ALF spaces and collapsing Ricci-flat metrics on the K3 surface

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The Kummer construction

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The Kummer construction as the prototypical example of the formation of orbifold singularities in sequences of Einstein 4–manifolds:

A sequence (M_i, g_i) of Einstein 4–manifolds with uniform bounds on Euler characteristic, total volume and diameter converges to an Einstein orbifold M_∞ with finitely many singularities.

The geometry around points where the curvature concentrates is modelled by ALE spaces (complete Ricci-flat 4–manifolds with maximal volume growth).

The Kummer construction on $\mathbb{T}^4 = \mathbb{T}^3 \times S^1_\ell$

In the Ricci-flat case if we drop the diameter bound collapsing can occur: $\operatorname{inj} g_i \to 0$ everywhere (Anderson, 1992) and the collapse occurs with bounded curvature outside a definite number of points (Cheeger–Tian, 2006).

Question: What is the structure of the points where the curvature concentrates?

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Page (1981): consider the Kummer construction along a 1–parameter family of "split" 4-tori $\mathbb{T}^4 = \mathbb{T}^3 \times S^1_\ell$ with a circle of length $\ell \to 0$.

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In this talk Page's suggestion is regarded as a simple case of a more general construction, in which the trivial product $\mathbb{T}^3 \times S^1$ is replaced by a non-trivial circle bundle over a punctured 3–torus and more general gravitational instantons of cubic volume growth appear as rescaled limits.

ALF gravitational instantons

Gravitational instantons: complete hyperkähler 4–manifolds with decaying curvature at infinity ($\|\mathrm{Rm}\|_{L^2} < \infty$. Stronger assumption: faster than quadratic curvature decay, $|\mathrm{Rm}| = O(r^{-2-\epsilon})$, $\epsilon > 0$)

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Hyperkähler $\Rightarrow$ Ricci-flat: $cr \leq \mathrm{Vol}(B_r) \leq Cr^4$.

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Moreover, if $a = 3$ then (M, g) is **ALF**:

There exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma < O(3)$ acting freely on S^2 such that $M \setminus K$ is the total space of a circle fibration over $(\mathbb{R}^3 \setminus B_R)/\Gamma$. The metric g is asymptotic to a Riemannian submersion

$$g = g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau}), \qquad \tau > 0.$$

If $\Gamma = \mathrm{id}$ then we say that M is an ALF space of **cyclic** type. If $\Gamma = \mathbb{Z}_2$ we say that M is an ALF space of **dihedral** type.

The Gibbons–Hawking ansatz

Gibbons–Hawking (1978): local form of hyperkähler metrics in dimension 4 with a triholomorphic circle action.

- U open subset of $\mathbb{R}^3$
- h positive harmonic function on U
- $P \to U$ a principal $U(1)$–bundle with a connection θ

(h, θ) satisfies the **monopole equation** $*dh = d\theta \iff$
$g^{\mathrm{gh}} = h\, g_{\mathbb{R}^3} + h^{-1}\theta^2$ is a hyperkähler metric on P

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Example: $x_1, \dots, x_n \in \mathbb{R}^3$, $k_i \in \mathbb{Z}_{>0}$, $\lambda \geq 0$,

$$h = \lambda + \sum_{i=1}^{n} \frac{k_i}{2\,|x - x_i|}$$

- Near x_i: smooth (orbifold) metric on $\mathbb{C}^2/\mathbb{Z}_{k_i}$.
- At infinity: ALE if $\lambda = 0$, ALF if $\lambda > 0$.

ALF spaces of cyclic type

In the previous example, the metrics with $k_i = 1$ for all i and $\lambda > 0$ are called *multi-Taub–NUT* metrics.

A multi-Taub–NUT metric with $n + 1$ "nuts" is also called an A_n ALF metric.

The A_0 ALF metric is the Taub–NUT metric.

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- Minerbe (2011): All ALF spaces of cyclic type are multi-Taub–NUT metrics.

In particular, an A_n ALF metric is asymptotic to the Gibbons–Hawking metric obtained from the harmonic function

$$h = 1 + \frac{n+1}{2\,|x|}.$$

ALF spaces of dihedral type

(M, g) is a D_m ALF space if up to a double cover g is asymptotic to the Gibbons–Hawking metric obtained from the harmonic function

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G. Chen – X. Chen (2015) classified ALF spaces of dihedral type:

- The D_0 ALF space is the Atiyah–Hitchin manifold (1988).
- The D_1 ALF metrics are the double cover of the Atiyah–Hitchin metric and its Dancer deformations (1993).
- The D_2 ALF spaces are Page's "periodic but nonstationary" gravitational instantons. Constructed by Hitchin (1984) and Biquard–Minerbe (2011).
- D_m, $m \geq 3$, constructed by Cherkis–Kapustin (1999) and Cherkis–Hitchin (2005), Biquard–Minerbe (2011) and Auvray (2012).

