The Particle Physics Interpretation of Particular SU(3) and G_2-holonomy Singularities

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Based on work done with:

SU(3)-holonomy: N. Lambert, M. Najjar, E. Svanes, J. Tian

G_2-holonomy: L. Foscolo

Motivation

M-theory on special holonomy spaces with particular kinds of singularities leads to a beautiful, geometric interpretation of particle physics phenomena, including models of real world particle physics

- One of the main raisons d'etre of this Simons Collaboration has been to understand this picture more deeply
- Constructions/Existence proof of e.g. G_2-hol spaces with conical singularities

- Much progress has been made both in the compact and non-compact cases
- Here, I will explain some recent progress in understanding conical G_2-spaces associated with unified particle physics models
- Also, if time, will discuss physics of some Asymp. Conical CY 3-folds

Outline

(1) Review G_2-holonomy singularities predicted by physics considerations

(3d families of Kronheimer ALE spaces, $X^4_{\Gamma_{ADE}}$, $\Gamma_{ADE} \in SU(2)$))

G_2-holonomy spaces: $X^7_{\Gamma_{ADE}} = \left\{ X^4_{\Gamma_{ADE}}(\{\omega_i\}_{S^2}) \,\middle|\, \forall \{\omega_i\}_{S^2} \right\}$

HK triple

- Claim: Cones over twistor spaces of $SO(3)$-inv S-D Einstein metrics on S^4 realise a particular series of $X^7_{D_K}$ (Hitchin), proving the conjecture in BSA/Witten

- Physically this is how one realises particles transforming as $\Lambda^2(\mathbb{C}^N)$ in $SU(N)$ gauge theories. eg the familiar $\underline{10}$ of $SU(5)$ Grand Unified theories receive this geometric interpretation in M-theory.
- For $N=3$ $\Lambda^2(\mathbb{C}^3)$ is the (anti) fundamental rep^n of $SU(3)$, so this is closely related to $\mathbb{R} \times WCP^3_{3,3,1,1}$ and another conjecture of BSA/Witten

2 5d Super Conformal Field Theories

- Arise from singular CY 3-folds
- Discuss briefly the metric
- Will study M-theory on $\mathbb{C}^3/\Gamma$ $\Gamma \in SU(3)$

Example of $\Gamma = \Delta(3N^3) = \mathbb{Z}_3 \ltimes (\mathbb{Z}_N \times \mathbb{Z}_N)$

Use results of Ito-Reid to predict the properties of infinite series of 5d SCFT's.

1 Particle Physics models from G_2-holonomy spaces.

(X^7, φ) generic notation for G_2-hol. space.

- Codim 4 orbifold singularities in X_7 are of type $\Gamma_{ADE} \in SU(2) \Leftrightarrow$ ADE gauge symmetries and fields localised there.
- Physics predicts that if Q_{ADE} is such a 3d orbifold locus, at points on Q_{ADE} where the ADE singularity increases rank,

in particular ways such that X^7 develops a codimension 7 singularity, that massless particles charged under ADE group are localised there.

(BSA/Witten, Atiyah/Witten).

- In particular, chiral fermions, (which are described by spinor fields in complex rep^{ts} of ADE) arise. These are crucial building blocks of the standard model of particle physics.

X^7 as a family of ALE spaces,

- $X^4_{\Gamma_{ADE}} \cong \widetilde{\mathbb{C}^2 / \Gamma_{ADE}}$ with asymptotically locally Euclidean, Hyperkähler metric.
  ω_i = HK triple

- Kronheimer: Moduli space of HK metrics $\cong$ HyperKähler quotient quiver variety based on affine Dynkin Diagram of ADE type.

- Explicitly, Kronheimer showed that for every -2-curve $\Sigma \in X^4_{\Gamma_{ADE}}$ and every triple $\alpha_i \in H^2(X,\mathbb{R})$ s.t. $\alpha \cdot \Sigma \neq 0$ $\exists$ an HK-triple, ω_i, with $[\omega_i] = \alpha_i$ exists.

