Complete non-compact manifolds with holonomy G2 and ALC asymptotics

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The asymptotic geometry of ALC manifolds

- (Σ^{n-2}, g_Σ) closed connected $\rightsquigarrow$ **cone** $\mathrm{C}(\Sigma) = \mathbb{R}_+ \times \Sigma$, $g_{\mathrm{C}} = dr^2 + r^2 g_\Sigma$
- $\pi : N^{n-1} \to \Sigma$ principal **circle bundle**, connection θ, $\ell > 0$ $(\ell = 1)$

$$\mathrm{BC}(\Sigma) = \mathbb{R}_+ \times N, \qquad g_{\mathrm{BC}} = dr^2 + r^2 g_\Sigma + \ell^2 \theta^2$$

- $\mathrm{Isom}(N)/\mathrm{Isom}^+(N) = \langle \iota \rangle$ standard involution $\iota^* \theta = -\theta$

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Definition: (M^n, g) complete non-cpct 1-ended is **ALC** of cyclic/dihedral type asymptotic to $\mathrm{BC}(\Sigma)$ with rate $\nu < 0$ if $\exists$ diffeo/double cover

$$f : (R, \infty) \times N \to M \setminus K \qquad \text{such that} \qquad |\nabla^k (g_{\mathrm{BC}} - f^* g)| = O(r^{\nu - k}).$$

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Example: 4d ALF hyperkähler metrics via the Gibbons–Hawking Ansatz

$$g_m = \left(m + \sum_{i=1}^n \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^n \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

G_2–manifolds and Calabi–Yau 3-folds

- smooth 7-manifold M endowed with a **G_2–structure** φ
  - φ a positive 3-form

$$\frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = g_\varphi(u, v)\, \mathrm{vol}_{g_\varphi}$$

  (M, φ) is a **G_2–manifold** if $d\varphi = 0 = d * \varphi$

  $\rightsquigarrow \mathrm{Hol}(g_\varphi) \subseteq G_2$ and $\mathrm{Ric}(g_\varphi) = 0$

- smooth 6-manifold B endowed with an **SU(3)–structure** (ω, Ω)
  - ω a non-degenerate 2-form
  - Ω a complex volume form $\rightsquigarrow$ almost complex structure J
  - compatibility: $\omega \wedge \Omega = 0$ and $4\,\omega^3 = 3\,\Omega \wedge \overline{\Omega}$ $\rightsquigarrow$ Riemannian metric $g_{\omega,\Omega}$

  (B, ω, Ω) is a **Calabi–Yau** 3-fold (CY) if $d\omega = 0 = d\Omega$

- (B, ω, Ω) CY 3-fold $\Longrightarrow$ $M = B \times S^1$, $\varphi = dt \wedge \omega + \mathrm{Re}\,\Omega$ G_2–manifold

Infinitely many ALC G2 manifolds

Theorem

- $(B, g_{\mathrm{CY}}, \omega, \Omega)$ **AC Calabi–Yau** 3**-fold**
- $M \to B$ **principal circle bundle** with $c_1(M) \cup [\omega] = 0 \in H^4(B)$

$\Longrightarrow$ $\mathbf{S}^1$**–invariant ALC $\mathbf{G_2}$–metric** g_ϵ on M $\forall\, \epsilon \ll 1$
with collapse with bounded curvature $(M, g_\epsilon) \xrightarrow{\epsilon \to 0} (B, g_{\mathrm{CY}})$

$$\varphi = \epsilon\, \theta \wedge \omega + h^{\frac{3}{4}} \mathrm{Re}\, \Omega,$$

$$*\varphi = -\epsilon\, \theta \wedge h^{\frac{1}{4}} \mathrm{Im}\, \Omega + \tfrac{1}{2} h\, \omega^2 \qquad g = \sqrt{h}\, g_B + \epsilon^2 h^{-1} \theta^2$$

$$d\omega = 0, \qquad d\left(h^{\frac{3}{4}} \mathrm{Re}\, \Omega\right) + \epsilon\, d\theta \wedge \omega = 0,$$

$$d\left(h^{\frac{1}{4}} \mathrm{Im}\, \Omega\right) = 0, \qquad \tfrac{1}{2} dh \wedge \omega^2 - \epsilon\, h^{\frac{1}{4}} d\theta \wedge \mathrm{Im}\, \Omega = 0.$$

The moduli space of ALC G2 metrics

Theorem (Joyce)
The moduli space of torsion-free G_2 structures on a closed smooth 7-manifold M is a smooth manifold of dimension $b_3(M)$ and the map $\varphi \mapsto ([\varphi], [*\varphi]) \in H^3(M) \times H^4(M)$ induces a Lagrangian immersion.

