The Gopakumar–Vafa finiteness conjecture

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This talk is based on `arxiv:2103.08221`.

It is joint work with **Aleksander Doan** (Columbia University and Trinity College Cambridge; former SCSHGAP PhD student), and **Eleny Ionel** (Stanford University).

It is a spin-off of ongoing joint work with Aleksander and has emerged out of discussions between Aleksander and Eleny at the 2020 SCSHGAP meeting "Geometry and Analysis of Moduli Spaces" at Imperial College London.

What is the Gopakumar–Vafa conjecture?

The Gopakumar–Vafa conjecture concerns the structure of Gromov–Witten theory of symplectic Calabi–Yau 3–folds.

It predicts that the Gromov–Witten invariants

$$\mathrm{GW}_{A,g}(X,\omega) \in \mathbf{Q}$$

of a symplectic Calabi–Yau 3–fold (X,ω) are encoded by counts of BPS states

$$\mathrm{BPS}_{A,g}(X,\omega) \in \mathbf{Z}$$

with

$$\mathrm{BPS}_{A,g}(X,\omega) = 0 \quad \text{for} \quad g \gg_A 1.$$

What is the Gopakumar–Vafa conjecture?

Gromov–Witten theory: nodal Riemann surfaces

A **nodal Riemann surface** is a closed Riemann surface (Σ, j) together with an involution ν whose *non*-fixed-point set S is finite and such that Σ/ν is connected.

Its **arithmetic genus** is

$$g(\Sigma, \nu) := 1 - \frac{1}{2}(\chi(\Sigma) - \#S).$$

Here is a nodal Riemann surface: of arithmetic genus 3:

What is the Gopakumar–Vafa conjecture?

Gromov–Witten theory: nodal J–holomorphic maps

Let (X, J) be an almost complex manifold. A **nodal J–holomorphic map** $u\colon (\Sigma, j, \nu) \to (X, J)$ is a smooth map $u\colon \Sigma \to X$ satisfying

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u \quad \text{and} \quad u \circ \nu = u.$$

The **reparametrisation** by $\phi \in \mathrm{Diff}(\Sigma)$ is

$$\phi_* u \coloneqq u \circ \phi^{-1}\colon (\Sigma, \phi_* j, \phi_* \nu) \to (X, J).$$

It is **stable** if

$$\mathrm{Aut}(u) \coloneqq \{\phi \in \mathrm{Diff}(\Sigma) : \phi_* u = u, \phi_* j = j, \phi_* \nu = \nu\}$$

is finite.

What is the Gopakumar–Vafa conjecture?

Gromov–Witten theory: $\overline{\mathscr{M}}_{A,g}(X,J)$

Let $A \in \mathrm{H}_2(X,\mathbf{Z})$ and $g \in \mathbf{N}_0$. The **moduli space of stable nodal J–holomorphic maps** representing A and of genus g is the set

$$\overline{\mathscr{M}}_{A,g}(X,J) := \{[u\colon\ (\Sigma,j,\nu) \to (X,J)] : u \text{ is stable}, u_*[\Sigma] = A, g(\Sigma,\nu) = g\}$$

equipped with the Gromov topology.

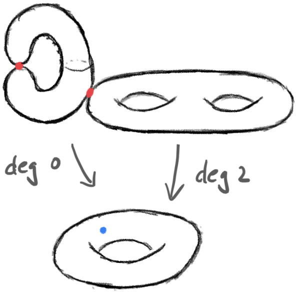
This space is quite complicated.

Let $[u\colon\ (\Sigma,j) \to (X,J)] \in \mathscr{M}_{A,g}(X,J)$. Consider the **nodal Hurwitz space**

$$\overline{\mathscr{H}}_{d,h}(\Sigma,j) := \overline{\mathscr{M}}_{d[\Sigma],h}(\Sigma,j).$$

Composition with u defines an inclusion

$$\kappa^u_{d,h}\colon\ \overline{\mathscr{H}}_{d,h}(\Sigma,j) \hookrightarrow \overline{\mathscr{M}}_{dA,h}(X,J).$$



What is the Gopakumar–Vafa conjecture?

