Special Kähler and related geometries

HyperKähler & quaternionic Kähler geometries

Def (M, g_M) is called HK, if

$$\mathrm{Hol}(g_M) \subset Sp(n) = \left\{ A : \mathbb{H}^n \to \mathbb{H}^n \text{ is } \mathbb{H}\text{-linear and orthogonal} \right\}$$

$n = \frac{1}{4} \dim_{\mathbb{R}} M$

Examples

a) $\mathbb{H}^n$, $\mathbb{T}^{4n}$

b) If $G \subset Sp(n)$, $\mathbb{H}^n /\!/\!/ G$

c) $Sp(1) \cong SU(2)$: If $n=4$,
HK $\iff$ Calabi-Yau
K3 surfaces are HK

d) Various moduli spaces

<u>Def</u> (N, g_N) is called QK, if

$$\mathrm{Hol}(g_N) \subset Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1) / \pm 1$$

<u>Remarks</u> • If $n = 1$, $Sp(1) \cdot Sp(1) = SO(4)$;
By defn, QK ≡ Einstein & self-dual.

• HK: (I_1, I_2, I_3) cx. str. ⟹ S^2-worth of cx. str.

QK: - only a rank 3 bundle $\mathcal{J}$ spanned by local alm. cx. str.
- rarely Kähler

(N, g_N) QK $\Rightarrow$

$\mathrm{scal}_N = \mathrm{const}$
- > 0 positive
- $= 0$ loc. HK (excluded)
- < 0 neg.

<u>Examples</u> a) $\mathbb{HP}^n$

b) Symmetric spaces (Wolf spaces), e.g. $Gr_2(\mathbb{C}^n)$

<u>Conj</u> (LeBrun - Salamon '94)
Any positive complete QK mfld is a Wolf space.

However, there should be plenty of neg. QK mflds.

The Swann bundle

$$\begin{array}{ccc} Sp(n)\cdot Sp(1) \hookrightarrow Fr_{QK} & \rightsquigarrow & Sp(1)/_{\pm 1} \hookrightarrow \mathcal{F} := Fr_{QK}/Sp(n) \\ \downarrow & & \cong \qquad\qquad \downarrow \\ N & & SO(3) \qquad\qquad N \end{array}$$

More concretely, $\mathcal{F}_n$ = orthonormal frames of $\mathcal{I}_n$

Swann bundle :

$$\begin{array}{c} \mathbb{H}^*/_{\pm 1} \hookrightarrow \mathcal{U}(N) := \mathcal{F} \times \mathbb{R}_{>0} \\ \downarrow \\ N \end{array}$$

Thm (Swann '91)

1) If N is a QK mfld, then $\mathcal{U}(N)$ is HK. Moreover, $\mathcal{U}(N)$ carries an action of

$$\mathbb{H}^* = Sp(1) \times \mathbb{R}_{>0}$$

isometric but rotates cx. str. (→ $Sp(1)$)

preserves cx. str. but scales the metric (→ $\mathbb{R}_{>0}$) $(*)$

2) If M is HK and carries a loc. free action of $\mathbb{H}^*$ s.t. $(*)$ holds, then

$$N := M/\mathbb{H}^* \text{ is QK.}$$

Cx. integrable systems

M HK mfld, $I := I_1$, $\underbrace{\omega := g(I_2\cdot,\cdot) + g(I_3\cdot,\cdot)\, i}_{\text{cx sympl. form}}$

Def An algebraic integrable system consists of:

- A hol. map $\pi : M \longrightarrow \Sigma$ s.t. $\forall \sigma \in \Sigma_0 := \Sigma \setminus \Delta$ $M_\sigma := \pi^{-1}(\sigma)$ is a compact Lagrangian submfld (a cx. torus)
- A smoothly varying family of polarizations $\rho_\sigma \in H^2(M_\sigma; \mathbb{Z})$

Ex a) Elliptic K3 b) Moduli spaces of Higgs bundles

Special Kähler geometry

The base $\Sigma_0 := \Sigma \setminus \Delta$ carries a very special str.

Def $(\Sigma_0, g, I, \omega, \nabla)$ is called special Kähler, if

- (Σ_0, g, I, ω) Kähler
- ∇ is flat symplectic torsion-free con-n on $T\Sigma_0$
- $(\nabla_X I) Y = (\nabla_Y I) X \quad \forall X, Y \in \mathfrak{X}(\Sigma_0)$

Further motivation:

- SK str. play a rôle in $N=2$ SYM theory.
- ——— // ——— // — in Mirror Symmetry.
- Natural str. on the moduli space of cx. str. of a CY 3-fold.

