Complete non-compact manifolds with holonomy G2 and ALC asymptotics

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The asymptotic geometry of ALC manifolds

- $(\Sigma^{n-2}, g_\Sigma)$ closed connected $\rightsquigarrow$ cone $C(\Sigma) = \mathbb{R}_+ \times \Sigma$, $g_C = dr^2 + r^2 g_\Sigma$
- $\pi : N^{n-1} \to \Sigma$ principal circle bundle, connection $\theta$, $\ell > 0$ ($\ell = 1$)
  $$BC(\Sigma) = \mathbb{R}_+ \times N, \quad g_{BC} = dr^2 + r^2 g_\Sigma + \ell^2 \theta^2$$
- $\text{Isom}(N)/\text{Isom}^+(N) = \langle \iota \rangle$ standard involution $\iota^* \theta = -\theta$
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Definition: \((M^n, g)\) complete non-cpct 1-ended is \textbf{ALC} of cyclic/dihedral type asymptotic to \(BC(\Sigma)\) with rate \(\nu < 0\) if \(\exists\) diffeo/double cover

\[f : (R, \infty) \times N \to M \setminus K \quad \text{such that} \quad |\nabla^k (g_{BC} - f^* g)| = O(r^{\nu-k}).\]
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**Example:** 4d ALF hyperkähler metrics via the Gibbons–Hawking Ansatz

\[ g_m = \left( m + \sum_{i=1}^n \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left( m + \sum_{i=1}^n \frac{1}{2|x - a_i|} \right)^{-1} \theta^2 \]
G₂–manifolds and Calabi–Yau 3-folds

- smooth 7-manifold $M$ endowed with a **G₂–structure** $\varphi$
  - $\varphi$ a positive 3-form
  
  $$\frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = g_\varphi(u, v) \text{vol}_{g_\varphi}$$

  $(M, \varphi)$ is a **G₂–manifold** if $d \varphi = 0 = d * \varphi$

  $\Rightarrow \text{Hol}(g_\varphi) \subseteq G_2$ and $\text{Ric}(g_\varphi) = 0$

- smooth 6-manifold $B$ endowed with an **SU(3)–structure** $(\omega, \Omega)$
  - $\omega$ a non-degenerate 2-form
  - $\Omega$ a complex volume form $\rightsquigarrow$ almost complex structure $J$
  - compatibility: $\omega \wedge \Omega = 0$ and $4 \omega^3 = 3 \Omega \wedge \overline{\Omega} \rightsquigarrow$ Riemannian metric $g_{\omega, \Omega}$

  $(B, \omega, \Omega)$ is a **Calabi–Yau 3-fold** (CY) if $d\omega = 0 = d\Omega$

- $(B, \omega, \Omega)$ CY 3-fold $\rightarrow M=B \times S^1$, $\varphi = dt \wedge \omega + \text{Re}\Omega$ **G₂–manifold**
Infinitely many ALC G2 manifolds

Theorem

- $(B, g_{CY}, \omega, \Omega)$ AC Calabi–Yau 3-fold
- $M \to B$ principal circle bundle with $c_1(M) \cup [\omega] = 0 \in H^4(B)$
- $\implies S^1$–invariant ALC G2–metric $g_{\epsilon}$ on $M \forall \epsilon \ll 1$
- with collapse with bounded curvature $(M, g_{\epsilon}) \xrightarrow{\epsilon \to 0} (B, g_{CY})$

\[
\varphi = \epsilon \theta \wedge \omega + h^3 \frac{1}{4} \text{Re} \Omega,
\]
\[
\ast \varphi = -\epsilon \theta \wedge h^3 \frac{1}{4} \text{Im} \Omega + \frac{1}{2} h \omega^2 \quad g = \sqrt{h} g_B + \epsilon^2 h^{-1} \theta^2
\]
\[
d\omega = 0, \quad d \left( h^3 \frac{1}{4} \text{Re} \Omega \right) + \epsilon d\theta \wedge \omega = 0,
\]
\[
d \left( h^4 \frac{1}{4} \text{Im} \Omega \right) = 0, \quad \frac{1}{2} dh \wedge \omega^2 - \epsilon h^4 d\theta \wedge \text{Im} \Omega = 0.
\]
The moduli space of ALC $G_2$ metrics