- All ALE HK metrics arise this way

- The -2 curves are the exceptional divisors in $\pi : X^4_\Gamma \longrightarrow \frac{\mathbb{C}^2}{\Gamma_{ADE}}$

Example 1 $\quad X^4_{r_{A_1}} = \dfrac{\widetilde{\mathbb{C}^2}}{\mathbb{Z}_2}$

Here $H_2(X) \cong \mathbb{Z}$ and there is one -2 curve, $\Sigma \cong S^2$

Hence the moduli space is given by the 3-periods $\left\{ S_i = \int_{S^2} \omega_i \right\} \cong \mathbb{R}^3$

The total space $X_7 = \left\{ X^4_{r_{A_1}}, S_i \right\}$

fibers $\dfrac{\widetilde{\mathbb{C}^2}}{\mathbb{Z}_2} \to X_7 \longrightarrow \mathbb{R}^3$

The S^2 collapses at the origin

where the fiber is $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$.

In fact one thinks physically as, starting with $SU(2)$ gauge symmetry at the origin and generically breaking it as $SU(2) \longrightarrow U(1)$ by a commuting triple of Higgs fields.

Physically this realises $U(1)$ with an "electron" and $X_7 = \mathbb{R}^+ \times \mathbb{CP}^3$.

Happily, $\exists$ a conical metric on $\mathbb{R}^+ \times \mathbb{CP}^3$, that of Bryant-Salamon.

<u>Example 2</u>

Consider $\Gamma_{A_N} = \mathbb{Z}_N$ and the Kronheimer family of ALE spaces in which $A_N \longrightarrow A_{N-1}$. This requires taking the 2-sphere represented by the <u>last</u> node of the Dynkin Diagram.

Hence X_7 is the universal family of partial resolutions of $\frac{\mathbb{C}^2}{\mathbb{Z}_{N+1}}$ where we resolve only the last node.

ie $\widetilde{\frac{\mathbb{C}^2}{\mathbb{Z}_{N+1}}} \hookrightarrow X_7 \longrightarrow \mathbb{R}^3$

Partial resol^{n}s

Here the generic fiber has an A_{N-1} singularity, whilst that over the origin has an A_N singularity.

$X^7 \cong \mathbb{R}^+ \times W\mathbb{CP}^3_{N,N,1,1}$ realises particles in the fundamental of $SU(N)$.

Here, we do not know if a G_2-holonomy metric exists. We will later solve the problem for $N=3$.

Particle physics models often require $\Lambda^2(\mathbb{C}^{k-2})$ rep$^{\underline{n}}$ of $SU(k-2)$,

These arise from $D_{k-2} \longrightarrow A_{k-3}$.

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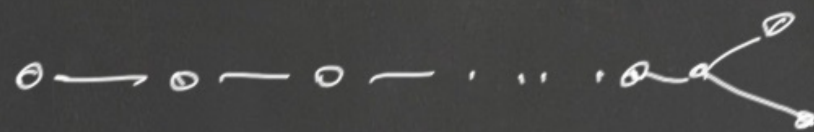


use this node.

In BSA/Witten we couldn't provide a useful description of X^7.

Here we will provide a description and show that a conical G_2-metric exists!

(This part is joint work with L. Foscolo)

In fact, the metrics are already in the literature.

Hitchin (Einstein metrics and Isomonodromic Transformation)

Constructed $SO(3)$-invariant cohomogeneity one Einstein metrics with $\int R > 0$ on S^4.
These have Self-Dual Weyl tensor.

Hitchin uses twistor method
to construct a compactification,
Z_k of $\dfrac{SL(2,\mathbb{C})}{\Gamma_{D_{k-2}}}$ which turns
out to be the twistor space of
of $S^4 \setminus \mathbb{RP}^2$.

- He was partly motivated to solve the
Isomonodromic problem for $\mathbb{P}^1 \setminus \{p_i\}$ by using the
natural flat connection on $\dfrac{SL(2,\mathbb{C})}{\Gamma_{D_{k-2}}}$ and

extending it to Z_k. Here the $\mathbb{P}^1$ intersects the "compactifying divisor" at the "punctures" p_i, providing the solution to the Isomonodromy problem.

