# Supersymmetry, Ricci flatness and the string/M landscape

Bobby Acharya (ICTP + KCL)

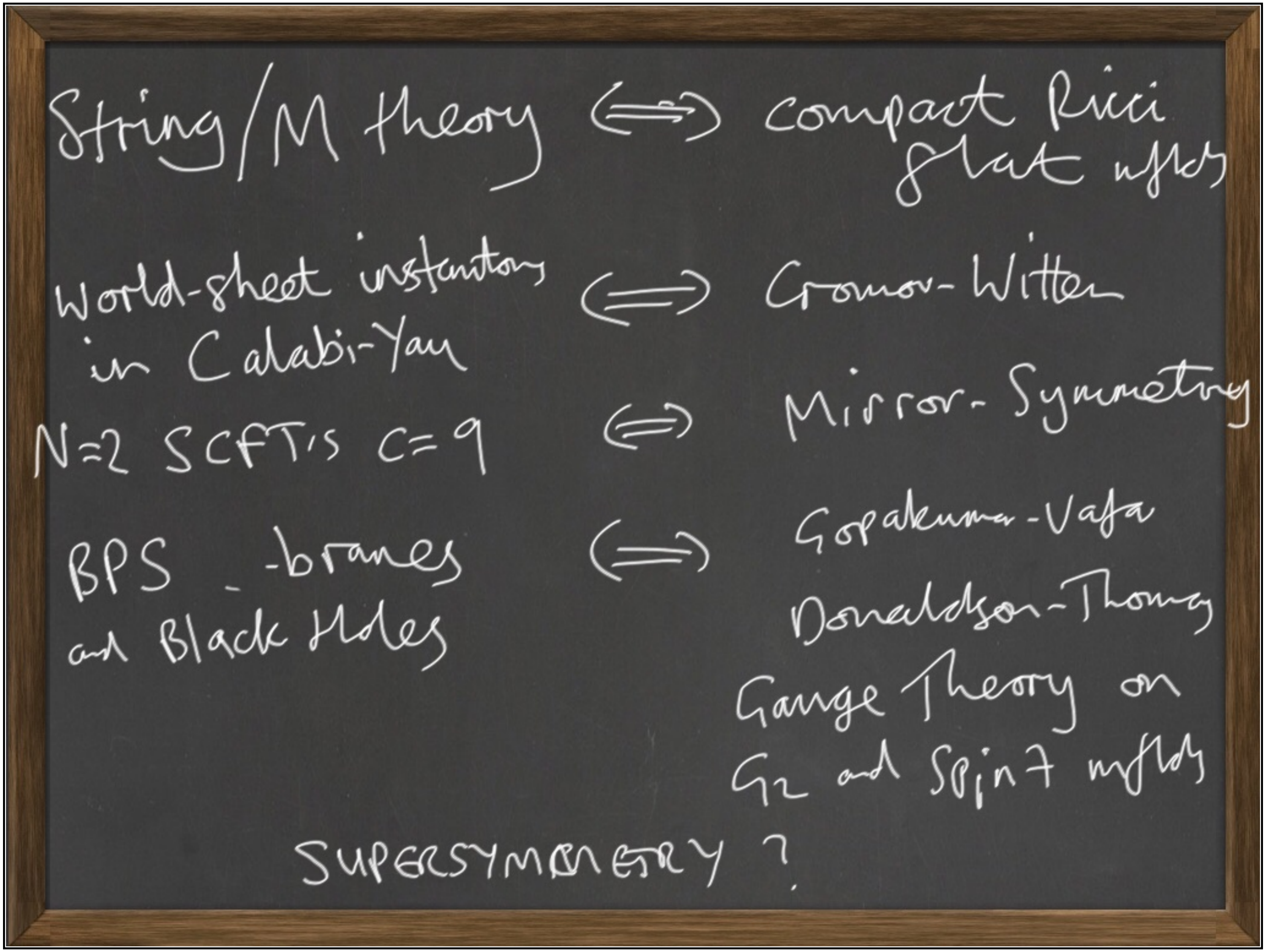
Simons Collaboration: Special Holonomy in Geometry, Analysis and Physics

Annual Meeting at the Simons Foundation

Sep 2018

Compact Ricci flat manifolds are remarkable objects!

