Infinitely many new families of cohomogeneity one G_2–manifolds

Lorenzo Foscolo

Heriot-Watt University

joint with Mark Haskins and Johannes Nordström

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Codimension-one collapse of G_2–metrics

- Sequence of 7-dimensional G_2–manifolds collapsing to Calabi–Yau 3-folds
- **Complete non-compact** G_2–manifolds with **non-maximal volume growth** and ALC asymptotics
- **Symmetries**: cohomogeneity one G_2–metrics
  - Bryant–Salamon (1989)
  - Brandhuber–Gomis–Gubser–Gukov (2001)
  - Cvetič–Gibbons–Lü–Pope (2001-2002)
  - Bogoyavlenskaya (2013)

TODAY: existence of all conjectural families of complete metrics and study of their limits $\rightsquigarrow$ new results not anticipated in the physics literature.

In particular, we proved the existence of **infinitely many new simply connected AC G_2–manifolds** and of infinitely many families of complete ALC G_2–metrics.

We also obtained the classification of complete $SU(2)^2 \times U(1)$–invariant G_2–metrics on simply connected manifolds.

ALC G_2–manifolds from AC Calabi–Yau 3-folds

- **A**symptotically **L**ocally **F**lat **hyperkähler** 4-**manifolds** (up to 2:1 cover)
  $S^1 \hookrightarrow M^4 \setminus K \longrightarrow \mathbb{R}^3$
  $g_M \approx g_{\mathbb{R}^3} + \ell^2\theta^2$
- **A**symptotically **L**ocally **C**onical **G_2–manifolds**
  $S^1 \hookrightarrow M^7 \setminus K \longrightarrow C(\Sigma)$
  - Sasaki–Einstein 5-manifold (Σ, g_Σ)
  - **Calabi–Yau cone** $C(\Sigma) = \mathbb{R}^+ \times \Sigma$, $g_C = dr^2 + r^2 g_\Sigma$

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Theorem (F.–Haskins–Nordström, 2017)

- $(B, g_0, \omega_0, \Omega_0)$ **A**symptotically **C**onical **Calabi–Yau 3-fold**
- $M \to B$ **principal circle bundle** with $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$

$\Longrightarrow$ **S^1–invariant ALC G_2–metric** g_ϵ on M $\forall\, \epsilon \ll 1$
with collapse with bounded curvature $(M, g_\epsilon) \xrightarrow{\epsilon \to 0} (B, g_0)$

Examples with $\mathbf{SU}(2)^2 \times \mathbf{U}(1)$ symmetry

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  $m, n = 1$: $\mathbb{C}_7$ family of ALC G_2–metrics (numerically CGLP, 2001)

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QUESTION: What happens when ϵ increases?

The Gibbons–Hawking Ansatz

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

($a_1, \ldots, a_n$ distinct: g_m complete; $a_1 = \cdots = a_n$: orbifold $\mathbb{C}^2/\mathbb{Z}_n$)

For $m > 0$, g_m is a hyperkähler **ALF** metric

- $m \to \infty$: **collapse** to $\mathbb{R}^3$ (with curvature blow-up at finitely many points)
- $m \to 0$: smooth convergence to an **ALE** metric asymptotic to $\mathbb{C}^2/\mathbb{Z}_n$
- By scaling get different picture of limit $m \to 0$

$$m\, g_m \underset{y = m\, x}{=} \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right) dy \cdot dy + \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right)^{-1} \theta^2$$

  so as $m \to 0$ we have convergence to **orbifold ALF** (Taub–NUT$/\mathbb{Z}_n$)

- **orbifold ALF + ALE** $\rightsquigarrow$ **smooth ALF** with $m \ll 1$

G_2–cones, AC G_2–manifolds and CS G_2–spaces

- **G_2–cone**: $C(\Sigma) = \mathbb{R}^+ \times \Sigma$, $g_C = dr^2 + r^2 g_\Sigma$
  $\mathrm{Hol}(g_C) \subseteq G_2$ iff Σ is a nearly Kähler 6-manifold
  - 4 homogeneous nearly Kähler 6-manifolds (Gray–Wolf, 1968):
    $S^3 \times S^3 = SU(2)^3/\Delta SU(2)$
  - **Free quotients** of $S^3 \times S^3$ (Cortés–Vásquez, 2015)
  - Inhomogeneous nK structures on S^6 and $S^3 \times S^3$ (F.–Haskins, 2017)

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  - **Bryant–Salamon** (1989): 3 examples of AC G_2–manifolds; in particular, AC G_2–metric on $M = S^3 \times \mathbb{R}^4$ asymptotic to the cone over the homogeneous $S^3 \times S^3$
  - Karigiannis–Lotay (2017): this is the **unique** AC G_2–manifold asymptotic to this cone
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- **C**onically **S**ingular **G_2–space**: $M \cap \mathcal{U}_p \simeq (0, \varepsilon) \times \Sigma$ and $g_M = g_C + O(r^\nu)$
  - except for cones themselves, no known examples of G_2–spaces with isolated conical singularities but otherwise complete

Main results

Theorem (F.–Haskins–Nordström, 2018)

- There exists a (unique up to scale) G_2–metric g_0 on $M_0 = (0, \infty) \times S^3 \times S^3$ such that
  - (M, g_0) has an **isolated conical singularity** modelled on the G_2–cone over the homogeneous nearly Kähler structure over $S^3 \times S^3$;
  - (M, g_0) has a **complete ALC end**.
- For every pair of coprime positive integers m, n the (simply connected) total space $M_{m,n}$ of the circle bundle over $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with first Chern class $(m, -n)$ carries a (unique up to scale) **complete AC** G_2–metric asymptotic to the cone over $S^3 \times S^3 / \mathbb{Z}_{2(m+n)}$.