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The **model** $\mathrm{BC}(\Sigma)$ for an ALC G_2 manifold

- Σ is a Sasaki–Einstein 5–manifold
  $\rightsquigarrow$ conical CY structure $\omega_C = d(\frac{1}{2}r^2\eta)$, Ω_C on $\mathrm{C}(\Sigma)$
- Hermitian–Yang–Mills connection θ: $d\theta \wedge \omega_C^2 = 0 = d\theta \wedge \Omega_C$

model *closed* positive 3-form

$$\varphi_{\mathrm{BC}} = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C - \frac{1}{2}r^2\eta \wedge d\theta$$

$\rightsquigarrow$ consider torsion-free ALC G_2 structures φ with $\varphi = \varphi_{\mathrm{BC}} + O(r^{-1-\delta})$

Moduli of ALC G2 metrics – consequences

Theorem
The moduli space of torsion-free ALC G_2 structures on M^7 of rate $\nu < -1$ is a smooth manifold with tangent space at φ the space $\mathcal{H}^3_\nu(M, g_\varphi)$ of closed and g_φ–coclosed 3-forms with $O(r^\nu)$ decay.

Similar deformation theory in non-compact AC setting studied by Karigiannis–Lotay in 2020

Assume $\mathcal{H}^3_\nu \longrightarrow H^3(M) \times H^4(M)$ is an immersion

- ν sufficiently close to -3
- Σ is a *regular* Sasaki-Einstein 5-manifold

Consequences:

- continuous symmetries (isometries that preserve the G_2 structure) of $\mathrm{BC}(\Sigma)$ extend to symmetries of $M \rightsquigarrow$ circle symmetry of cyclic ALC G_2 manifolds

  cf. classification of 4d ALF hyperkähler spaces of cyclic type (Minerbe)
- (M, φ) cyclic ALC and $\mathrm{BC}(\Sigma) \to \mathrm{C}(\Sigma)$ flat circle bundle $\Rightarrow \mathrm{Hol}(g_\varphi) \subsetneq G_2$

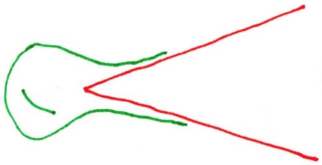
  $\gamma =$ harmonic 1-form dual to Killing field generating circle action

$$\|d\gamma\|^2_{L^2} = -\int_M d\gamma \wedge d\gamma \wedge \varphi = 0 \Longrightarrow \nabla\gamma = 0$$

Dihedral ALC manifolds?

Biquard–Minerbe: 4d ALF hyperkähler spaces of dihedral type

- dihedral ALF orbifold Taub–NUT/Γ
  - Taub–NUT metric on $\mathbb{R}^4$: $(1+|x|^{-1})\,dx\cdot dx + (1+|x|^{-1})^{-1}\theta^2$, $d\theta = \mathrm{vol}_{S^2}$
  - Γ = binary dihedral group acting on Taub–NUT as a hyperkähler symmetry
- desingularize by gluing in an ALE metric asymptotic to $\mathbb{R}^4/\Gamma$ at ∞



Want an *applicable* G_2 **analogue** of this construction

- conically singular ALC G_2 space (M_0, φ_0)
  singular points $p_1, \ldots, p_k$ modelled on G_2 cones $\mathrm{C}(N_1), \ldots, \mathrm{C}(N_k)$
- asymptotically conical G_2 manifolds (M_i, φ_i) asymptotic to $\mathrm{C}(N_i)$ at ∞