Thm (Cecotti – Ferrara – Girardello '89)

Σ_0 is SK $\Rightarrow$ $T^*\Sigma_0$ is HK

Rem The correspondence $\Sigma_0 \rightsquigarrow T^*\Sigma_0$ is known as a c-map.

Assume Σ_0 admits an action of $\mathbb{C}^*$ s.t.

$$\mathbb{C}^* = U(1) \times \mathbb{R}_{>0}$$

preserves SK str. → $U(1)$; preserves I but scales the metric → $\mathbb{R}_{>0}$

Thm If Σ_0 admits a $\mathbb{C}^*$-action as above, then one can associate with Σ_0 a QK mfld $\mathcal{N}(\Sigma_0)$.

↪ new explicit examples of QK mflds [Cortes – Saha – Thung]

Thm (Hwang '07) If M and Σ are proj., then $\Sigma \cong \mathbb{C}P^n$ & Δ is a hypersurface.

Thm (Lu '99) A SK metric is incomplete unless flat.

The holomorphic cubic form

$\pi^{1,0} : T_{\mathbb{C}}\Sigma \longrightarrow T^{1,0}_{\mathbb{C}}\Sigma$ can be thought of as

$\pi^{1,0} \in \Omega^{1,0}(\Sigma ; T_{\mathbb{C}}\Sigma)$

$\boxed{H} := -\,\omega(\pi^{1,0}, \nabla\pi^{1,0}) \in H^0(\mathrm{Sym}^3 T^*M)$

Fact $\boxed{H} \equiv 0 \iff \nabla = \nabla^{LC}$

$\boxed{H}$ measures how far ∇ is from ∇^{LC}.

SK structures on Riemann surfaces

Local questions

- z any holomorphic coord. on Σ
- $\boxed{H} = \boxed{H}_0(z)\, dz^3$; $g = e^{-u} |dz|^2$

$$\boxed{\Delta u = 16 \left| \boxed{H}_0 \right|^2 e^{2u}} \qquad (*)$$

Thm (H'15) Any SK on D or D^* determines a solution of $(*)$. Conversely, any solution of $(*)$ determines a SK str.

Rem ∇ can be computed explicitly in terms of u and $\boxed{H}_0$.

Local examples

a) $g_n := -C \cdot r^{n+1} \ln r \, |dz|^2$, $\boxed{H}_{0,n} := C z^n$

is a solution of $(*)$ $\forall n \in \mathbb{Z}$

$n = -1$: $g_{-1} = -\ln r \, |dz|^2 \longleftrightarrow$ Ooguri-Vafa metric

b) $g^c_\beta = r^\beta |dz|^2$, $\boxed{H}_0 \equiv 0$, $\beta \in \mathbb{R}$ (flat cone)

Assume $\boxed{H}_0$ is holomorphic on D^* and $(**)$
$n := \mathrm{ord}_0 \boxed{H}_0 \in \mathbb{Z}$.

Thm (H'15, Callies-H'18) Let $(g, \boxed{H}_0)$ be any SK str. on D or D^* s.t. $(**)$ holds.
Then (g, ∇) is modelled either on
(g_n, ∇^n) or $(g^c_\beta, \nabla^{LC}_\beta)$ for some $\beta < n+1$.

Model	Singularity		
$g^c_\beta = r^\beta \lvert dz\rvert^2$	conical	$\beta > -2$ →	Loc. Conical
		$\beta = -2$ →	Asympt. Cylindr.
		$\beta < -2$ →	Asympt. Conical
$g^n = - r^{n+1} \ln r \lvert dz\rvert^2$	logarithmic		$(n = \operatorname{ord} \Phi)$

Rem) There are explicit examples of SK str. with $\mathcal{F}$ being non-meromorphic.

2) $\mathrm{Hol}(S^1; \nabla) \in SL(2;\mathbb{Z}) \iff$ Log-sing. or conical with $\beta \in \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z}$

3) If $\pi: M \longrightarrow D$ is an alg. int. system, then

Kodaira type	Type of isol. singularity	β
I_0 or I_0^*	(conical ?) / log	$\beta \in \mathbb{Z}$
I_b or I_b^*	log	$n = b - 1$?
II or II^*	conical	$\beta = \frac{6k \pm 1}{3}$, $k \in \mathbb{Z}$
III or III^*	conical	$\beta = \frac{1}{2} + k$, $k \in \mathbb{Z}$
IV or IV^*	conical	$\beta = \frac{6k \pm 2}{3}$, $k \in \mathbb{Z}$

- From now on Ξ is assumed to be meromorphic
- For any SK str. we have the period map

$$\tau : \Sigma_0 \longrightarrow \mathbb{H} = \{ \operatorname{Im} z > 0 \}$$

defined locally only

$$g_{\mathbb{H}} := \frac{1}{(\operatorname{Im} z)^2} |dz|^2$$ with const. neg. curv.

