Examples of non-compact G_2-manifolds

based on joint work w/

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§. Motivation from physics

M-theory on $\mathbb{R}^d \times \underset{\text{non-cpct}}{M^{11-d}}$ $\quad$ $d=5$ $M = CY3$-fold
$d=4$ $M = G_2$-mfld

metric $g_{\mathbb{R}^d} + g_M$

"gauge" 3-form field C: $\quad$ curvature $dC = G \in \Omega^4(\mathbb{R}^d \times M)$

$dG = 0 = d^*G$

"gauge transformations" $C + d\Lambda$, $\Lambda \in \Omega^2(\mathbb{R}^d \times M; U(1))$

Kaluza-Klein reduction $\quad g_{\mathbb{R}^d} + \varepsilon^2 g_M \quad \varepsilon \to 0$

$$C = \sum_i C^i_{\mathbb{R}^d} \wedge C^{3-i}_M$$

$\uparrow$ closed & coclosed

Rmk: (M, g_M) Ric $= 0$ complete $\Rightarrow$ no decaying harmonic functions & 1-forms

$$\rightsquigarrow \quad C = \sum_{\varsigma \in \mathcal{H}^3(M)} \Phi_\varsigma \varsigma + \sum_{\sigma \in \mathcal{H}^2(M)} A_\sigma \wedge \sigma$$

- moduli space of vacua:

$(\Phi_\varsigma)_{\varsigma \in L^2\mathcal{H}^3(M)}$ $\quad$ scalar $L^2\mathcal{H}^3(M)$-valued fields

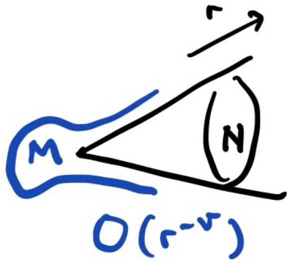
$(A_\sigma)_{\sigma \in L^2\mathcal{H}^2(M)}$ $\quad$ abelian connection w/ $\mathfrak{h} = L^2\mathcal{H}^2(M)$

- parameters:

non-L^2 closed & coclosed 3-form on M $\longleftrightarrow$ coupling constant

non-L^2 $\qquad$ " $\qquad$ 2-form " $\longleftrightarrow$ flavour symmetry

§. AC & ALC manifolds



M = (partial) desingularisation of a cone $C(N^{11-d-1})$

asymptotically conical (AC)

- $d=5$, i.e. M $CY3$

→ CY cones: K-stable canonical singularities (Collins-Szekelyhidi)

→ AC CYs : crepant resolutions & versal deformations (Conlon-Hein)

- $d=4$, i.e. M G_2

→ G_2 cones: few known examples

$C(\mathbb{CP}^3)$, $C(\mathbb{F}_3)$ $\quad$ $C(S^3 \times S^3)$ $\quad$ inhomogeneous $C(S^6)$, $C(S^3 \times S^3)$ (FH) $\quad$ $C(S^3 \times S^3/\Gamma)$

→ AC G2s : few known examples

$\Lambda^- S^6$, $\Lambda^- \mathbb{CP}^3$ (Bryant-Salamon) $\quad$ $S^3 \times \mathbb{R}^4$ $\quad$? $\quad$ $\Gamma = \Gamma_{2, m+n, m+2\mathbb{Z}}$ (F.-Haskins-Nordström)

<u>Rmk</u> extend symmetries of C to classify (Karigiannis-Lotay)

<u>Thm</u> (F.-Haskins-Nordström)

$(B^6, \omega_0, \Omega_0)$ AC CY3 orbifold asymptotic to CY cone $C(\Sigma^5)$

$M^7 \to B$ (smooth) principal S^1-orbibundle w/ $c_1(M) \cup [\omega_0] = 0 \in H^4_{orb}(B; \mathbb{R})$

