Collapsing co-associative fibrations

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PLAN

- Collapsing fibrations.
- 1. Digression: deforming singularities;
  2. Digression: boundary value problem

Collapsing Two general contexts.

Context 1: G_2-geometry
M^7 closed, oriented 7-manifold. $\mathcal{M}$ the moduli space of G_2-structures on M. We are interested in the “boundary”of $\mathcal{M}$. *i.e.* if $\phi_i \in \mathcal{M}$ what can we say about

$$\lim_{i'}(M, \phi_{i'})?$$

(For a subsequence i'.)

Context 2: General Riemannian geometry

(N_i, g_i) Riemannian manifolds with $\mathrm{Ricci} = 0$ (say). Normalise to $\mathrm{Diameter} = 1$.

Non-collapsed case $\mathrm{Volume}(N_i, g_i) \geq \nu > 0$. Relatively well-understood (Cheeger-Colding . . .);

Collapsing case $\mathrm{Volume}(N_i, g_i) \to 0$. Much current activity.

Picture of collapsing that one might expect/hope for.

$$f : N \to B$$

where $\mathrm{dimB} = k < \mathrm{dim}N$, a fibration over a dense open subset $B_0 \subset B$. Diameter of typical fibre $O(\epsilon)$, $\epsilon \to 0$.

1. For a point $b \in B_0$ the rescaled limit is a translation-invariant structure on $X_b \times \mathbf{R}^k$ $(X_b = f^{-1}(b))$.
2. The variation of the structure on X_b as b varies over B_0 satisfies an equation: the "adiabatic limit".
3. There is some appropriate extension of the discussion to the *singular fibres*, over $B \setminus B_0$.

There are (at least) three things one can try to do.

1. Establish that collapsing happens in this way.
2. Extract a sensible/plausible adiabatic limit in dimension k.
3. Prove reconstruction/gluing results:

$$\text{adiabatic solution} \Rightarrow \text{genuine solution for } \epsilon << 1.$$

Model: Gross-Wilson (2000). Study Calabi-Yau metrics on elliptically fibred $K3$ surface. Fixed complex structure, vary Kähler class so that volume of fibre tends to 0.

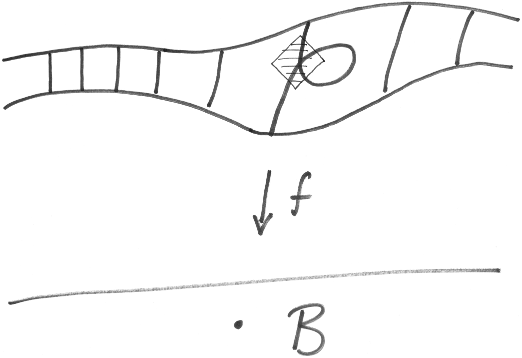
$$f : N \to S^2 = \mathbf{CP}^1$$

a *Lefschetz fibration*, 24 singular fibres with ordinary double points. Local model for f:

$$f(z_1, z_2) = z_1^2 + z_2^2.$$

General fibres are elliptic curves, genus 1.

$$B_0 = S^2 \setminus \{24 \text{ points}\}.$$


f
B

1. Away from the singular fibres the rescaled limit is

$$\text{flat } 2 - \text{torus} \times \mathbf{R}^2.$$

2. The variation of the flat structure gives a *holomorphic map*

$$\mu : B^0 \to \text{moduli space of flat } 2 - \text{tori}.$$

   Also, the limiting metric on B_0 satisfies $\mathrm{Ricci} = \mu^*(\Omega_{WP})$ (Weil-Petersen form).

3. The behaviour around the 24 critical points is modelled on the *Ooguri-Vafa metric*.

Now we go on to our main topic: *collapsing co-associative Kovalev-Lefschetz fibrations* of G_2-manifolds. We focus on item (2): extraction of a sensible/plausible adiabatic limit.

Set-up. $f : M^7 \to B$ for a 3-manifold B. Fibres $X_b = f^{-1}(b)$.

$L \subset B$ a link: $B_0 = B \setminus L$.

At a critical point, over a point of L, the model for f is $f_0 : \mathbf{C}^3 \times \mathbf{R} \to \mathbf{C} \times \mathbf{R}$:

$$f_0(z_1, z_2, z_3, t) = (z_1^2 + z_2^2 + z_3^2, t).$$

Recall that a G_2-structure on M can be given by a 3-form ϕ which is “positive” at each point.