Main Result

Main Theorem (F., 2016)

For every collection of 8 ALF spaces of dihedral type $M_1, \dots, M_8$ and n ALF spaces of cyclic type $N_1, \dots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

there exists a sequence $\{g_\epsilon\}_{\epsilon>0}$ of Kähler Ricci-flat metrics on the K3 surface such that:

- As $\epsilon \to 0$ the metric g_ϵ collapses to the flat orbifold $\mathbb{T}^3/\mathbb{Z}_2$ with bounded curvature outside $8+n$ points. The first 8 points are the fixed points of the involution on $\mathbb{T}^3$.
- An ALF space of dihedral type arises as a rescaled limit of the sequence close to one of the fixed points of the involution on $\mathbb{T}^3$.
- An ALF space of cyclic type arises as a rescaled limit of the sequence close to one of the other n points.

Remarks:

- Hitchin (1974) showed that the unique non-flat Ricci-flat 4–manifolds with $2\chi + 3\tau = 0$ are the K3 surface with a hyperkähler metric, an Enriques surface (the quotient of a K3 surface by an involution) with a Kähler Ricci-flat metric and the quotient of an Enriques surface by an anti-holomorphic involution without fixed points.

  Luft–Servje (1984) showed that the only flat orientable 3–manifolds admitting an involution with finitely many fixed points are $\mathcal{G}_1 = \mathbb{T}^3$, $\mathcal{G}_2 = \mathbb{T}^3/\mathbb{Z}_2$ and $\mathcal{G}_6 = \mathbb{T}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

  Working equivariantly with respect to a finite group action the Main Theorem yields the existence of sequences of Ricci-flat metrics on Enriques surfaces and their quotients by an anti-holomorphic involution collapsing to $\mathcal{G}_2/\mathbb{Z}_2$ and $\mathcal{G}_6/\mathbb{Z}_2$ respectively.

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- If M_j is a D_{m_j} ALF gravitational instanton and N_i is an A_{k_i-1} ALF gravitational instanton then the Euler characteristic constraint is equivalent to

$$\sum_{j=1}^{8} m_j + \sum_{i=1}^{n} k_i = 16.$$

The GH ansatz over a punctured 3–torus

$(\mathbb{T}, g_{\mathbb{T}})$ a flat 3–torus with standard involution $\tau : \mathbb{T} \to \mathbb{T}$
$\mathrm{Fix}(\tau) = \{q_1, \ldots, q_8\}$
Assign an integer weight $m_j \in \mathbb{Z}_{\geq 0}$ to each q_j

Choose further distinct $2n$ points $\pm p_1, \ldots, \pm p_n$
Assign an integer weight $k_i \geq 1$ to each pair $\pm p_i$

Denote by $\mathbb{T}^*$ the punctured torus

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If $\sum m_j + \sum k_i = 16$ then there exists a harmonic function h on $\mathbb{T}^*$ with prescribed singularities:

$$h = \frac{k_i}{2\,\mathrm{dist}(\pm p_i, \cdot)} + O(1) \qquad h = \frac{2m_j - 4}{2\,\mathrm{dist}(q_j, \cdot)} + O(1)$$

Furthermore, there exists a principal $U(1)$–bundle $M^{\mathrm{gh}} \to \mathbb{T}^*$ with a connection θ such that $d\theta = *dh$

Background S^1–invariant hyperkähler metrics

For each $\epsilon > 0$ obtain an (incomplete) hyperkähler metric

$$g_\epsilon^{\mathsf{gh}} = (1 + \epsilon\, h)\, g_{\mathbb{T}} + \epsilon^2 (1 + \epsilon\, h)^{-1} \theta^2$$

over the open set M_ϵ^{gh} where $1 + \epsilon\, h > 0$.

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Key fact: For $\epsilon > 0$ sufficiently small $1+\epsilon\, h > 0$ outside of balls of radius $\propto \epsilon$ around the points q_j with $m_j = 0, 1$.

Moreover, g_ϵ^{gh} descends to a quotient $M_\epsilon^{gh}/\mathbb{Z}_2$.

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Moreover, g_ϵ^{gh} descends to a quotient $M_\epsilon^{gh} / \mathbb{Z}_2$.

Goal: For ϵ sufficiently small extend $(M_\epsilon^{gh} / \mathbb{Z}_2, g_\epsilon^{\mathrm{gh}})$ to a complete metric on a 4–manifold M_ϵ by gluing

- an $A_{k_i - 1}$ ALF space in a neighbourhood of $\pm p_i$
- a D_{m_j} ALF space in a neighbourhood of q_j

Definite triples

Donaldson, 2006:

Let (M^4, μ_0) be an oriented 4–manifold with volume form μ_0.