Since the '80's they have had lots of significant roles and applications in string/M theory


String/M theory ⟺ compact Ricci flat mflds
World-sheet instantons in Calabi-Yau ⟺ Gromov-Witten
N=2 SCFT's c=9 ⟺ Mirror-Symmetry
BPS -branes and Black Holes ⟺ Gopakumar-Vafa
Donaldson-Thomas
Gauge Theory on G2 and Spin7 mflds
SUPERSYMMETRY ?

These remarkable properties seem to be related to SUPERSYMMETRY.

All of the physics on LHS is supersymmetric

Compact, simply connected Ricci flat manifolds of **special holonomy**: remarkable, rather magical objects.

Are they the **only** such Ricci flat manifolds?

All known examples are of this type.

Supersymmetry manifests itself as a parallel spinor: $\boxed{\nabla_{g_X}\Psi = 0}$

↑ Levi-Civita of $g_X$

# Berger/Simons Holonomy Classification

| Dim(X) | Hol($g_X$) | Name |
|---|---|---|
| $n$ | $SO(n)$ | Riemannian |
| $n=2k$ | $U(k)$ | Kähler |
| $n=2k$ | $SU(k)$ | Calabi-Yau |
| $n=4k$ | $Sp(1)\cdot Sp(k)$ | Quaternionic Kähler |
| $n=4k$ | $Sp(k)$ | HyperKähler |
| $n=7$ | $G_2$ | Exceptional |
| $n=8$ | $Spin(7)$ | Exceptional |

- $X$ irreducible, $\pi_1(X)=0$

# Ricci flatness and Holonomy

| $\mathrm{Dim}(X)$ | $\mathrm{Hol}(g_X)$ | Ricci flat? $\mathrm{Ric}(g_X)=0$? |
|---|---|---|
| $n$ | $SO(n)$ | |
| $n=2k$ | $U(k)$ | NO |
| $n=2k$ | $SU(k)$ | YES |
| $n=4k$ | $Sp(1)\cdot Sp(k)$ | NO |
| $n=4k$ | $Sp(k)$ | YES |
| $n=7$ | $G_2$ | YES |
| $n=8$ | $Spin(7)$ | YES |

# Holonomy and Ricci flatness.

| Dim(X) | Hol($g_X$) | Name | Parallel Spinor? $\nabla\Psi=0$ | Ricci flat? $Ric=0$. |
|---|---|---|---|---|
| $n$ | $SO(n)$ | Riemannian | ✗ | ??? |
| $n=2k$ | $U(k)$ | Kähler | ✗ | ✗ |
| $n=2k$ | $SU(k)$ | Calabi-Yau | ✓ | ✓ |
| $n=4k$ | $Sp(1)\cdot Sp(k)$ | Quaternionic Kähler | ✗ | ✗ |
| $n=4k$ | $Sp(k)$ | HyperKähler | ✓ | ✓ |
| $n=7$ | $G_2$ | Exceptional | ✓ | ✓ |
| $n=8$ | $Spin(7)$ | Exceptional | ✓ | ✓ |

The known compact simply connected Ricci flat manifolds are

supersymmetric $\Leftrightarrow \nabla_i \eta = 0$

Would not be true for generic holonomy Ricci flat.

## Main Question:

Are there compact, simply connected Ricci flat manifolds with *generic* $SO(n)$ Holonomy?

($\pi_1(X) = 0$ as we use Cheeger-Gromoll splitting theorem).

(Will also discuss finite $\pi_1(X)$ later)

Let's consider this question from a physical point of view and explore the consequences.

Physically, a compact, Ricci flat manifold, X, with dim (X) ≤ 10 provides a <u>model</u> for the extra dimensions in string/M theory.

$$\text{Spacetime} = M^{D,1} \equiv X_n \times \mathbb{R}^{D-n,1}$$

$$g(M) = g(X_n) + \text{flat}$$

- "Integrating over $X$", retaining zero modes and taking a low energy limit gives a physical theory of gravity and matter in asymptotically flat spacetime.
- This is the low energy limit of string/M theory on $X$.