Similar desingularization in compact G_2 setting studied by Karigiannis in 2009

An example of a dihedral ALC G_2 manifold

Theorem

- ALC G_2 metric on $M_0 = \mathbb{R}_+ \times S^3 \times S^3$ with conical singularity modelled on C
- AC G_2 manifolds $M_{m,n}$ for all $m, n \in \mathbb{Z}_{>0}$ coprime asymptotic to $\mathrm{C}/\mathbb{Z}_{2(n+m)}$ with rate -3

- symmetries of ALC G_2 metric: $\mathrm{SU}(2)^2 \times \mathrm{N}$, $\mathrm{N} = \mathrm{U}(1) \rtimes \mathbb{Z}_2$
  - cyclic $\mathbb{Z}_4 \subset \mathrm{U}(1) \subset \mathrm{N}$
  - **dihedral** $1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$ in $1 \to \mathrm{U}(1) \to \mathrm{N} \to \mathbb{Z}_2 \to 1$
- symmetries of G_2 cone C: $\mathrm{SU}(2)^2 \times \mathrm{SU}(2)$ so $\mathrm{C}/\mathbb{Z}_4^{cyclic} \simeq \mathrm{C}/\mathbb{Z}_4^{dihedral}$
  $\rightsquigarrow$ use $M_{1,1}$ to desingularise $M_0/\mathbb{Z}_4^{cyclic}$ and $M_0/\mathbb{Z}_4^{dihedral}$

$\Rightarrow$ existence of **dihedral ALC** G_2 manifolds

Desingularizing conically singular ALC G_2 spaces

Theorem

- (M_0, φ_0) conically singular ALC G_2 space with singularities $\{p_i\}_{i=1}^k$ modelled on cones $\mathrm{C}(N_1), \ldots, \mathrm{C}(N_k)$
- AC G_2 manifold (M_i, φ_i) asymptotic to $\mathrm{C}(N_i)$ with rate $\nu_i \leq -3$
- **Topological conditions**
  - $([\varphi_1|_{\partial M_1}], \ldots, [\varphi_k|_{\partial M_k}])$ lies in the image of

$$H^3(M_0) \to \bigoplus_{i=1}^k H^3(N_i) \oplus H^3(\partial_\infty M_0) \to \bigoplus_{i=1}^k H^3(N_i)$$

  - $([*\varphi_1|_{\partial M_1}], \ldots, [*\varphi_k|_{\partial M_k}], 0)$ lies in the image of

$$H^4(M_0) \to \bigoplus_{i=1}^k H^4(N_i) \oplus H^4(\partial_\infty M_0)$$

$\Rightarrow$ existence ALC G_2 desingularizations (M, φ_t) of (M_0, φ_0) for $t \ll 1$.

Ingredients of the proof

- (M_i, φ_i) AC G_2 manifold asymptotic to $\mathrm{C}(N_i)$ with rate $\nu_i \leq -3$

$$\varphi_i = \varphi_{\mathrm{C}} + \xi_i + d\zeta_i, \qquad [\xi_i] \in H^3(\partial M_i)$$
$$*\varphi_i = *\varphi_{\mathrm{C}} + \eta_i - *\xi_i + d\theta_i, \qquad [\eta_i] \in H^4(\partial M_i)$$

- Necessary topological conditions guarantee existence of closed (and coclosed) forms ξ_0 and ζ_0 extending ξ_i and ζ_i to M_0

$\rightsquigarrow$ **closed** G_2 structure φ'_t on $M = M_0 \sharp \bigsqcup_i M_i$ with *small torsion* $|d^*\varphi'_t| \ll 1$

- Joyce: look for torsion-free G_2 structure $\varphi'_t + d\sigma$ via iteration scheme

$$\triangle \sigma_j = d^* \chi_{j-1}$$

- In non-compact ALC setting
    - Require φ'_t to have small and *decaying* torsion

$$d\chi = 0, \qquad d^*\chi = d^*\varphi'_t, \qquad \|\chi\|_{L^2} + \|d^*\chi\|_{C^0} \ll 1$$

    - double indicial root -2 for $\triangle : \Omega^2 \to \Omega^2$ on a 6d cone
      $\rightsquigarrow$ can solve $\triangle \sigma_j = d^*\chi_{j-1}$ with $d\sigma_j \in L^2$ but non-optimal decay for σ_j