Positive 3-forms

For all $v \in TM, v \neq 0$:

$$i_v(\phi) \wedge i_v(\phi) \wedge \phi > 0.$$

The conditions for a torsion-free G_2-structure are

$$d\phi = 0 \quad , d *_\phi \phi = 0. \quad (****)$$

(Here $*_\phi$ is the $*$ operator of the metric determined by ϕ.)
We assume that the fibres X_b of f are “co-associative” i.e. $\phi|_{X_b} = 0$.

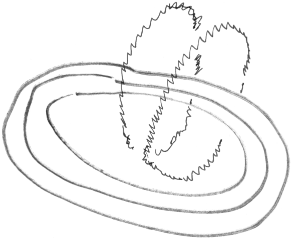
REMARK A special case is $M = N \times S^1, B = S^2 \times S^1$ and all data S^1-invariant. Then we are studying *Calabi-Yau metrics on Lefschetz-fibred complex 3-folds*.

Calabi-Yau geometry $\subset G_2$-geometry.

Kovalev's examples (topologically):

$$S^3 = (D_1^2 \times S^1) \cup_{S^1 \times S^1} S^1 \times D_{22}.$$

- Take two Lefschetz fibrations (with boundary) $g_i : N_i \to D_i^2$, trivial on the boundaries with fibre X.
- Take
$$M = (N_1 \times S^1) \cup (S^1 \times N_2),$$
gluing the boundaries $S^1 \times S^1 \times X$ by interchanging the S^1 factors.
- Then $g_1 \times \mathrm{id}$ and $\mathrm{id} \times g_2$ glue to give a KL fibration $f : M \to S^3$.



Kovalev's link L

One can argue that, in a collapsing sequence of this kind the re-scaled limit over a point $b \in B_0$ should be given by

$$\text{hyperkähler metric on} X_b \times \mathbf{R}^3.$$

If $\omega_1, \omega_2, \omega_3 \in \Omega^2(X_b)$ give the hyperkähler structure, the model positive 3-form on $X_b \times \mathbf{R}^3$ is

$$\sum_{i=1}^{3} \omega_i \, dt_i - dt_1 dt_2 dt_3.$$

The only compact hyperkähler 4-manifolds are $K3$ surfaces and tori. We restrict attention to the K3 case. Recall that, with the cup-product form,

$$H^2(X_b) = \mathbf{R}^{3,19}.$$

Local Torelli for K3 surfaces. The moduli of hyperkähler structures are locally given by the “periods”

$$\mathrm{Span}(\omega_1, \omega_2, \omega_3) \in \mathrm{Gr}_3^+(\mathbf{R}^{3,19}).$$

So for a small open set $U \subset B_0$ we have a map

$$\mu_U : U \to \mathrm{Gr}_3^+(\mathbf{R}^{3,19}).$$

One can argue that the adiabatic limit of the torsion free condition (****) is the condition that μ is the Gauss map of a parametrised *maximal positive submanifold* $h_U : U \to \mathbf{R}^{3,19}$.

Terminology

"positive" submanifold in $\mathbf{R}^{p,q}$: the tangent space is a maximal positive subspace at each point.

"maximal" positive submanifold in $\mathbf{R}^{p,q}$: stationary for the volume functional.

c.f Hitchin's variational characterisation of the torsion-free condition for G_2 structures.

Remark

If $\mu : U \to \mathrm{Gr}_3^+(\mathbf{R}^{3,19})$ is the Gauss map of a maximal positive submanifold then the classical equations of submanifold theory give:

- μ is a *harmonic* map;
- The induced Ricci curvature is

$$\mathrm{Ricci} = \mu^*(g_{\mathrm{Gr}}).$$

Construction of h_U.
Over $f^{-1}(U)$ we can write $\phi = d\sigma$. By the co-associative condition ($\phi|_{X_b} = 0$) the restriction of σ to each fibre is a closed 2-form. Then we define

$$h_U(b) = [\sigma|_{X_b}] \in H^2(X_b).$$

This map h_U is independent of the choice of σ, up to the addition of a constant.

Global formulation

The cohomology along the fibres gives a sheaf $\mathcal{V} = R^2 f_*(\mathbf{R})$ over B. The Leray spectral sequence gives a a natural map $H^3(M) \to H^1(B, \mathcal{V})$ and the image of $[\phi]$ gives a class $\chi \in H^1(B; \mathcal{V})$.