Z_k can be described as the projectivisation of a series of rank 2 vector bundles over $\mathbb{CP}^2$, labelled by k.

$SO(3,\mathbb{C})$ acts on Z_k with dense, open orbit $SO(3,\mathbb{C}) / \Gamma_{D_{k-2}}$

After identifying the required real structure and real twistor lines one can show that Z_k (minus a suitable set) is the twistor spasse of a SD Einstein orbifold. This turns out to be S^4 with a degree $k-2$ orbifold sing. along $\mathbb{RP}^2$.

Bryant-Salamon: if (M^4, g_{SDE}) is a complete S-D Einstein orbifold with +ve scalar curvature, then $\exists$ a G_2-holonomy metric on $\Lambda^2_-(M^4)$.
This is "complete" as an orbifold and is Asymptotically conical.
At ∞ the cone is $\mathbb{R}^+ \times Z(M)$ where $Z(M^4)$ is the twistor space.

Singularities of X_7

Hitchin's metrics on S^4 have an orbifold singularity along an embedded $\mathbb{RP}^2$ of the form (locally) $\frac{\mathbb{C}}{\mathbb{Z}_{k-2}} \times \mathbb{RP}^2$.

In Z_k, this $\mathbb{RP}^2$ lifts to an S^2 along which we have a cod^{n} 4, A_{k-3} singularity.

In $\Lambda^{2-}_{-}(S^4)$, this becomes an A_{K-3} singularity along $\frac{\mathbb{R}\times S^2}{\mathbb{Z}_2}$ (non trivial line bundle over the $\mathbb{RP}^2$)

$\Rightarrow$ A_{K-3} gauge symmetry along $\frac{\mathbb{R}\times S^2}{\mathbb{Z}_2}$.

* Monodromy of $\frac{\mathbb{C}^2}{\mathbb{Z}_{K-2}}$ around $\pi_1\left(\frac{\mathbb{R}\times S^2}{\mathbb{Z}_2}\right)$

$\Rightarrow$ Hints that A_{K-3} is broken by this monodromy.

But to what?

Two possibilities: $SO(K-2)$ or $Sp\left[\frac{K-3}{2}\right]$.

On the other hand, when $\Lambda^2_-(S^4)$ is degenerated to $\mathbb{R}^+ \times \mathbb{Z}_k(S^4)$ we see the A_{k-3} gauge symmetry along $\mathbb{R}^3$ and a conical singularity at the origin.

This is exactly as before!

Is this describing A_{K-3} gauge symmetry plus particles in some representation of A_{K-3} ?

YES! In fact the fiber over the origin is precisely $\frac{\mathbb{C}^2}{\Gamma_{D_{k-2}}}$.

Hence $\mathbb{R}^+ \times Z_k(S^4) \cong X_7(D_{k-2} \to A_{k-3})$ precisely the space we discussed before!

According to that discussion, we expect particles transforming as $\Lambda^2(\mathbb{C}^{k-3})$ of A_{k-3}.

This proves the conjecture in (BSA/Witten).

To see that $\mathbb{R}^+ \times Z_k(S^4) \cong X^7(D_{k-2} \to A_{k-3})$,

$Z_k(S^4)$ contains (a Lagrangian) $L = \dfrac{S^3}{\Gamma_{D_{k-2}}}$

$\left(Z_k \setminus \begin{pmatrix}\text{Anti canonical}\\ \text{divisor}\end{pmatrix} \cong \dfrac{SO(3,\mathbb{C})}{\Gamma_{D_{k-2}}} \right)$

Complement of $L = Z\left(\dfrac{\widetilde{\mathbb{C}^2}}{\Gamma_{D_{k-2}}}\right) = S^2 \times \dfrac{\widetilde{\mathbb{C}^2}}{\Gamma_{D_{k-2}}}$

Here $\dfrac{\widetilde{\mathbb{C}^2}}{\Gamma_{D_{k-2}}}$ is the partial resol$^{\underline{n}}$ with A_{k-3}-sing

Supersymmetry requiring both scalar fields (ϕ) and spinor fields to be localised at the codim^{n} 7 singularity.

if $\phi \in \Lambda^2(\mathbb{C}^{K=2})$, then a generic $\langle\phi\rangle \neq 0$ can be thought of as a symplectic form, $\langle\phi\rangle = a\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$

This breaks $A_{K-3} \rightarrow Sp\left(\left[\frac{K-3}{2}\right]\right)$.

