The Gopakumar–Vafa finiteness conjecture

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2021-09-14

SCSHGAP Progress and Open Problems 2021

based on joint work with **Aleksander Doan** and **Eleny Ionel**

arXiv: 1809.04731, 1910.12338, 2103.08221

At the 2018 SCSHGAP meeting "Gauge Theory and Special Holonomy" at Imperial College London, Aleksander told me about the following question (which he learned about from Aleksey Zinger).

Theorem (Castelnuovo 1889)

If C is an irreducible, non-degenerate curve of degree d in $\mathbf{C}P^n$, then

$$\text{genus}(C) \leqslant \gamma(d,n) \sim d^2/n. \quad (1)$$

Are there analogues of Castelnuovo's genus bound in symplectic geometry?

Let (X, ω) be a closed symplectic manifold.

Denote by $\mathscr{J}$ the space of ω–tamed almost complex structures on X.

Let $J \in \mathscr{J}$. Let (Σ, j) be a Riemann surface. A **pseudo-holomorphic map** $u\colon (\Sigma, j) \to (X, J)$ is a smooth map satisfying

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u.$$

It is **simple** if there is an $x \in \Sigma$ with $u^{-1}(u(x)) = \{x\}$.

For $A \in \mathrm{H}_2(X, Z)$ and $g \in \mathbf{N}_0$ set

$$\mathscr{M}^{\mathrm{si}}_{A,g}(J) := \{[u\colon (\Sigma, j) \to (X, J)] : u \text{ simple}, u_*[\Sigma] = A, \mathrm{genus}(\Sigma) = g\}.$$

For $J \in \mathscr{J}$ is the **Castelnuovo number**

$$\gamma_A(J) := \sup\big\{g \in \mathbf{N}_0 : \mathscr{M}^{\mathrm{si}}_{A,g}(J) \neq \varnothing\big\}$$

finite? Can it be bounded effectively?

What about

$$\gamma_A := \sup_{J \in \mathscr{J}} \gamma_A(X, J)?$$

Theorem (McDuff 1992)

If $\dim X = 4$, *then*

$$\gamma_A \leqslant \frac{1}{2}(A \cdot A + \langle c_1(X,\omega), A\rangle).$$

Proposition (Gromov's h-principle; T.-J. Li 2005)

If $\dim X \geqslant 6$, *then*

$$\gamma_A = \infty.$$

Proposition (transversality and index theorem)

There is a comeager subset $\mathscr{J}^* \subset \mathscr{J}$ *such that for every* $J \in \mathscr{J}^*$ *the following holds:*

1. *if* $\dim X \geqslant 8$, *then*

$$\gamma_A(J) \leqslant \frac{2\langle c_1(X,\omega), A\rangle}{\dim X - 6} + 1.$$

2. *If* $\dim X = 6$, *then* $\mathscr{M}^{\mathrm{si}}_{A,g}(J) = \varnothing$ *unless*

$$\langle c_1(X,\omega), A\rangle \geqslant 0.$$

This leaves us with

$$\dim X = 6$$

and $A \in \mathrm{H}_2(X, \mathbf{Z})$ with

(CY)
$$\langle c_1(X,\omega), A\rangle = 0$$

or

(Fano)
$$\langle c_1(X,\omega), A\rangle > 0.$$

The questions for (CY) and (Fano) have a different flavour.

The remainder of this talk assumes that (X, ω) is a **symplectic Calabi–Yau** 3**–fold**:

$$\dim X = 6 \quad \text{and} \quad c_1(X,\omega) = 0.$$

Theorem (Doan and Walpuski 2018)

There is a comeager subset $\mathscr{J}^\dagger \subset \mathscr{J}$ such that for every $J \in \mathscr{J}^\dagger$ and $A \in \mathrm{H}_2(X, \mathbf{Z})$

$$\# \coprod_{g \in \mathbf{N}_0} \mathscr{M}^{\mathrm{si}}_{A,g}(J) < \infty; \quad \textit{hence:} \quad \gamma_A(J) < \infty.$$

$\mathscr{J}^\dagger$ is the subset of **super-rigid** almost complex structures. $\mathscr{J}^\dagger$ is comeager (Wendl 2019).