**Theorem** (Joyce)
The moduli space of torsion-free $G_2$ structures on a closed smooth 7-manifold $M$ is a smooth manifold of dimension $b_3(M)$ and the map

$\varphi \mapsto ([\varphi], [\star \varphi]) \in H^3(M) \times H^4(M)$ induces a Lagrangian immersion.
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The **model** $BC(\Sigma)$ for an ALC $G_2$ manifold

- $\Sigma$ is a Sasaki–Einstein 5-manifold
  - $\rightsquigarrow$ conical CY structure $\omega_C = d(\frac{1}{2} r^2 \eta)$, $\Omega_C$ on $C(\Sigma)$
- Hermitian–Yang–Mills connection $\theta$: $d \theta \wedge \omega^2_C = 0 = d \theta \wedge \Omega_C$

model **closed** positive 3-form

$$\varphi_{BC} = \theta \wedge \omega_C + \text{Re} \Omega_C - \frac{1}{2} r^2 \eta \wedge d \theta$$

$\rightsquigarrow$ consider torsion-free ALC $G_2$ structures $\varphi$ with $\varphi = \varphi_{BC} + O(r^{-1-\delta})$
**Theorem**

The moduli space of torsion-free ALC $G_2$ structures on $M^7$ of rate $\nu < -1$ is a smooth manifold with tangent space at $\varphi$ the space $H^3_\nu(M, g_\varphi)$ of closed and $g_\varphi$–coclosed 3-forms with $O(r^\nu)$ decay.

Similar deformation theory in non-compact AC setting studied by Karigiannis–Lotay in 2020

Assume $H^3_\nu \longrightarrow H^3(M) \times H^4(M)$ is an immersion

- $\nu$ sufficiently close to $-3$
- $\Sigma$ is a regular Sasaki-Einstein 5-manifold

**Consequences:**

- **continuous symmetries** (isometries that preserve the $G_2$ structure) of $BC(\Sigma)$ extend to symmetries of $M \twoheadrightarrow$ circle symmetry of cyclic ALC $G_2$ manifolds
  
  cf. classification of 4d ALF hyperkähler spaces of cyclic type (Minerbe)

- $(M, \varphi)$ cyclic ALC and $BC(\Sigma) \rightarrow C(\Sigma)$ flat circle bundle $\Rightarrow$ $\text{Hol}(g_\varphi) \subsetneq G_2$

  $\gamma = \text{harmonic 1-form dual to Killing field generating circle action}$

\[ \|d\gamma\|^2_{L^2} = -\int_M d\gamma \wedge d\gamma \wedge \varphi = 0 \implies \nabla \gamma = 0 \]
Dihedral ALC manifolds?

Biquard–Minerbe: 4d ALF hyperkähler spaces of dihedral type
- dihedral **ALF orbifold** Taub–NUT/Γ
  - Taub–NUT metric on \(\mathbb{R}^4\): \((1 + |x|^{-1}) \, dx \cdot dx + (1 + |x|^{-1})^{-1} \theta^2, \, d\theta = \text{vol}_{S^2}\)
  - \(\Gamma = \) binary dihedral group acting on Taub–NUT as a hyperkähler symmetry
- desingularize by gluing in an **ALE** metric asymptotic to \(\mathbb{R}^4/\Gamma\) at \(\infty\)

Want an *applicable* **G**\(_2\) **analogue** of this construction
- **conically singular** ALC **G**\(_2\) space \((M_0, \varphi_0)\)
  - singular points \(p_1, \ldots, p_k\) modelled on **G**\(_2\) cones \(C(N_1), \ldots, C(N_k)\)
- **asymptotically conical** **G**\(_2\) manifolds \((M_i, \varphi_i)\) asymptotic to \(C(N_i)\) at \(\infty\)

Similar desingularization in compact **G**\(_2\) setting studied by Karigiannis in 2009
An example of a dihedral ALC $G_2$ manifold