$\Rightarrow \forall\ \epsilon \ll 1$ $\exists$ S^1-inv. ALC G_2-holonomy metric (M, φ_ϵ)

$$M \setminus K \underset{O(r^{-\nu})}{\approx} \left(\begin{array}{c} S^1_\epsilon \hookrightarrow BC(\Sigma) \\ \downarrow \\ C(\Sigma) \end{array}, \ g^\infty_{ALC} = g_{C(\Sigma)} + \epsilon^2 \theta^2 \right)$$

§. Cohomogeneity-1 examples

$C(\Sigma)$ conifold $\qquad \Sigma = SU(2)\times SU(2)/\Delta U(1)$

$B =$ small resolution $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ $\rightsquigarrow \left(S^3\times\mathbb{R}^4, \varphi_{\epsilon,\lambda}\right)$ $\qquad \mathbb{D}_7$ family

$\mathrm{Vol}(S^3)=\lambda^3$

$B =$ $K_{\mathbb{P}^1\times\mathbb{P}^1}$ asymptotic to $C/\mathbb{Z}_2$ $\rightsquigarrow \left(M_{m,n}, \varphi_{\epsilon,\lambda}\right)$

$c_1(M) = (m,-n)$

$[\omega_0] = \lambda^2(m,n) \qquad m,n>0$

$M_{1,1} = \mathbb{C}_7$

<u>Rmk</u> (Topology of $M_{m,n}$)

- $M_{m,n}$ retracts onto $S \subset \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m,-n)$ $\qquad S \simeq S^2\times S^3/\mathbb{Z}_{\gcd(m,n)}$

- $\partial M_{m,n} = (S^3\times S^3)/\Gamma_{2,m+n,m+2\mathbb{Z}}$ $\qquad (S^3\times S^3/\mathbb{Z}_{n+m})/\mathbb{Z}_2$

$\Gamma_{2,m+n,m+2\mathbb{Z}} \simeq \mathbb{Z}_{2(n+m)}$ (if $\gcd(m,n)=1$)

$$H^2_c(M_{m,n};\mathbb{Z}) \to H^2(M_{m,n};\mathbb{Z}) \twoheadrightarrow H^2(\partial M_{m,n};\mathbb{Z}) \to \underset{0}{\underset{\shortparallel}{H^3_c(M_{m,n};\mathbb{Z})}}$$

§. From ALC to AC

(M, φ_ϵ) Can deform away from $\epsilon \ll 1$?

<u>Rmk</u> (4d ALF spaces)

$\rightarrow$ formation of isolated orbifold singularities, e.g. multiTN

Fix $[\omega_i]$

$\rightarrow$ AH, $\widetilde{\mathrm{AH}}$ have no ALE limit

TN w/ negative mass

Use $SU(2)\times SU(2)\times U(1)$ symmetry:

$(S^3 \times \mathbb{R}^4, \varphi_{\epsilon,\lambda})$ $\qquad \frac{\epsilon}{\lambda} \longrightarrow +\infty$

D_7 family

$\rightarrow$ $\lambda = 1$, $\epsilon \rightarrow \infty$: Bryant–Salamon AC $S^3 \times \mathbb{R}^4$

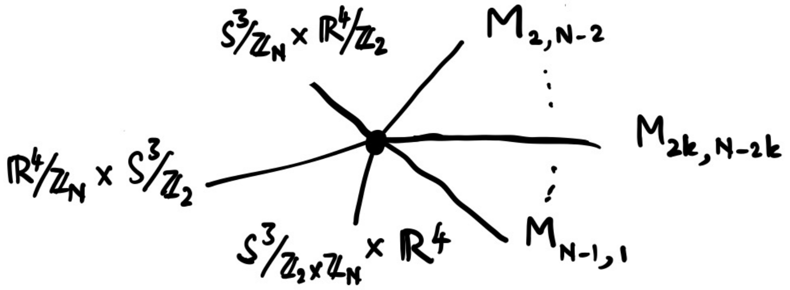
$\rightarrow$ $\lambda \rightarrow 0$, $\epsilon = 1$: conically singular (cs) ALC $C(S^3 \times S^3)$

$$(M_{m,n}, \varphi_\epsilon) \xrightarrow[\epsilon \to \infty]{} \text{new AC } G_2\text{-metric asymptotic to } C\left(S^3 \times S^3 / \Gamma_{2,m+n,m+2\mathbb{Z}}\right)$$