Gromov–Witten theory: $\mathrm{GW}_{A,g}(X,\omega)$

$\overline{\mathscr{M}}_{A,g}(X,J)$ carries a virtual fundamental class (VFC)

$$[\overline{\mathscr{M}}_{A,g}(X,J)]^{\mathrm{vir}} \in \mathrm{H}_{\mathrm{vdim}(A,g)}(\overline{\mathscr{M}}_{A,g}(X,J),\mathbf{Q})$$

with

$$\mathrm{vdim}(A,g) := (\dim X - 6)(1-g) + 2\langle c_1(X,J), A\rangle.$$

A closed symplectic manifold (X,ω) is a **symplectic Calabi–Yau** 3**–fold** if $\dim X = 6$ and $c_1(X,\omega) = 0$, i.e., $\mathrm{vdim} = 0$.

Its **Gromov–Witten invariants** are

$$\mathrm{GW}_{A,g}(X,\omega) := \int_{[\overline{\mathscr{M}}_{A,g}(X,J)]^{\mathrm{vir}}} \mathbf{1} \in \mathbf{Q}$$

for an ω–tame almost complex structure J. They are independent of the choice of J.

Gopakumar and Vafa 1998 argued that topological string amplitudes in type IIA string theory (= Gromov–Witten invariants) on Calabi–Yau 3–folds are related to counts of BPS states in M–theory (= ?) by the **marvelous formula**

$$\text{(GV)} \quad \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{GW}_{A,g}(X,\omega)\cdot t^{2g-2}q^A = \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{BPS}_{A,g}(X,\omega)\cdot \sum_{k=1}^{\infty}\frac{1}{k}(2\sin(kt/2))^{2g-2}q^{kA}.$$

A **geometric** definition of $\mathrm{BPS}_{A,g}(X,\omega)$ has not been found yet. However:

- If A primitive, then $\mathrm{BPS}_{A,g}(X,\omega)$ is the count of J–holomorphic embeddings for a generic ω–tame almost complex structure J (Zinger 2011).
- Bai and Swaminathan 2021 have made excellent progress towards a definition of $\mathrm{BPS}_{A,g}(X,\omega)$ if A is twice a primitive class (inspired by Taubes 1996: Gr).
- The are numerous attempts to define $\mathrm{BPS}_{A,g}$ in algebraic geometry using sheaf-theory, but (GV) fails or is not (yet) known to hold (Hosono, Saito, and Takahashi 2001 ×; Kiem and Li 2016 ×; Maulik and Toda 2018 ?).

What is the Gopakumar–Vafa conjecture?

Turning (GV) into a definition leads to:

Conjecture (Bryan and Pandharipande 2001; "The Gopakumar–Vafa conjecture")

Let (X, ω) *be a symplectic Calabi–Yau 3–fold. The numbers* $\mathrm{BPS}_{A,g}(X, \omega)$ *defined by*

$$\text{(GV)} \quad \sum_{A \neq 0} \sum_{g=0}^{\infty} \mathrm{GW}_{A,g}(X, \omega) \cdot t^{2g-2} q^A = \sum_{A \neq 0} \sum_{g=0}^{\infty} \mathrm{BPS}_{A,g}(X, \omega) \cdot \sum_{k=1}^{\infty} \frac{1}{k} (2 \sin(kt/2))^{2g-2} q^{kA}$$

satisfy:

(integrality) $\mathrm{BPS}_{A,g}(X, \omega) \in \mathbf{Z}$.

(finiteness) $\mathrm{BPS}_{A,g}(X, \omega) = 0$ *for* $g \gg_A 1$.

The Gopakumar–Vafa conjecture holds

Theorem (Ionel and Parker 2018)

The Gopakumar–Vafa integrality conjecture holds.

Theorem (Doan, Ionel, and Walpuski 2021)

The Gopakumar–Vafa finiteness conjecture holds.

The proof relies on an upgrade of Ionel and Parker's **cluster formalism** using **geometric measure theory**.

Can the **BPS Castelnuovo number**

$$\gamma_A^{\mathrm{BPS}}(X,\omega) := \sup\{g \in \mathbf{N}_0 : \mathrm{BPS}_{A,g}(X,\omega) \neq 0\} < \infty$$

be bounded effectively (in terms of A, $c_1(X,\omega)$, ...)?