So this is compatible with the smoothing of the conical singularity.

Anomaly

Fermions in $\Lambda^2(\mathbb{C}^N)$ of $SU(N)$ are "anomalous".

Witten (Anomaly Cancellation in G_2-manifolds) explained how to calculate such anomalies.

The local $\frac{\mathbb{C}^2}{\mathbb{Z}_{K-2}}$ singularities admit a certain $U(1)$ action. There is a corresponding $U(1)$ connection along $\mathbb{R}^3$ whose 1st Chern class on the bounding S^2 is the anomaly.

The computation of this degree gives $d = 4 - (k-2)$

This is precisely the anomaly coefficient of the $\Lambda^2(\mathbb{C}^{k-2})$ rep^n of $SU(k-2)$.

Note that, since $A_3 = D_3$,

we have $X_7(D_3 \longrightarrow A_2)$

$$\cong X_7 (A_3 \to A_2)$$

So, it is natural to expect that this is $\mathbb{R}^+ \times W\mathbb{CP}^3_{(3,3,1,1)}$

Hence we would establish that $\exists$ a G_2-holonomy cone metric on $\mathbb{R}^+ \times W\mathbb{CP}^3_{(3,3,1,1)}$

(2) M theory on Calabi-Yau Orbifolds $\frac{\mathbb{C}^3}{\Gamma}$.

Work done with

N. Lambert, M. Najjar,
E. Svanes and J. Tian

5d Super Conformal Field Theories

$\Updownarrow$

Codimension Six Singularities of Calabi-Yau 3-folds.

90's : Seiberg, Morrison/Seiberg
Intriligator/Morrison/Seiberg

plus:

Lots of work since then, many hundreds of papers.....

Including, by some of our collaborators here (Schafer-Nameki et al)

- Because these theories have "enough" supersymmetry, one can use "topology" plus analyticity of the field theory to make statements about the 5d SCFT's. eg phase structure, moduli space even count BPS states.

- But, what about the Calabi-Yau metric?

Calabi-Yau metrics for 5d SCFT's

- Very little discussion in physics literature
- Physicists tend to assume that some version of the Calabi-Yau theorem will hold in these singular, non-compact settings.....

- 5d SCFT's do not contain gravity.
- Decoupling limit: think of a <u>compact</u> CY 3-fold which develops a singularity via a divisor collapsing to a point.
- physics analysis shows there are new, zero mass degrees of freedom at the singularity

- So lets decouple gravity by

$$G_N \longrightarrow 0$$

- We might then be left with a non-trivial QFT in the limit

- But $G_N \longrightarrow 0$ means rescaling the region away from the singularity and pushing it to <u>infinity</u>.

- The result ought to be a non-compact CY 3-fold
- Metric? As we go off to 'infinity', $r \to \infty$, the CY metric must look like

$$g_{CY} = dr^2 + r^\alpha g(\Sigma_5) + \dots$$

So, we could define metrics with given asymptotic behaviour.
But are there different sensible choices for α?

Only the conical case $\alpha = 2$ has the scaling symmetry

$$r \longrightarrow \lambda r, \quad \text{Ricci} = 0$$

which would be inherited by the 5d QFT

Secondly, the full spacetime $(CY_3 \times \mathbb{R}^{4,1}, g)$ should be complete as a metric space.

Otherwise one has degrees of freedom at the boundary of the complete region.