<u>Rmk</u>: asymptotic cone only depends on $m+n$, $m \pmod 2$

$n+m = N$ odd

$n \equiv 0 \pmod 2$



Rmk : CS ALC # AC $\rightsquigarrow$ ALC (FHN)

$\to$ CS ALC metric on $C(S^3 \times S^3)$ + Bryant-Salamon AC $\begin{cases} \mathbb{D}_7^{\pm} \\ \mathbb{B}_7 \end{cases}$

$\to$ $\Gamma_{2,2,1+2\mathbb{Z}} \simeq \mathbb{Z}_4 \hookrightarrow \mathrm{Aut}(\text{CS ALC}) = SU(2)^2 \times N(U(1))$

"cyclic" $\mathbb{Z}_4 \subset U(1) \subset N(U(1))$ or "dihedral" $\mathbb{Z}_4 \subset N(U(1))$

choices conjugate in $\mathrm{Aut}(\text{cone}) = SU(2)^3$

$\rightsquigarrow$ $\begin{cases} \text{CS ALC on } C(S^3 \times S^3)/\mathbb{Z}_4^{\text{cyclic}} + \text{AC } M_{1,1} = \mathbb{C}_7 \\ \text{CS ALC on } C(S^3 \times S^3)/\mathbb{Z}_4^{\text{dihedral}} + \text{AC } M_{1,1} = \mathbb{A}_7 \end{cases}$

§. AC vs. ALC : Hodge theory

(M^7, φ) complete AC or ALC, $\mathrm{Hol}(g) = G_2$

<u>Rmk</u>: AC: $\partial M \not\simeq (S^6, g_{rd}) \iff \mathrm{Hol}(g) = G_2$

$\partial M \simeq (S^6, g_{rd}) \iff M = \mathbb{R}^7$

ALC: $\left(\begin{array}{c} S^1 \hookrightarrow \partial M \\ \downarrow \\ \Sigma^5 \end{array} \right) / \mathbb{Z}_2$ dihedral

$\left.\begin{array}{l} \text{dihedral} \\ \text{cyclic} + \\ c_1(\partial M) \neq 0 \in H^2(\Sigma; \mathbb{R}) \end{array}\right\} \Rightarrow \mathrm{Hol}(g) = G_2$

- $\pi_1(M)$ finite & $\tilde{M}$ is still AC/ALC

  every decaying harmonic 1-form vanish

- $L^2\mathcal{H}^2(M) \simeq H^2_c(M; \mathbb{R})$ (Hausel–Hunsicker–Mazzeo)

<u>Rmk</u>: 4d abelian gauge theory from M-theory on $\mathbb{R}^4 \times M$

w/ $\mathrm{Lie}(T) = H^2_c(M; \mathbb{R})$ & $T = H^2_c(M; \mathbb{R}) / \Lambda$

$$H^2_c(M; \mathbb{Z}) \subseteq \Lambda \subseteq H^2(M; \mathbb{Z}) \cap H^2_c(M; \mathbb{R})$$

$$0 = H^1(\partial M; \mathbb{Z}) \longrightarrow H^2_c(M; \mathbb{Z}) \longrightarrow H^2(M; \mathbb{Z}) \longrightarrow H^2(\partial M; \mathbb{Z})$$