Over $B_0 \subset B$ the sheaf $\mathcal{V}$ can be viewed as a flat vector bundle V with fibre $\mathbf{R}^{3,19}$. The monodromy around a small circle linking L is the reflection r_δ in a vanishing cycle δ:

$$r_\delta(\alpha) = \alpha + (\delta.\alpha)\delta.$$

This means that if (B, L) is regarded as an orbifold, V extends to a flat orbifold vector bundle over B.

The class $\chi \in H^1(B; \mathcal{V})$ defines a lift of V to a flat affine orbifold bundle V_χ.

The global version of the map h_U is a section h of V_χ.

There is a global version of the condition that the image defines a maximal positive submanifold. Around a point of L we are studying solutions with “branch points” as in classical minimal surface theory.

Model: in $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ the graph of the multivalued function $\mathrm{Re}(z^{3/2})$.

Now we arrive at precise questions.
Given such an orbifold bundle V_χ does it admit a maximal positive section? Is the solution unique up to diffeomorphism?
These questions make sense, independent of the existence of any M etc. But we hope that they should be related to existence and uniqueness questions for torsion-free G_2-structures.

Going back to the REMARK we have versions of these questions in the Calabi-Yau 3-fold setting.

The fibres over L carry singular Calabi-Yau metrics, with singularities modelled on $\mathbf{C}^2/\pm 1$.
Yang Li (2017): There is a Calabi-Yau metric on $\mathbf{C}^3$ with tangent cone at infinity

$$\left(\mathbf{C}^2/\pm 1\right) \times \mathbf{C}$$

.
(With subsequent more general results of Székelyhidi, Conlon-Rochon.)
One expects that Li's metric (times $\mathbf{R}$) gives the model for the behaviour of the collapsing sequence around the critical points of f.

Digression 1; deformations of singularities

For the “maximal positive section” set-up to be a sensible adiabatic limit of the G_2 equations (****) one needs to have a Fredholm deformation theory. Away from L this is standard (the linearised equation is a Laplace-type equation for a normal vector field). The problem is to incorporate deformations of the singular set (work in progress).

Very similar questions arise in the study of singular solutions of the G_2*-instanton equation.* (Yuanqi Wang, work in progress). Here we have a 1-dimensional submanifold $\Gamma \subset M^7$ and a connection defined over $M \setminus \Gamma$ with singularity modelled on a singular Hermitian Yang-Mills connection over $\mathbf{C}^3 \setminus \{0\}$, corresponding to a reflexive sheaf on ^{3} (cf. Bando-Siu).

Recall our REMARK. There has been much recent progress in understanding such singular solutions in gauge theory, and the relation to algebraic geometry. (Jacob-Walpuski, Chen-Sun).

ALSO to singular solutions of equations of Seiberg-Witten type over 3-manifolds (Taubes, Haydys, Walpuski, Doan), which are related to G_2-geometry via a programme of Haydys-Walpuski. Takahashi obtains a Fredholm deformation problem for some solutions of this kind.

Digression 2; boundary-value problems

The “indefinite Plateau problem”.

Suppose that a submanifold $\Sigma^{p-1} \subset \mathbf{R}^{p,q}$ is the boundary of some positive submanifold, is it the boundary of a maximal positive submanifold?

(Bartnik-Simon(1982) Yes, if $q = 1$.)

One would like to see this question (with $(p, q) = (3, 19)$ as the adiabatic limit of an existence question for torsion-free G_2-structures. Given M^7 compact with boundary and a closed positive form ρ on ∂M is there a solution ϕ of the G_2-equations (****) with $\phi|_{\partial M} = \rho$?
(A motivation for this is to get interesting PDE and analysis problems without complicated geometry and topology.)

Necessary foundations: a Fredholm theory for this boundary value problem.
The linearised equation can be set up as follows.
Recall that on a G_2-manifold there is a decomposition

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}.$$

On the other hand on the boundary there is a decomposition

$$\Lambda^2 = \Lambda^2_{\partial M} \oplus \Lambda^1_{\partial M}.$$

These are related as follows. The boundary ∂M has an $SU(3)$ structure so there is an 8-dimensional summand

$$\Lambda^{1,1}_0 \subset \Lambda^2_{\partial M}$$

and this lies in Λ^2_{14}.

Now we have the PDE equation, for sections α, ρ of Λ^2_{14}:

$$\Delta \alpha = \rho$$

with boundary conditions

$$\alpha|_{\partial M, 8} = 0, d^* \alpha|_{\partial M} = 0.$$

(Here ρ is given and α is to be found.)
This is a linear elliptic boundary value problem and leads to a Fredholm theory for the nonlinear problem (cf. work of Fine, Lotay & Singer for hyperkähler 4-manifolds).