## Quantum Gravity and Stability

- $g(X)$ is part of a fluctuating quantum field $\hat{g}(X) = g(x) + \delta g(x) + \ldots$
- Perturbative stability: $\delta g$ small requires that the spectrum of linearised Einstein equations is semi-positive

- Perturbative stability

so spectrum of Lichnerowicz Laplacian $-\Delta_L \delta g$ (on transverse, traceless modes) is semi-positive

$$\Delta_L \delta g = -\nabla^2 \delta g - \text{Riem} * \delta g$$

$$\text{Riem} * \delta g \simeq R_a{}^c{}_b{}^d \, \delta g_{cd}$$

Action $S = \int \sqrt{g} R$ Potential $V(g_x) = -\int \sqrt{g_x} R(g_x)$

- Minimally we require perturbative stability. More generally:
- Suppose that the background $g(x)$ with $Ric(g(x)) = 0$ is a <u>good</u> approximation to a <u>quantum</u> string/M theory <u>state</u>

$\Rightarrow$ Quantum gravity in Minkowski spacetime

So, if $g(X)$ is "suitably stable", we will obtain a consistent model of quantum gravity plus matter.

Stable Ricci flat $X$ $\Longleftrightarrow$ Quantum gravity in Minkowski space

For $\pi_1(X) = 0$ we have:

This physical model is supersymmetric

$\iff$ $(X, g(X))$ has parallel spinors

i.e. special holonomy

These are <u>always</u> perturbatively stable (due to susy) and often suitably stable in the full theory.

> We conjecture: for compact, simply connected $X$, there are <u>no</u> <u>stable</u> Ricci flat metrics with generic $SO(n)$ holonomy.

$\Rightarrow$ All Minkowski spacetime physics in the string landscape is supersymmetric.

T. Banks (2000) has conjectured (from a very different point of view) that the only quantum theories of gravity in Minkowski space are exactly supersymmetric.

Other remarks:

String Landscape vs Swampland (Vafa)

↑ All "consistent" string/M theory vacua

↑ Everything else

Which Ricci flat manifolds "give" us points in the Landscape??

Related ideas:

(Ooguri-Vafa) In the context of anti-de Sitter (AdS) spacetime have conjectured that non-supersymmetric AdS is in the swampland

- Instabilities also important

. . .

- Elaborating on this suggests that Ricci flat metric cones of generic holonomy are unstable.

⇒ expect +ve scalar curvature Einstein 5-manifolds and 7 manifolds to have instabilities when they are <u>not</u> Sasaki-Einstein or weak $G_2$.

⇒ Examples studied by Ooguri and Spodyneiko that have "Witten Instabilities".

Horowitz, Polchinski, Ogera for $S^5/\Gamma$.

We will return to these sorts of examples later

## Ricci flat metrics on simply connected manifolds

If $\dim(X) = 1, 2, 3$ , straightforward

So conjecture is non-trivial for $\dim(X) \geq 4$.

In certain dimensions one can prove that "many" manifolds <u>cannot</u> have Ricci flat metrics

- Topological obstructions
  - Hitchin-Thorpe in $n=4$
  - Seiberg-Witten (see LeBrun)
  - Index obstructions using harmonic spinors (Hitchin)
- Explicit attempts have "failed"
  eg Brendle-Kapouleas

- $\dim(X) = 4$

Simply connected compact, 4-manifolds are of the topological types:

Non-spin: $Y(p,q) \equiv p\,\mathbb{CP}^2 \# q\,\overline{\mathbb{CP}^2}$

Spin: $X(m,n) \equiv m\,K3 \# n\, S^2 \times S^2$ (plus reversed orientation)

(modulo $\frac{11}{8}$ conjecture)

## Hitchin-Thorpe Inequality:

$\chi$: Euler
$\tau$: sign

If $(X, g_X)$ is Einstein then

$$2\chi \geq 3|\tau|$$

Equality iff $X = K3$ $(\pi_1(X) = 0)$.

So, for $q \gg p$ $Y(p,q)$ has no Einstein metrics.

Le Brun has used Seiberg-Witten theory to construct manifolds which satisfy Hitchin-Thorpe but can not be Einstein.