- So, complete, asymptotically
conical, Calabi-Yau metrics
($\alpha = 2$)
seem to be the natural candidates

- Note: $\exists$ algebraic CY 3-fold
singularities which do NOT
have conical metrics
(Collins / Székelyhidi)

We consider M theory on $\frac{\mathbb{C}^3}{\Gamma} \times \mathbb{R}^{4,1}$

with flat metric and $\Gamma \subset SU(3)$.

90's: Roan, Ito, Markusevich, Reid

...

Showed <u>all</u> $\frac{\mathbb{C}^3}{\Gamma}$ admit

crepant resolutions $\widetilde{\frac{\mathbb{C}^3}{\Gamma}}$.

<u>Van Coevering</u>: these <u>all</u> admit

AC Calabi-Yau metrics !

In the physics literature on 5d SCFT's, mostly the case when Γ is Abelian has been studied.

Many of these give rise to "interesting SCFT's."

What about non-Abelian Γ?

A class of non-Abelian groups:

- Let H be a diagonal, Abelian finite subgroup of $SU(3)$, so elements are of the form $h = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ with $h^N = \mathbb{1}$, $\alpha\beta\gamma = 1$.

- Extend H by $\mathbb{Z}_3$: $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The groups $G = \langle H, T \rangle$ generated by H and T are known as "trihedral".

Ito showed that one way to provide a crepant resolution of $\frac{\mathbb{C}^3}{G}$ is to first construct a T-invariant resolution of $\frac{\mathbb{C}^3}{H}$

and then resolve the quotient by T ie

$$\widetilde{\mathbb{C}^3/G} = \widetilde{\left(\widetilde{\mathbb{C}^3/H}\right)}\Big/T.$$

Physical Interpretation:

- 5d theory from $\mathbb{C}^3/H$ has a $T = \mathbb{Z}_3$ symmetry
- Gauging this symmetry gives the 5d theory from $\mathbb{C}^3/G$.

Example Gauging T-symmetry of 5d T_N theories.

- 5d T_N theories arise from a particular $H \cong \mathbb{Z}_N \times \mathbb{Z}_N$:

$$H = \left\langle \begin{pmatrix} \eta & & \\ & \bar{\eta} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \eta & \\ & & \bar{\eta} \end{pmatrix}, \begin{pmatrix} \eta & & \\ & 1 & \\ & & \bar{\eta} \end{pmatrix} \right\rangle$$

where $\eta^N = 1$.

T_N theories (Gaiotto-Maldacena) are:

- Interacting SCFT's
- Have global symmetry group
  $SU(N)^3$
- Have a Coulomb branch which corresponds to crepant resolutions $\mathbb{C}^3/\mathbb{Z}_N\times\mathbb{Z}_N$ with rank of gauge group = # compact divisors

- The 3 $SU(N)$ global symmetries arise from the 3 lines of A_{N-1} singularities in $\frac{\mathbb{C}^3}{\mathbb{Z}_N \times \mathbb{Z}_N}$.

- When we gauge the T symmetry the 3 lines are permuted so $SU(N)^3 \to SU(N)$

- Further, T introduces additional global symmetries as it also has fixed points.

This is either:

$N \neq 3k$ 1 line of A_2-singularities

$N = 3k$ 3 lines of A_2-singularities

So flavour symmetry group is (at least):

$$G_F = \begin{cases} SU(N) \times SU(3), & N \neq 3k \\ SU(N) \times SU(3)^3, & N = 3k \end{cases}$$