<u>Def</u> $\tilde{g} := \tau^* g_{\mathbb{H}}$ the associated hyperbolic metric (with isolated singularities)

Thm (H-Xu'18) Let Ξ be a meromorphic cubic diff. on a RS Σ (may be non-compact). Let $\tilde{g}$ be a hyperb. metric on $\Sigma \setminus \{p_1, \dots, p_k\}$. Then $\exists!$ SK str. on $\Sigma \setminus (\mathrm{supp}\,\Xi \cup \{p_1, \dots, p_k\})$ whose ass. hyp. metr. & ass. cubic form are $\tilde{g}$ and Ξ respectively.

Rem The correspondence of the above thm is explicit:

$$\left.\begin{aligned} \tilde{g} &= e^{2v} |dz|^2 \\ \Xi &= \Xi_0 \, dz^3 \end{aligned}\right\} \Rightarrow \left\{ g_{SK} = 4 |\Xi_0| e^{-v} |dz|^2 \right.$$

In particular, one can also determine the singularity of the SK str. at $\mathrm{supp}\,\Xi \cup \{p_1, \dots, p_k\}$.

The Seiberg-Witten curve

$$M := \{ y^2 = (x-1)(x+1)(x-u) \} \subset \mathbb{C}^2_{(x,y)} \times \mathbb{C}_u \subset \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$$

$\pi : M \longrightarrow \mathbb{P}^1$

holomorphic Lagrangian fibration
with 3 singular fibers at $1, -1, \infty$

Can show: • $\mathrm{ord}_{\pm 1} \boxed{H} = -1$ & $\mathrm{ord}_\infty \boxed{H} = -4$

$$\Longrightarrow \quad \boxed{H} = \frac{1}{u^2-1} du^3$$

• the associated hyperbolic metric has
cusps at $1, -1, \infty$ $\Longrightarrow$

$$\Rightarrow \quad \tilde{g} = \frac{4\pi^2}{|u^2-1|^2} \left(\int_{\mathbb{C}} \frac{\mathrm{vol}_\xi}{|\xi^2-1|\,|\xi-u|} \right)^{-2} |dz|^2$$

$$\Downarrow$$

$$\boxed{g_{SW} = \left(\int_{\mathbb{C}} \frac{\mathrm{vol}_\xi}{|\xi^2-1|\,|\xi-z|} \right) |dz|^2}$$

The Seiberg-Witten SK metric on $\mathbb{P}\setminus\{1,-1,\infty\}$.

Q: What is known about the HK metric on the SW curve?

Global questions

The moduli space

Fix a RS Σ.

$$\underline{p} := (p_1, \dots, p_k), \quad \underline{\beta} := (\beta_1, \dots, \beta_\ell, \beta_{\ell+1}, \dots, \beta_k)$$

$p_1, \dots, p_k$: pairwise distinct pts on Σ; $\beta_1, \dots, \beta_\ell \in \mathbb{R}$; $\beta_{\ell+1}, \dots, \beta_k \in \mathbb{Z}$

The moduli space of SK structures with prescribed sing.:

$$\mathcal{M}_k^\ell(\underline{p}, \underline{\beta}) := \{ (g, \nabla) \text{ SK str. on } \Sigma \setminus \underline{p} \text{ s.t. }$$

Ξ is meromorphic, $\Xi \not\equiv 0$,

for $j \leq \ell$ the singularity near p_j is modelled on $g^c_{\beta_j} = r^{\beta_j} |dz|^2$,

for $j \geq \ell+1$ — // — // — // — // $g_{\beta_j} = -r^{\beta_j} \log r \, |dz|^2 \}$

$\overline{\qquad\qquad\qquad\qquad}$ $\mathbb{R}_{>0}$

$$\mathcal{R}^\ell_k(\underline{p},\underline{b}) := \{ g \mid \exists \nabla \text{ with } [g,\nabla] \in \mathcal{M}^\ell_k(\underline{p},\underline{b}) \} / \mathbb{R}_{>0}$$