- decaying but non-L^2 closed & coclosed 2-forms

  AC: $\quad \mathrm{im}\, H^2(M) \longrightarrow H^2(\partial M)$

  ALC: $\quad \mathrm{im}\, H^2(X) \longrightarrow H^2_+(\Sigma)$

  - "cyclic" $\mathrm{im}\, H^2(M) \to H^2(\partial M) \oplus \mathbb{R}$
  - "dihedral" $\mathrm{im}\, H^2(M) \to H^2(\partial M)$

$$X = M \cup \Sigma / \mathbb{Z}_2$$

• $$L^2\mathcal{H}^3(M) = \begin{cases} \text{AC} & H^3_c(M) = \operatorname{im} H^3_c(M) \to H^3(M) \oplus \operatorname{im} H^2(\partial M) \to H^3_c(M) \\ \text{ALC} & \operatorname{im} H^3_c(M) \to H^3(M) \end{cases}$$

broken symmetry
& Goldstone boson

• decaying closed & coclosed 3-form from topology

$$\begin{array}{lcccc} \text{AC} & <-4 & -4 & -3 & \\ & \operatorname{im}\big(H^3_c(M) \to H^3(M)\big) & \oplus \operatorname{im}\big(H^2(\partial M) \to H^3_c(M)\big) & \oplus \operatorname{im}\big(H^3(M) \to H^3(\partial M)\big) & \\ & & & \parallel & \\ & & & \operatorname{im}\big(H^3(X) \to H^3_+(\Sigma)\big) & \oplus \operatorname{im}\big(H^3(M) \to H^2_-(\Sigma)\big) \\ \text{ALC} & <-3 & -3 & -3 & -2 \end{array}$$

<u>Rmk</u>: in good cases (e.g. Σ regular or toric) no other decaying closed & coclosed 3-forms

$\rightsquigarrow$ $\mathcal{M}^{ALC}_{-1-\delta}(M) \longrightarrow H^3(M;\mathbb{R}) \oplus H^4(M;\mathbb{R})$ immersion (FHN)

w/ image of dim $b_3(M) + \dim \operatorname{im} H^2(\partial M) \to H^3_c(M)$

Moreover, if $\begin{array}{c} S^1 \hookrightarrow M \\ \downarrow \\ B \end{array}$ dimension of ALC G_2-moduli coincides

w/ dimension of "$c_1(M)$-compatible" CY deformations of B:

→ change $[\omega_0]$ so that $[\omega_0] \cup c_1(M) = 0$ preserved

→ change $\operatorname{Re}\Omega$ by decaying closed & coclosed 3-form

§. Other examples of ALC G_2-manifolds

- $B = K_D$ D Kähler-Einstein del Pezzo

  ω_0 Calabi AC CY metric, $[\omega_0] \in H^2_c(B)$

  $c_1(M)|_D$ primitive $(1,1)$-class on $(D, \omega_0|_D = \omega_{KE})$

$\rightsquigarrow$ $b_2(D) - 1$ dim'l family of ALC G_2-metrics

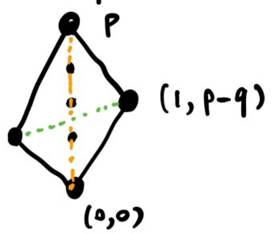
- B = small resolution of cA_p-sing. $\quad xy + z^{p+1} + w^{p+1} = 0$

$\rightsquigarrow$ M^7 w/ $b_2(M) = p-1$, $b_3(M) = p$, $b_4(M) = b_5(M) = 0$

$$\partial M \simeq \#_{p-1}(S^2 \times S^4) \# \#_p (S^3 \times S^3)$$

(Rmk: "non-topological" deformations of cplx str. of B)

- B = small partial resolution of $Y^{p,q}$ cone



$\rightsquigarrow$ $M = S^3 \times \mathbb{R}^4$

<u>Q</u> : possible AC limit? Bryant-Salamon?

<u>Rmk</u> : $SU(2) \times T^2$ - symmetry

- B = crepant resolution of conifold$/\mathbb{Z}_p$ (Acharya-F.-Najjar-Svanes)

$\rightsquigarrow$ M w/ $\partial M \simeq S^3 \times S^3 / \Gamma_{p,N,q+p\mathbb{Z}} = (S^3 \times S^3/\mathbb{Z}_N)/\mathbb{Z}_p$

<u>Rmk</u>: duals of M-th. uplift of IIA on $T^* S^3/\mathbb{Z}_p$ w/ N D6-branes
$q+p\mathbb{Z}$ RR 1-form flux