A **definite triple** is a triple $\underline{\boldsymbol{\omega}} = (\omega_1, \omega_2, \omega_3)$ of 2–forms on M such that $\mathrm{Span}(\omega_1, \omega_2, \omega_3)$ is a 3–dimensional positive definite subspace of $\Lambda^2 T_x^* M$ at every point $x \in M$. Equivalently the matrix Q defined by

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Every definite triple $\underline{\boldsymbol{\omega}}$ defines a Riemannian metric $g_{\underline{\boldsymbol{\omega}}}$:

the conformal class is given by $\Lambda^+_{g_{\underline{\boldsymbol{\omega}}}} T^* M = \mathrm{Span}(\omega_1, \omega_2, \omega_3)$

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hyperkähler if it is closed and Q is constant.

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hyperkähler if it is closed and Q is constant.

Donaldson's question: does every compact 4–manifold with a closed definite triple also admit a hyperkähler structure?

Deforming approximately hyperkähler triples

By gluing ALF spaces to the Gibbons–Hawking hyperkähler manifold M_ϵ^{gh} obtain a *closed* definite triple $\underline{\boldsymbol{\omega}}_\epsilon$ which is approximately hyperkähler: the matrix $Q_\epsilon = (\det Q)^{-\frac{1}{3}} Q$ is closer and closer to the identity as $\epsilon \to 0$.

Question: How to deform $\underline{\boldsymbol{\omega}}_\epsilon$ into a genuine hyperkähler triple?

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Look for a triple $\underline{\boldsymbol{\eta}}$ of closed 2–forms such that

$$\tfrac{1}{2}(\omega_i + \eta_i) \wedge (\omega_j + \eta_j) = \delta_{ij}\, \mu_{\underline{\boldsymbol{\omega}}_\epsilon}.$$

Rewrite this as

$$\underline{\boldsymbol{\eta}}^+ = \mathcal{F}(Q_\epsilon - \mathrm{id} + \underline{\boldsymbol{\eta}}^- * \underline{\boldsymbol{\eta}}^-)$$

and look for $\underline{\boldsymbol{\eta}}$ of the form $\underline{\boldsymbol{\eta}} = d\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}}$ for

- a triple of 1–forms $\underline{\mathbf{a}}$ with $d^*\underline{\mathbf{a}} = 0$, and
- a triple $\underline{\boldsymbol{\zeta}}$ of harmonic self-dual forms with respect to $g_{\underline{\boldsymbol{\omega}}_\epsilon}$.

The linearised equation

We want to apply the Implicit Function Theorem to deform $\underline{\boldsymbol{\omega}}_\epsilon$ to a genuine hyperkähler structure.

Need to control the norm of the inverse of the linearised operator.

The linearised operator in our problem is

$$d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+.$$

Consider this operator with respect to the collapsing Gibbons–Hawking metric g_ϵ^{gh}.

As $\epsilon \to 0$ $d^* + d^+$ converges to the Dirac operator of the flat 3–torus $\mathbb{T}^3$ acting on $\mathbb{Z}_2$–invariant spinors (functions and 1–forms).

There is **no** $\mathbb{Z}_2$–invariant kernel!

Parameter count

The Main Theorem yields a full-dimensional family of Ricci-flat metrics on the K3 surface:

- Flat 3–tori have 6 moduli
- $3n$ parameters determine the position of $p_1, \ldots, p_n$
- The monopole (h, θ) on the punctured torus has 4 moduli (the constant ϵ and a point in the dual torus $\hat{\mathbb{T}}$ parametrising flat connections on $\mathbb{T}^3$)
- The moduli space of A_{k_i-1} ALF metrics has dimension $3(k_i - 1)$
- The moduli space of D_{m_j} ALF metrics has dimension $3m_j$

$$6 + 3n + 4 + 3\sum_{i=1}^{n}(k_i - 1) + 3\sum_{j=1}^{8} m_j = 58$$

Stable minimal surfaces and holomorphicity

Wirtinger's Inequality: every holomorphic submanifold of a Kähler manifold is volume minimising in its homology class.

Micallef (1984): Every stable minimal surface in a flat 4–torus must be holomorphic with respect to some complex structure compatible with the metric.

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Micallef–Wolfson (2006): there exists a hyperkähler metric g on the K3 surface and a homology class α such that the area minimiser in α decomposes into the sum of branched minimal immersions, not all of which can be holomorphic with respect to some complex structure compatible with the metric.

Non-holomorphic strictly stable minimal spheres

An immediate consequence of the Main Theorem is the existence of a (simpler) counterexample:

The D_1 ALF space double-cover of the Atiyah–Hitchin manifold contains a strictly stable minimal sphere $[\Sigma]$ with $[\Sigma] \cdot [\Sigma] = -4$.

By the adjunction formula Σ cannot be holomorphic.

Consider the metric g_ϵ given by the Main Theorem using a configuration of ALF spaces containing the rotationally symmetric D_1 ALF space.

Use an Implicit Function Theorem for minimal immersions due to White to deform the strictly stable minimal sphere Σ.