong} \not{S} \otimes L \implies \Lambda^2 E \cong \Lambda^2(\not{S} \otimes L) \cong L^2,$$

which is a contradiction.

Theorem (H'16)

Assume (A, Ψ) solves $\not{D}_{A,B}\Psi = 0, \mu(\Psi) = 0$ over $M \setminus Z$. Then there is an extra infinitesimal structure (θ, or) on Z such that $[Z, \theta, or] \in H_1(M, \mathbb{Z})$ is well-defined. Moreover,

$$[Z, \theta, or] = \mathrm{PD}(c_1(L^2)).$$