The bounds are not effective. The proof is by contradiction and uses compactness and regularity results from geometric measure theory (Federer and Fleming 1960; De Lellis, Spadaro, and Spolaor 2018).

Proposition (Doan, Ionel, and Walpuski 2021)

There is a comeager subset $\mathscr{J}^\ddagger \subset \mathscr{J}$ such that for every **compact** *$K \subset \mathscr{J}^\ddagger$ and* **primitive** *$A \in \mathrm{H}_2(X, \mathbf{Z})$*

$$\sup_{J \in K} \gamma_A(J) < \infty.$$

$\mathscr{J}^\ddagger$ is the subset of those $J \in \mathscr{J}$ for which every simple J–holomorphic map is an embedding.

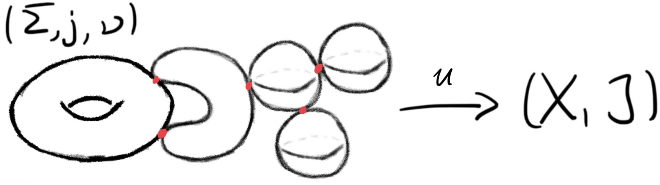
The proof uses (a simple version of) Allard's regularity theorem.

Let $I_{A,g}$ be an invariant that "counts" J–holomorphic "curves" representing $A \in \mathrm{H}_2(X, \mathbf{Z})$ and of genus $g \in \mathbf{N}_0$. What can we say about the I **Castelnuovo number**

$$\gamma_A^I := \sup\{g \in \mathbf{N}_0 : I_{A,g} \neq 0\}?$$

To define $I_{A,g}$ one needs a compactification of $\mathscr{M}^{\mathrm{si}}_{A,g}(J)$ together with a deformation theory.

The **moduli space of stable nodal pseudo-holomorphic curves**

$\overline{\mathscr{M}}_{A,g}(J) := \Big\{$  $\Big\}$

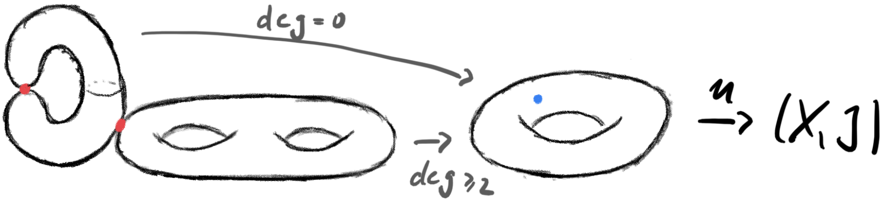
leads to the **Gromov–Witten invariants**

$$\mathrm{GW}_{A,g} := \int_{[\overline{\mathscr{M}}_{A,g}(J)]^{\mathrm{vir}}} \mathbf{1} \in \mathbf{Q}.$$

The question is hopeless for $I = \mathrm{GW}$.

Let (Σ, j) be a Riemann surface. Consider the **nodal Hurwitz space**

$$\overline{\mathscr{H}}_{d,h}(\Sigma, j) := \overline{\mathscr{M}}_{d[\Sigma],h}(\Sigma, j).$$



If $[u\colon (\Sigma, j) \to (X, J)] \in \overline{\mathscr{M}}_{A,g}$, then composition with u defines an inclusion

$$\overline{\mathscr{H}}_{d,h}(\Sigma, j) \hookrightarrow \overline{\mathscr{M}}_{dA,h}(J).$$

This leads to the appearance of **ghosts** and **multiple covers**.