**Theorem**

- ALC $G_2$ metric on $M_0 = \mathbb{R}_+ \times S^3 \times S^3$ with conical singularity modelled on $C$
- AC $G_2$ manifolds $M_{m,n}$ for all $m, n \in \mathbb{Z}_{>0}$ coprime asymptotic to $C/\mathbb{Z}_2^{(n+m)}$ with rate -3

**Symmetries**

- Symmetries of ALC $G_2$ metric: $SU(2)^2 \times N$, $N = U(1) \times \mathbb{Z}_2$
  - Cyclic $\mathbb{Z}_4 \subset U(1) \subset N$
  - Dihedral $1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$ in $1 \to U(1) \to N \to \mathbb{Z}_2 \to 1$

- Symmetries of $G_2$ cone $C$: $SU(2)^2 \times SU(2)$ so $C/\mathbb{Z}_4^{cyclic} \simeq C/\mathbb{Z}_4^{dihedral}$
  - $\rightsquigarrow$ use $M_{1,1}$ to desingularise $M_0/\mathbb{Z}_4^{cyclic}$ and $M_0/\mathbb{Z}_4^{dihedral}$

$\Rightarrow$ existence of **dihedral ALC $G_2$ manifolds**
Desingularizing conically singular $ALC \ G_2$ spaces

**Theorem**

- $(M_0, \varphi_0)$ conically singular $ALC \ G_2$ space with singularities $\{p_i\}_{i=1}^k$ modelled on cones $C(N_1), \ldots, C(N_k)$
- AC $G_2$ manifold $(M_i, \varphi_i)$ asymptotic to $C(N_i)$ with rate $\nu_i \leq -3$
- **Topological conditions**
  - $([\varphi_1|_{\partial M_1}], \ldots, [\varphi_k|_{\partial M_k}])$ lies in the image of
    \[
    H^3(M_0) \to \bigoplus_{i=1}^k H^3(N_i) \oplus H^3(\partial_\infty M_0) \to \bigoplus_{i=1}^k H^3(N_i)
    \]
  - $([*\varphi_1|_{\partial M_1}], \ldots, [*\varphi_k|_{\partial M_k}], 0)$ lies in the image of
    \[
    H^4(M_0) \to \bigoplus_{i=1}^k H^4(N_i) \oplus H^4(\partial_\infty M_0)
    \]

$\Rightarrow$ existence $ALC \ G_2$ desingularizations $(M, \varphi_t)$ of $(M_0, \varphi_0)$ for $t \ll 1$. 
Ingredients of the proof

- \((M_i, \varphi_i)\) AC \(G_2\) manifold asymptotic to \(C(N_i)\) with rate \(\nu_i \leq -3\)

\[
\varphi_i = \varphi_C + \xi_i + d\zeta_i, \quad [\xi_i] \in H^3(\partial M_i)
\]

\[
*\varphi_i = *\varphi_C + \eta_i - *\xi_i + d\theta_i, \quad [\eta_i] \in H^4(\partial M_i)
\]

- Necessary topological conditions guarantee existence of closed (and coclosed) forms \(\xi_0\) and \(\zeta_0\) extending \(\xi_i\) and \(\zeta_i\) to \(M_0\)

\(\rightsquigarrow\) **closed** \(G_2\) structure \(\varphi'_t\) on \(M = M_0 \amalg \bigsqcup_i M_i\) with **small torsion** \(|d^*\varphi'_t| \ll 1\)

- Joyce: look for torsion-free \(G_2\) structure \(\varphi'_t + d\sigma\) via iteration scheme

\[
\triangle \sigma_j = d^* \chi_{j-1}
\]

- In non-compact ALC setting
  - Require \(\varphi'_t\) to have small and **decaying** torsion

\[
d\chi = 0, \quad d^*\chi = d^*\varphi'_t, \quad \|\chi\|_{L^2} + \|d^*\chi\|_{C^0} \ll 1
\]

- double indicial root \(-2\) for \(\triangle : \Omega^2 \to \Omega^2\) on a 6d cone

\(\rightsquigarrow\) can solve \(\triangle \sigma_j = d^* \chi_{j-1}\) with \(d\sigma_j \in L^2\) but non-optimal decay for \(\sigma_j\)