Proof of the Gopakumar–Vafa conjecture

The strategy

- Verify the a local version of the conjecture.
- Decompose (a truncation) of

$$\mathrm{GW}(X,\omega) := \sum_{A\neq 0}\sum_{g=0}^{\infty} \mathrm{GW}_{A,g}(X,\omega)\cdot t^{2g-2}q^A$$

  into finitely many parts

$$S_1 + \cdots + S_I.$$

- By deformation express S_i as

$$S_i = \text{an admissible local contribution} + \text{higher order terms}.$$

- Decompose and deform the higher order terms until they are admissible (or truncated).

Proof of the Gopakumar–Vafa conjecture

Gromov–Witten contributions

If $\mathcal{S} \subset \overline{\mathscr{M}}_{A,g}(X,J)$ is open and closed, then its **Gromov–Witten contribution** is

$$\mathrm{GW}_{A,g}(\mathcal{S},J) \coloneqq \int_{[\overline{\mathscr{M}}_{A,g}(X,J)]^{\mathrm{vir}}} \mathbf{1}_{\mathcal{S}}.$$

Set

$$\overline{\mathscr{M}}(X,J) \coloneqq \coprod_{A\neq 0} \coprod_{g=0}^{\infty} \overline{\mathscr{M}}_{A,g}(X,J).$$

If $\mathcal{S} \subset \overline{\mathscr{M}}(X,J)$ is open and closed, then its **Gromov–Witten contribution** is

$$\mathrm{GW}(\mathcal{S},J) \coloneqq \sum_{A\neq 0} \sum_{g=0}^{\infty} \mathrm{GW}_{A,g}(\overline{\mathscr{M}}_{A,g}(X,J) \cap \mathcal{S}, J) \cdot t^{2g-2} q^{A}.$$

Proof of the Gopakumar–Vafa conjecture

Super-rigid curves

A J–**holomorphic curve** $C \subset X$ is a subset which is the image of a non-constant (simple) J–holomorphic map.

An embedded J–holomorphic curve $C \subset X$ is **super-rigid** if

$$\kappa_C \colon \overline{\mathscr{H}}(C) \hookrightarrow \overline{\mathscr{M}}(X, J) \quad \text{with} \quad \overline{\mathscr{H}}(C) := \coprod_{d=1}^{\infty} \coprod_{g=0}^{\infty} \overline{\mathscr{H}}_{d,g}(C)$$

is open and closed.

Super-rigidity is equivalent to the family of normal Cauchy–Riemann operators

$$\underline{\mathfrak{d}}_{C,J}^{N} := \left(\pi^* \mathfrak{d}_{C,J}^{N}\right)_{[\pi] \in \overline{\mathscr{H}}(C)}$$

having trivial kernels: "equivariant transversality".

Theorem (Zinger 2011)

If C is super-rigid, then

$$\mathrm{GW}_{d[C],g}(\operatorname{im}\kappa^C_{d,g}, J) = \int_{[\overline{\mathscr{H}}_{d,g}(C,j)]^{\mathrm{vir}}} e(\operatorname{coker}\underline{\mathfrak{d}}^N_{C,J}). \tag{Z}$$

Theorem (Wendl 2019; resolution of the super-rigidity conjecture)

For a generic ω–tame almost complex structure J every J–holomorphic curve C is embedded and super-rigid.

Unfortunately, determining the Gromov–Witten contributions of super-rigid curves using (Z) is quite non-trivial.

Indeed, the proof of the Gopakumar–Vafa conjecture does not rely on Wendl's breakthrough.

Theorem (Lee 2009; Ionel and Parker 2018)

Let $C \subset X$ be an embedded J–holomorphic curve. Let $\mathscr{L}$ be a spin structure on C, i.e.,

$$\mathscr{L}^2 = K_C.$$

($NC \cong \mathscr{L} \oplus \mathscr{L}$ as complex vector bundles.) There is an ω–tame almost complex structure J_C with respect to which C is J_C–holomorphic and super-rigid, and

$$\mathfrak{d}^N_{C,J_C} = \mathfrak{d} \oplus \mathfrak{d} \quad \textit{with} \quad \mathfrak{d} = \bar{\partial}_{\mathscr{L}} + \mathfrak{a}.$$

The Gromov–Witten contribution

$$\mathrm{GW}(\mathrm{im}\,\kappa_C, J_C)$$

can be determined.