Theorem (Zinger 2011)

If A is primitive, then

$$\#_{A,g} := \text{signed count of } \mathscr{M}^{\mathrm{si}}_{A,g}(J)$$

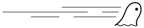
is independent of the choice of $J \in \mathscr{J}^{\ddagger}$.

The proof uses Zinger's localisation formula and Pandharipande 1999.

An alternative proof establishes that ghosts do not appear in the closure of

$$\mathscr{M}^{\mathrm{si}}_{A,g}(\mathscr{J}^{\ddagger}) \subset \overline{\mathscr{M}}_{A,g}(\mathscr{J}^{\ddagger})$$

using ideas going back to the Ionel and Zinger.



Corollary (Doan and Walpuski 2019)

If A is primitive, then

$$\gamma^{\#}_{A} < \infty.$$

Gopakumar and Vafa 1998 argued that the Gromov–Witten invariants of a Calabi–Yau 3–fold are related to counts of BPS states in M–theory by the **marvelous formula**

$$\text{(GV)} \qquad \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{GW}_{A,g}\cdot t^{2g-2}q^{A} = \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{BPS}_{A,g}\cdot \sum_{k=1}^{\infty}\frac{1}{k}(2\sin(kt/2))^{2g-2}q^{kA}.$$

A direct geometric definition of $\mathrm{BPS}_{A,g}$ has not been found yet; however ...

If A is primitive, then $\mathrm{BPS}_{A,g} = \#_{A,g}$ (Zinger 2011).

Bai and Swaminathan 2021 have made excellent progress towards a definition of $\mathrm{BPS}_{A,g}$ if A is twice a primitive class (inspired by Taubes 1996: Gr).

The are numerous attempts to define $\mathrm{BPS}_{A,g}$ in algebraic geometry using sheaf-theory, but (GV) fails or is not (yet) known to hold: Hosono, Saito, and Takahashi 2001 ×; Kiem and J. Li 2016 ×; Maulik and Toda 2018 **?**.

Conjecture (Bryan and Pandharipande 2001; "The Gopakumar–Vafa conjecture")

The numbers $\mathrm{BPS}_{A,g}$ *defined by*

$$\text{(GV)} \qquad \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{GW}_{A,g}\cdot t^{2g-2}q^A = \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{BPS}_{A,g}\cdot \sum_{k=1}^{\infty}\frac{1}{k}(2\sin(kt/2))^{2g-2}q^{kA}$$

satisfy:

(integrality) $\mathrm{BPS}_{A,g} \in \mathbf{Z}$.

(finiteness) $\mathrm{BPS}_{A,g} = 0$ *for* $g \gg_A 1$*; i.e.:* $\gamma_A^{\mathrm{BPS}} < \infty$.

Theorem (Ionel and Parker 2018)

The Gopakumar–Vafa integrality conjecture holds.

Theorem (Doan, Ionel, and Walpuski 2021)

The Gopakumar–Vafa finiteness conjecture holds.

A J–**holomorphic curve** C is a subset which is the image of a simple J–holomorphic map u.

An embedded J–holomorphic curve is **super-rigid** if the inclusions

$$\overline{\mathscr{H}}_{d,g}(C) \hookrightarrow \overline{\mathscr{M}}_{d[C],g}(J)$$

are open and closed.

Theorem (Zinger 2011)

If C is super-rigid, then is Gromov–Witten contributions are

(Z) $$\mathrm{GW}_{d[C],g}(C, J) = \int_{[\overline{\mathscr{H}}_{d,g}(C,j)]^{\mathrm{vir}}} e(\text{obstruction bundle}).$$

If integrality and finiteness were known for the contribution

$$\mathrm{GW}(C, J) := \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \mathrm{GW}_{d[C],g}(C, J) \cdot t^{2g-2} q^{d[C]}$$

of an arbitrary super-rigid curve C, then the Gopakumar–Vafa conjecture would follow from Wendl's resolution of the super-rigidity conjecture our earlier work.