__Thm__ (H–Xu '18)
Let Σ be a cmpt RS, $\text{genus}_k(\Sigma) = \gamma$.

$$\mathcal{M}^\ell_k(\underline{p},\underline{b}) \neq \emptyset \implies \begin{cases} 4(\gamma-1) < \sum_{j=1} \beta_j \\ \sum_{j=1}^{k} [\beta_j] \leq 6(\gamma-1) + k - \ell \end{cases} \quad (\ast\ast\ast)$$

$$\underbrace{\mathcal{M}^\ell_k(\underline{p},\underline{b}) \subset}_{\text{open dense}} S^{2N+1}, \quad \underbrace{\mathcal{R}^\ell_k \subset}_{\text{Zarisski open}} \mathbb{CP}^N$$

For $\Sigma = \mathbb{P}^1$, $\mathcal{M}^\ell_k(\underline{p},\underline{b}) \neq \emptyset \iff (\ast\ast\ast)$

__Rem__ $D := -\sum_{1}^{\ell} [\beta_j] p_j - \sum_{\ell+1}^{k} (\beta_j - 1) p_j$

Embedding: $\mathcal{R}^\ell_k \to \mathbb{P}(H^0(K^3_\Sigma + D))$, $[g,\nabla] \mapsto [\boxed{H}]$

Rem In the special case $\ell = k$ (conical sing. only),

$$\mathcal{M}_k^k \cong S^{2N+1} \quad \text{and} \quad \mathcal{R}_k^k \cong \mathbb{C}P^N$$

In particular, both are compact.

If $\ell < k$ (there is at least one log-sing.)

$$\overline{\mathcal{M}_k^\ell(\underline{\beta}, \underline{b})} := \bigcup_{m \in \mathbb{Z}_{\geq 0}^{k-\ell}} \mathcal{M}_k^\ell(\underline{\beta}, \underline{b} + (0, m))$$

$\cong S^{2N+1}$

only finitely many strata because of $\sum [\beta_j] + |m| \leq 6(\gamma-1) + k - \ell$

with $\mathcal{M}_k^\ell(\underline{\beta}, \underline{b})$ being the top stratum

Gravitational instantons

Q: Does there exist a gravitational inst. with a single singular fiber?

$$T^2 \hookrightarrow M^4 \quad \text{(complete) HK mfld}$$
$$\downarrow \pi \qquad \text{s.t. } M_0 = \pi^{-1}(0) \text{ is the}$$
$$\mathbb{C} \qquad \text{only singular fiber}$$

A: No (at least assuming that Ξ is meromorphic)

Follows from

Prop Any SK str. on $\mathbb{P}^1$ with isolated singularities and meromorphic $\Xi \neq 0$ has at least 3 singular pts.

Q': Does there exist a gravitational inst. with 2 sing. fibers?

Which pairs of sing. fibers can occur? Not all

Further comments / questions

- Characterization of SK structures with $\mathrm{Hol}(\nabla) \subset SL(2;\mathbb{Z})$ (work in progress)

- HK metrics on elliptic K3 surfaces?

$$\left\{ \begin{array}{c} T^2 \hookrightarrow K3 \\ \downarrow \\ \mathbb{P}^1 \supset \Delta \end{array} \right\} \longrightarrow \underbrace{\mathrm{Sym}^{24}\mathbb{P}^1 = \mathbb{P}^{24}}_{\text{codimension 5 variety}}$$

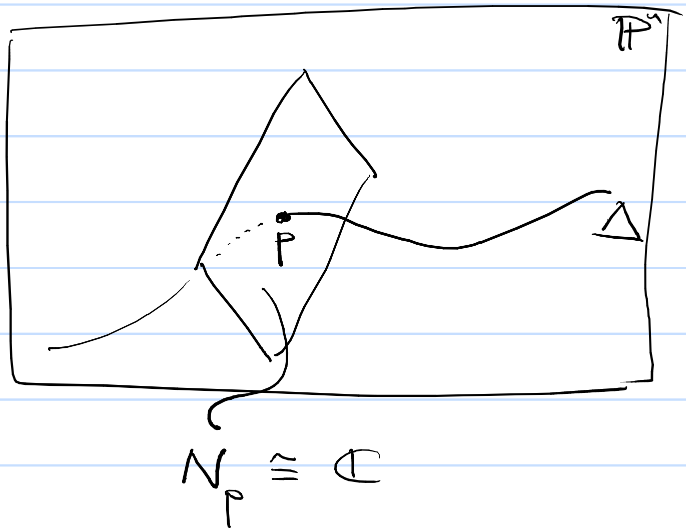
Δ consists of 24 pts generically

Q: Can one describe [the codimension 5 variety]?

- Higher dimensions ?

$$M \downarrow \mathbb{P}^n \supset \Delta$$

assume a smooth hypersurface



Q: Is it true that singularities of SK structures in higher dimensions are modeled on 1-dimensional singularities ?