- Physically this means gauging T can only be done if new states are introduced

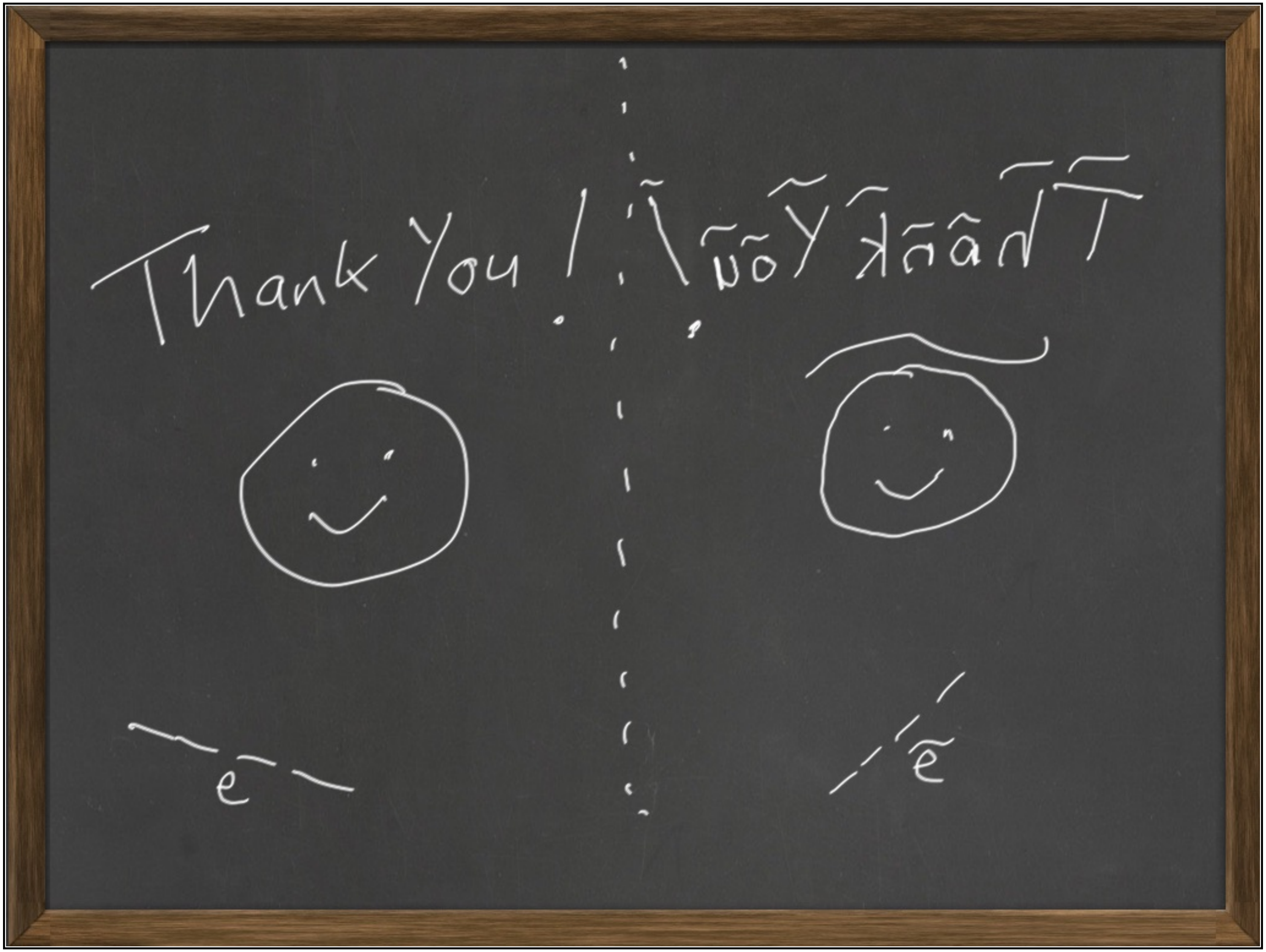
- In fact, using a construction of T_N theories in F-theory due to Benini, Benvenuti, Tachikawa we can show that when $N = 3K$, the gauged T- theory actually has symmetry

$$SU(N) \times E_6 \; !$$

One can also compute the rank of the Coulomb branch gauge group, using a theorem of Ito-Reid, ...

$$\mathrm{rk}(\mathrm{Gauge}) = \begin{cases} \frac{1}{6}(N^2-3N+2) & N=3K \\ \frac{1}{6}(N^2-3N+6) & N=3K \end{cases}$$

So we predict an infinite series of 5d SCFT's with the above properties.


Thank You!