Proof of the Gopakumar–Vafa conjecture

The local Gopakumar–Vafa conjecture holds for J_C

For $g \in \mathbf{N}_0$ set

(BP) $$G_g(q,t) := \log\left(1 + \sum_{d=1}^{\infty}\sum_{\mu\vdash d}\prod_{\square\in\mu}(2\sin(h(\square)\cdot t/2))^{2g-2}q^d\right).$$

Here $\mu \vdash d$ indicates that the sum is taken over all partitions μ of d, $\square \in \mu$ indicates that $\square$ is a box in the Young diagram of μ, and $h(\square)$ denotes the hook length of $\square$.

Theorem (Bryan and Pandharipande 2008; Ionel and Parker 2018)

The Gromov–Witten contribution of $\operatorname{im}\kappa_C$ *is*

$$\mathrm{GW}(\operatorname{im}\kappa_C, J_C) = G_{g(C)}(q^{[C]}, t).$$

Theorem (Pandharipande and Thomas 2009; Ionel and Parker 2018)

The series $G_g(q,t)$ *satisfies integrality and finiteness.*

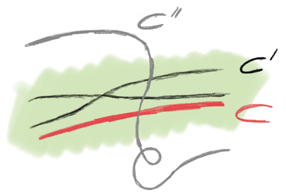
The **mass** of a J–holomorphic curve C is

$$\mathbf{M}(C) := \mathrm{area}(C) = \langle [\omega], [C] \rangle =: \mathbf{M}([C]).$$

Fix a **mass threshold** $\Lambda > 0$.

A Λ**–cluster** is a triple $\mathcal{O} = (\mathcal{U}, J, C)$ consisting of:

- an open subset $\mathcal{U} \subset \{\text{closed subsets of } X\}$,
- an ω–tame almost complex structure J, and
- an embedded J–holomorphic curve C, the **core**,



such that:

- There is no J–holomorphic curve C' with $\mathbf{M}(C') \leqslant \Lambda$ and $C' \in \partial\mathcal{U}$.
- There is an $A \in \mathrm{H}_2(X, \mathbf{Z})$ such that every J–holomorphic curve C' with $\mathbf{M}(C') \leqslant \Lambda$ and $C' \in \mathcal{U}$ there is a $k \in \mathbf{N}$ with $[C'] = kA$.
- C is the unique such J–holomorphic curve with $[C] = A$, i.e., $k = 1$.

(⋆) Using tools from geometric measure theory (Federer and Fleming 1960; Allard 1972; De Lellis, Spadaro, and Spolaor 2018) one can prove the following three results.

Proposition (Cluster contribution)

Let $\mathscr{O} = (\mathscr{U}, J, C)$ be a Λ–cluster. There is an open neighborhood $\mathscr{V}$ of J such that for every $J' \in \mathscr{V}$

$$\overline{\mathscr{M}}_\Lambda(\mathscr{U}, J') := \left\{[u] \in \overline{\mathscr{M}}(X, J') : \mathbf{M}(u_*[\Sigma]) \leqslant \Lambda, \operatorname{im} u \in \mathscr{U}\right\}$$

is open and closed; moreover: its contribution to $\mathrm{GW}(X, \omega)$ is independent of $J' \in \mathscr{V}$.

For a Λ–cluster $\mathscr{O} = (\mathscr{U}, J, C)$ set

$$\mathrm{GW}_\Lambda(\mathscr{O}) := \mathrm{GW}(\overline{\mathscr{M}}_\Lambda(\mathscr{U}, J), J).$$

Proposition (Cluster decomposition)

There is a finite set $\{\mathscr{O}_i = (\mathscr{U}_i, J_i, C_i) : i \in I\}$ *of* Λ*–clusters such that*

$$\mathrm{GW}(X, \omega)_\Lambda = \sum_{i \in I} \mathrm{GW}_\Lambda(\mathscr{O}_i).$$

Here $(\cdot)_\Lambda$ *denotes the* Λ*–truncation.*

Theorem (Cluster isotopy)