## Harmonic Spinors

(Lichnerowicz, Hitchin)

- Let $S \longrightarrow X$ be the spin bundle ($S^+ \oplus S^-$)
- $\not{D}_g$ denotes the Dirac operator on $(X, g_X)$
- Lichnerowicz formula

$$\not{D}_g^2 = \nabla_g^2 + \frac{1}{4} R_g$$

where $R_g$ is the scalar curvature

If $Ric(g) = 0$, $\not{D}_g^2 = \nabla_g^2$ so harmonic spinors are parallel.

$\Rightarrow$ A Ricci flat manifold has harmonic spinors iff $g_X$ has special holonomy.

Can use this to classify scalar flat compact manifolds in $4k$ dimensions.
(Hitchin, Futaki)

In $\dim(X) = 4$ spin category,

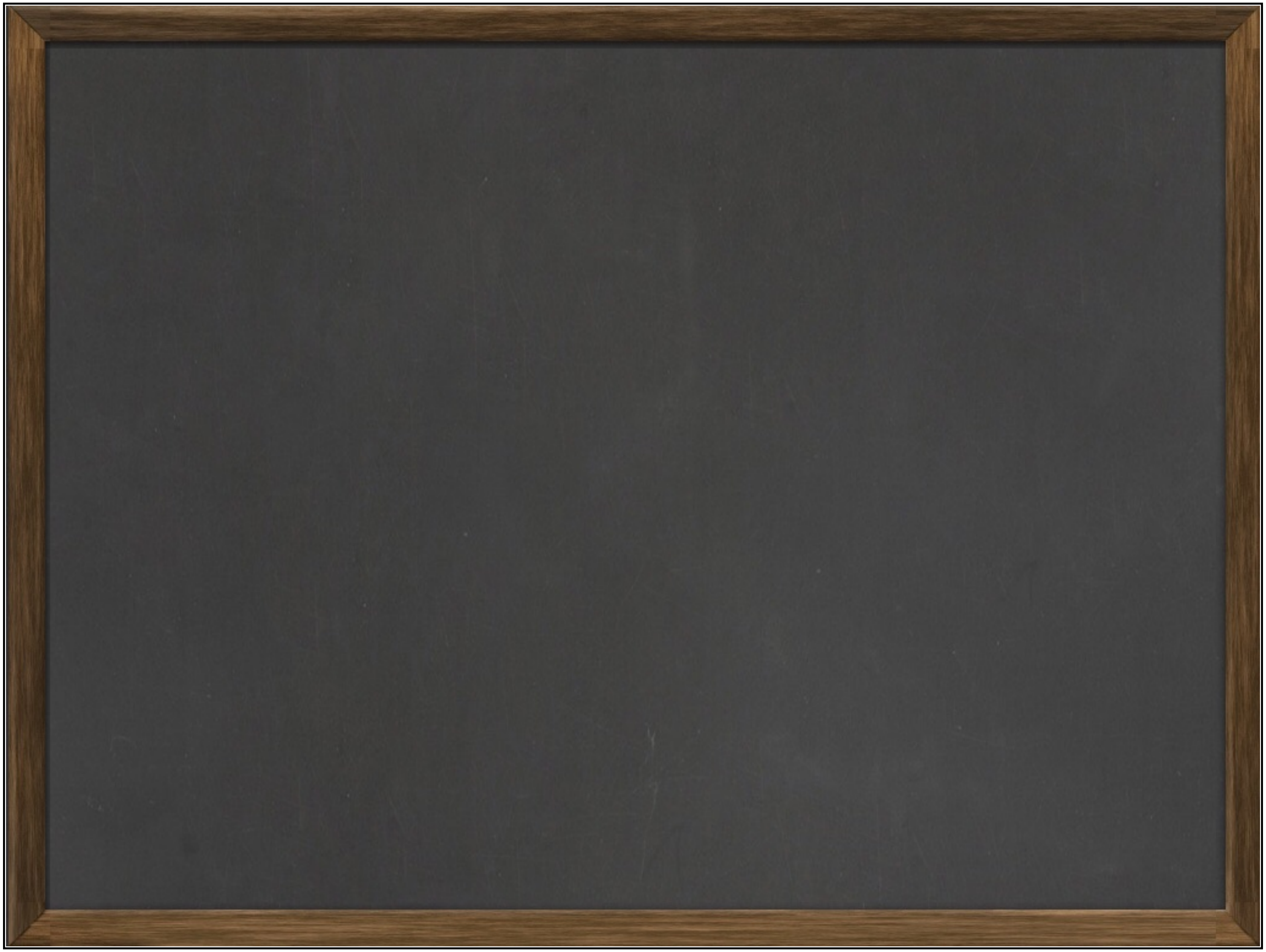
$$X(m,n) = m\,K3 \# n(S^2 \times S^2)$$

and all of these satisfy Hitchin-Thorpe

$$\mathrm{ind}\left(\not{D}_{g_X}(X(m,n))\right) = 2m \quad \forall n .$$

$\Rightarrow$ We obtain: $X(m \geq 1, n > 0)$ does not admit Ricci flat metrics.

Proof: The $2m$ harmonic spinors would be parallel, implying $SU(2)$ holonomy which only happens when $X = K3$.



- Can $S^2 \times S^2$ be Ricci flat?

This would be very interesting to settle, even for the standard smooth structure.

- Could the case $X = K3 \# (S^2 \times S^2) = X(1,1)$ be useful?
- By the above there are no Ricci flat metrics.
- Gluing?
- <u>Kapovitch-Lott</u>: if $X$ is cpt, spin 4-mfld with $\hat{A}(X) \neq 0$ and $\mathrm{vol}(X,g) \geq v$, $\mathrm{diam}(X) \leq 1$ and $|\mathrm{Ric}(X,g)| \leq \varepsilon \Rightarrow X \cong K3$.

So $K3 \# S^2 \times S^2$ has no <u>almost</u> Ricci flat metric with $\mathrm{diam}(X) \leq 1$.

- They used general theory of Cheeger,
  Cheeger-Naber,
  Anderson ..

<u>$\dim X = 8$</u>. If $X$ is not a product

If $\hat{A}(X) = 1, 2, 3$ ; Spin(7), SU(4), Sp(2)

Holonomy

are the only possibilities for

Ricci flat manifolds.

So if $\hat{A}(X) > 4$, no Ricci flat metric exists.

# Explicit Constructions

- Can one explicitly construct/prove that a Ricci flat metric exists?