Theorem (Bryan and Pandharipande 2008; Pandharipande and Thomas 2009; Lee 2009; Ionel and Parker 2018)

For every embedded J–holomorphic curve C, there is a $J_C \in \mathscr{J}$ such that:

1. *C is J_C–holomorphic.*
2. *C is super-rigid with respect to J_C.*
3. $\mathrm{GW}(C, J_C) = t^{2\mathrm{genus}(C)-2} q^{[C]} + \cdots$ *and satisfies integrality and finiteness.*

An upgrade of Ionel and Parker's **cluster formalism** using geometric measure theory proves that: for every $\Lambda > 0$ there is a finite set $\{(C_i, J_{C_i}) : i \in I\}$ such that

$$\text{(E)} \qquad \sum_{\mathbf{M}(A) \leqslant \Lambda} \sum_{g=0}^{\infty} \mathrm{GW}_{A,g} \cdot t^{2g-2} q^A = \sum_{i \in I} \pm \mathrm{GW}(C_i, J_{C_i})_\Lambda.$$

Here $(\cdot)_\Lambda$ indicates truncation to those q^A with $\mathbf{M}(A) := \langle [\omega], A \rangle \leqslant \Lambda$.

This implies the Gopakumar–Vafa conjecture—both integrality and finiteness. ■

The cluster formalism is built upon a compactification of the space $\mathscr{C}_\Lambda(J)$ of embedded J–holomorphic curves C with $\mathbf{M}(C) \leqslant \Lambda$.

Ionel and Parker use

$$\coprod_{\mathbf{M}(A)\leqslant\Lambda}\ \coprod_{g\leqslant\Gamma} \overline{\mathscr{M}}_{A,g}(J).$$

Therefore, their version of (E) requires a **genus truncation**: finiteness $\times$.

It is tempting to take the closure of $\mathscr{C}_\Lambda(J) \subset \mathscr{K}$, the set of closed subset of X equipped with the topology induced by the Hausdorff distance.

This does not induce the appropriate topology on $\mathscr{C}_\Lambda(J)$ because of multiple cover phenomena.

A *J*–**holomorphic cycle** is a finite formal sum

$$C = \sum_{i \in I} m_i C_i$$

with $m_i \in \mathbf{N}$ and C_i a J–holomorphic curve.

The **homology class** and **mass** of C are

$$[C] = \sum_{i \in I} m_i [C_i], \quad \mathbf{M}(C) = \sum_{i \in I} m_i \mathbf{M}(C_i).$$

The **current** $\delta_C \in \mathrm{I}_2(X) \subset \mathrm{Hom}(\Omega^2(X), \mathbf{R})$ of C is

$$\delta_C(\alpha) := \sum_{i \in I} m_i \int_{C_i} \alpha.$$

Denote by $\overline{\mathscr{C}}_\Lambda(J)$ the set of J–holomorphic cycles with the coarsest topology such that the map $C \mapsto \delta_C$ is continuous with respect to the weak-$*$ topology.

Theorem (Federer and Fleming 1960; De Lellis, Spadaro, and Spolaor 2018; Doan and Walpuski 2018)

$\overline{\mathscr{C}}_\Lambda(J)$ is compact.

Theorem (Allard 1972; Doan, Ionel, and Walpuski 2021)

$\mathscr{C}_\Lambda(J) \subset \overline{\mathscr{C}}_\Lambda(J)$ is open and the inclusion is a topological embedding.

There is a natural continuous map

$$\mathfrak{z}\colon \coprod_{\mathbf{M}(A)\leqslant\Lambda} \coprod_{g=0}^{\infty} \overline{\mathscr{M}}_{A,g}(J) \to \overline{\mathscr{C}}_\Lambda(J).$$

It "contracts" the nodal Hurwitz spaces.

Upshot: $\overline{\mathscr{C}}_\Lambda(J)$ is a tighter compactification (and genus-agnostic).

The End