If $\mathscr{O}_0 = (\mathscr{U}_0, J_0, C)$ *and* $\mathscr{O}_1 = (\mathscr{U}_1, J_1, C)$ *are* Λ*–clusters* **with the same core**, *then there is a finite set* $\{\mathscr{O}_i = (\mathscr{U}_i, J_i, C_i) : i \in I\}$ *of* Λ*–clusters such that*

$$\mathrm{GW}_\Lambda(\mathscr{O}_0) = \pm\mathrm{GW}_\Lambda(\mathscr{O}_1) + \sum_{i \in I} \pm\mathrm{GW}_\Lambda(\mathscr{O}_i)$$

and

$$[C_i] = d_i[C] \quad \textit{with} \quad \mathbf{d_i} \geqslant \mathbf{2}.$$

There is a finite set $\{\mathscr{O}_i = (\mathscr{U}_i, J_i, C_i) : i \in I\}$ of Λ–clusters with $J_i = J_{C_i}$ and

(E) $$\mathrm{GW}(X,\omega)_\Lambda = \sum_{i \in I} \pm \mathrm{GW}_\Lambda(\mathscr{O}_i) = \sum_{i\in I} \pm G_{g(C_i)}(q^{[C_i]}, t)_\Lambda.$$

Since $G_g(q,t) = t^{2g-2}q + \cdots$, (E) for arbitrarily large Λ implies the following.

Theorem

There are unique coefficients $e_{A,g}(X,\omega)$ such that

$$\mathrm{GW}(X,\omega) = \sum_{A\neq 0}\sum_{g=0}^{\infty} e_{A,g}(X,\omega)\cdot G_g(q^A,t);$$

moreover, they satisfy:

(integrality) $e_{A,g}(X,\omega) \in \mathbf{Z}$.

(finiteness) $e_{A,g}(X,\omega) = 0$ *for* $g \gg_A 1$.

Since the $G_g(q^A,t)$ satisfy integrality and finiteness, this proves the Gopakumar–Vafa conjecture. ∎

What makes the cluster formalism work?

Gromov compactness

The cluster formalism requires a compactification of the space of embedded pseudo-holomorphic curves (the cores).

Theorem (Gromov compactness)

Denote by $\mathscr{J}$ the space of ω–tame almost complex structures. The map

$$(\mathrm{pr}_{\mathscr{J}}, [\cdot], g) \colon \overline{\mathscr{M}} := \coprod_{J \in \mathscr{J}} \overline{\mathscr{M}}(X, J) \to \mathscr{J} \times \mathrm{H}_2(X, \mathbf{Z}) \times \mathbf{N}_0$$

is proper with respect to the Gromov topology on $\overline{\mathscr{M}}$.

The genus component g is essential.

Ionel and Parker's original cluster formalism is based on Gromov compactness and, therefore, requires a **genus threshold** in addition to the mass threshold.

What makes the cluster formalism work?

Pseudo-holomorphic cycles

Taubes 1996 (SW = Gr) lights the way forward.

A J–**holomorphic cycle** is a J–holomorphic integral 2–dimensional current

$$\delta \in \mathbf{I}_2(X) \subset \mathrm{Hom}(\Omega^2(X), \mathbf{R})$$

without boundary. It can be thought of as a formal finite sum

$$C = m_1 C_1 + \cdots + m_I C_I$$

of distinct J–holomorphic curves C_i with $m_i \in \mathbf{N}$ (**De Lellis, Spadaro, and Spolaor 2018**).

The **space of** J**–holomorphic cycles** is the set

$$\mathscr{C}(X, J) = \{\delta \in \mathbf{I}_2(X) : \delta \text{ is } J\text{–holomorphic}\}$$

equipped with the weak–$*$ topology. (This agrees with Taubes' geometric topology.)

What makes the cluster formalism work?

Federer and Fleming's compactness theorem

Theorem (Federer and Fleming 1960)

The map

$$(\mathrm{pr}_{\mathscr{J}}, [\cdot])\colon \mathscr{C} := \coprod_{J \in \mathscr{J}} \mathscr{C}(X, J) \to \mathscr{J} \times \mathrm{H}_2(X, \mathbf{Z})$$

is proper.

There is a natural continuous map

$$\mathfrak{z}\colon \overline{\mathscr{M}} \to \mathscr{C}.$$

It "contracts" the nodal Hurwitz spaces.

Both $\overline{\mathscr{M}}$ and $\mathscr{C}$ contain the space of embedded J–holomorphic curves as an open subset (trivially and using Allard 1972 respectively).

Upshot: $\mathscr{C}$ is a tighter compactification (and genus-agnostic).

The End