- Could try gluing non-compact known Ricci flat metrics to obtain compact ones
- Idea of Page: "non-supersymmetric Kummer construction" was investigated by Brendle-Kapouleas

Kummer construction of K3:

a) Flat, singular orbifold $T^4/\pm\mathbb{1}$

b) Remove the 16 singular pts

c) Glue in 16 copies of $T^*S^2$

This gives a manifold, diffeomorphic to K3 if $T^*S^2$'s all have same orientation. Since $T^*S^2$ admits a hyperkähler Ricci flat metric (Eguchi-Hanson) one can try and prove the existence of a hyperkähler metric in this limit.

d) This was done by Topinalo, see Donaldson for a very clear description of this gluing construction.

• If we reverse orientation of any of the $T^*S^2$'s in this gluing, we no longer get K3 but another 4-manifold. •

Brendle-Kapouleas studied this

## Brendle Kapouleas

They took 8 Eguchi-Hansons and
8 Eguchi Hanson's
arranged in a "checkerboard"
configuration.

| | | | |
|---|---|---|---|
| + | − | + | − |
| − | + | − | + |
| + | − | + | − |
| − | + | − | + |

8 SD + 8 ASD.

In the limit of large $T^4/\mathbb{Z}_2$ and small Eguchi-Hanson's, they showed perturbatively that **no** Ricci flat metric exists.

Interesting that even though a given E-H is **far** from any given $\overline{E\text{-}H}$, the interactions between them in the perturbative Einstein equations are non-trivial.

---

Non compact model worth studying:

$$\frac{\mathbb{R}^3 \times S^1}{\pm \mathbb{1}} + \text{Eguchi-Hanson} + \text{Eguchi-Hanson}$$

# Witten Instability of $\mathbb{R}^4 \times S^1$ (1983)

- Witten gave alternative proof of Schoen-Yau +ve mass theorem for $\mathbb{R}^n$ using spinors
- Around same time he showed that flat $\mathbb{R}^4 \times S^1$ is unstable
- $\exists$ a Euclidean instanton (based on Euclidean Schwarzschild)
- $\Rightarrow$ A bubble can be nucleated which eats the space from inside (v. quickly)
- $S^1$ has odd spin structure.