- Infinitely many new geometric transitions in G_2–geometry
- Desingularise (M_0, g_0) using the AC metrics to produce families of complete ALC metrics

Step 1: the ODE system

Work on $(0, \infty) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/K_0$, where $K_0 = \{1\}$ or $K_0 = \mathbb{Z}_{2(n+m)}$

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- **closed SU**$(2)^2 \times$ **U**(1)**–invariant** G_2–structure

$$\varphi = p\, e_1 \wedge e_2 \wedge e_3 + q\, e_1' \wedge e_2' \wedge e_3' + d\left(a\,(e_1 \wedge e_1' + e_2 \wedge e_2') + b\, e_3 \wedge e_3'\right)$$

$$p, q \in \mathbb{R} \qquad a', b' > 0 \qquad F(a,b) = 4a^2(b-p)(b+q) - (b^2+pq)^2 > 0$$

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- If $a=b$ additional isometric right action of $\triangle\mathrm{SU}(2)$ and φ is automatically coclosed
  - The solution with $p=0=q$ is the **G_2–cone** over the homogeneous nearly Kähler structure over $S^3\times S^3$
  - Only when $(p,q)=(r_0^3,-r_0^3),(-r_0^3,0)$ or $(0,r_0^3)$ for some $r_0>0$, φ extends smoothly at $s=0$ over a 3-sphere $\rightsquigarrow$ complete **Bryant–Salamon** AC G_2–metric on $\mathrm{S}^3\times\mathbb{R}^4$

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- In general coclosedness of $\varphi \iff (a,b)$ satisfies the 2nd-order ODE (the Euler–Lagrange equation for Hitchin's volume functional)

$$2F(a'b'' - b'a'') - a'b'\,(a'F_a - 2b'F_b) = 0$$

Step 2: local solutions

- **local solutions in a neighbourhood of the singular orbit** $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ where K is one of

$$\{1\} \times \mathrm{SU}(2), \qquad K_{m,n} = \{(e^{i\theta_1}, e^{i\theta_2}) \mid e^{i(m\theta_1 + n\theta_2)} = 1\}$$

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- 1-parameter family of **local CS solutions** with rate $\nu_0 = \frac{\sqrt{145}-7}{2} \approx 2.5$

$$a(s) = s^3 \left(1 + c\, s^{\nu_0} + \dots\right), \qquad b(s) = s^3 \left(1 - 2c\, s^{\nu_0} + \dots\right)$$

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- for all $p, q \in \mathbb{R}$ a 1-parameter family of **AC ends**

$$s^{-3} a(s) = 1 + O(s^{-3}), \qquad s^{-3} b(s) = 1 + O(s^{-3}),$$
$$b(s) - a(s) = c\, s^{-\nu_0 - 4} + \ldots$$

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$$s^{-3}a(s)=1+O(s^{-3}),\qquad s^{-3}b(s)=1+O(s^{-3}),$$
$$b(s)-a(s)=c\,s^{-\nu_0-4}+\dots$$

This uses standard analysis of singular IVP of the form

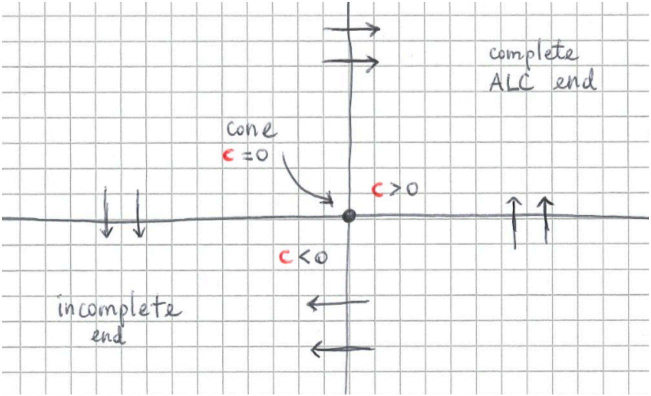
$$sy'=L(y)+Q(s,y),\qquad y(0)=0.$$

Solutions as generalised power series in powers of $s,s^{\nu_1},\dots,s^{\nu_k}$, where $\nu_1,\dots,\nu_k$ are eigenvalues of the linearisation L satisfying **non-resonance** conditions.

Step 3: complete ALC ends

ALC end: $a(s) \approx s^3$, $b(s) \approx \ell s^2$ for some $\ell > 0$

Look at sign of $a - b$ and $a'b - ab'$:



local CS sol's: $a(s) = s^3\,(1 + c\,s^{\nu_0} + \dots)$, $b(s) = s^3\,(1 - 2c\,s^{\nu_0} + \dots)$

Step 4: AC metrics & comparison