## Witten

Euclidean Schwarzschild $d=5$

$$ds^2 = \frac{dr^2}{1-\frac{R^2}{r^2}} + r^2 d\Omega_3^2 + \left(1-\frac{R^2}{r^2}\right) d\phi^2$$

$d\Omega_3^2$ ↑ round $S^3$

$\phi$: $S^1$ $(2\pi R)$

$r: (R, \infty)$  At $r = \infty$, radius of $S^1 = R$.

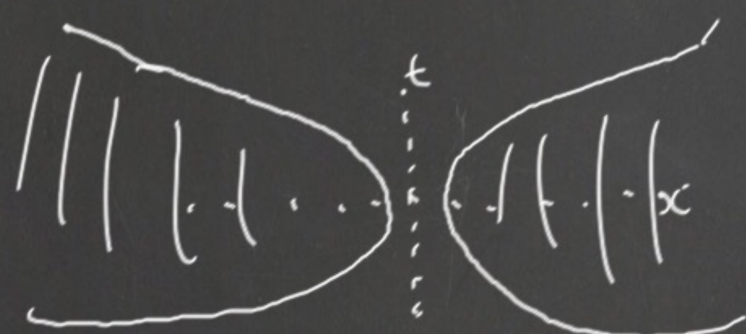
At $r \to R$ $S^1$ collapses and the geometry "ends".

Analytically continuing along a polar angle in $S^3$ gives a metric asymptotic to flat $\mathbb{R}^{3,1} \times S^1$ with an expanding "bubble of nothing".

$$ds^2 = \frac{dr^2}{(1-R^2/r^2)} + r^2(d\theta^2 + \cos^2\theta\, d\Omega_2^2) + \left[1-R^2/r^2\right) d\phi^2$$

$$d\bar{s}^2 = \frac{dr^2}{(1-R^2/r^2)} + r^2(-dt^2 + \cosh^2 t\, d\Omega_2^2) + (1-R^2/r^2)\, d\phi^2$$

Bubble of <u>nothing</u> which eats space from inside out. (Witten)




<u>Nothing</u> inside.

## Compact Ricci flat mflds with $\pi_1$ finite

- eg Enriques surface $\cong K3/\mathbb{Z}_2$

These admit Ricci flat metrics coming from Ricci flat metrics on K3 which are $\mathbb{Z}_2$ invariant.

- These are perturbatively stable (as HK metrics on K3 are)

- But these metrics have no parallel spinors, so no supersymmetry

We conjecture that: Compact $(X, g_X)$ Ricci flat manifolds without parallel spinors and finite $\pi_1(X)$ are unstable.

(with Narain)

For Kummer Enriques:

$$X^4 = T^4 / \mathbb{Z}_2 \times \mathbb{Z}_2$$

String theory here is "exactly solvable".

In the small volume limit, there are negative eigenvalue modes, which are strings wound around the $S^1 \subset X^4$ which generates $\pi_1(X^4) = \mathbb{Z}_2$.

So, there is an instability at small volume.

Physically, in $X^4 \times \mathbb{R}^{5,1}$, make a small bubble in $\mathbb{R}^{5,1}$ inside of which this mode is excited.

This bubble will grow and contain the "true" vacuum.

At large volume there could be a Witten instability

This would be a Ricci flat Lorentzian 10-manifold $(M^{9,1}, g(M))$ asymptotic to $(X^4 \times \mathbb{R}^{5,1})$.

$$ds^2 = \frac{dr^2}{\left(1-\frac{R^2}{r^2}\right)} + r^2\left(-dt^2 + \cosh^2 t \, d\Omega_2^2\right) + \left(1-\frac{R^2}{r^2}\right) d\phi^2$$

$$+ \tilde{R}^2 \left(dy_1^2 + dy_2^2 + dy_3^2\right) + dx_1^2 + dx_2^2$$

$$+ \text{ small corrections.} \qquad (\phi, y_i) \in T^4 / \mathbb{Z}_2 \times \mathbb{Z}_2$$

(Work in progress with Narain)

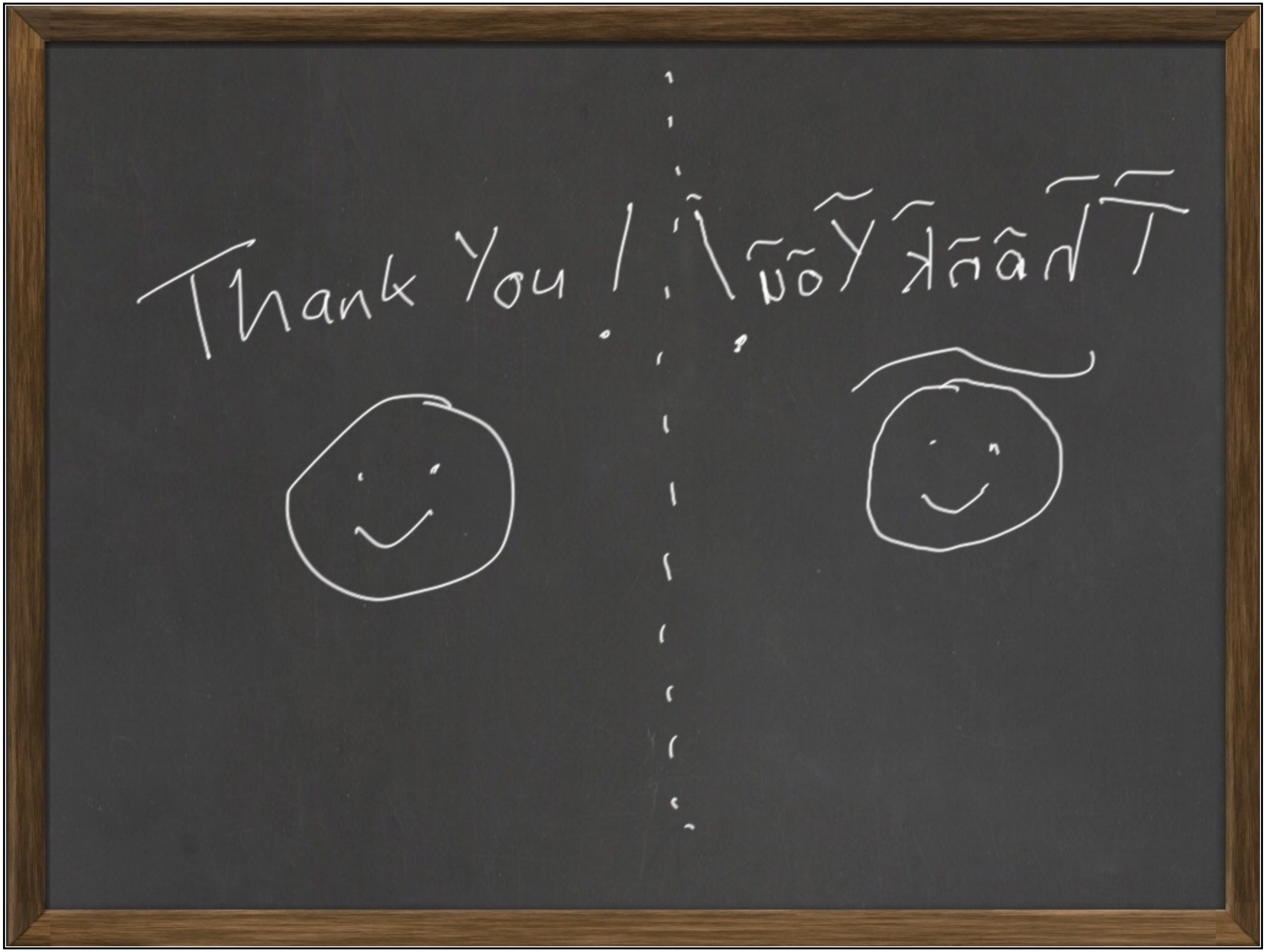
In summary:

- lots to do in simply connected case
  - hard problems in general so any progress is welcome
- non-simply connected case: looks as if Witten instabilities might be a general phenomenon

# Remarks

- If the conjecture were true it could be regarded as a sort of prediction that supersymmetric particles exist at some unspecified scale <u>below</u> the string scale.

- If it were not true, then, since special holonomy manifolds exist, one can still have supersymmetry


Thank You !