Families of Special Holonomy Metrics defined by Algebraic Curvature Conditions

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$\eta : TB \to \mathbb{R}^n$, $\theta : TB \to \mathfrak{h}$, and $R : B \to K(\mathfrak{h})$ is the curvature function, where $K(\mathfrak{h})$ is the H-representation

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Second Bianchi: $\mathrm{d}R = -\theta.R + R'(\eta)$. where

$$R' : B \to K^{(1)}(\mathfrak{h}) \subset \mathrm{Hom}\big(\mathbb{R}^n, K(\mathfrak{h})\big)$$

represents the covariant derivative of the curvature.

Example: SU(2) ⊂ SO(4)

$$\begin{pmatrix} \mathrm{d}\eta_0 \\ \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & 0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

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$$K\big(\mathfrak{su}(2)\big) = S^2_0(\mathbb{R}^3) \simeq \mathbb{R}^5 \qquad \text{and} \qquad K^{(1)}(\mathfrak{su}(2)) \simeq \mathbb{C}^6 \simeq S^5(\mathbb{C}^2)$$

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É. Cartan (1926): SU(2)-holonomy depends on 2 functions of 3 variables.

Basic holonomy problem: For a given subgroup $H \subset \mathrm{SO}(n)$ how to classify, up to local diffeomorphism, the 'solutions' to the structure equations

$$\begin{aligned} \mathrm{d}\eta &= -\theta \wedge \eta \\ \mathrm{d}\theta &= -\theta \wedge \theta + R(\eta \wedge \eta) \end{aligned}$$

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Example: $H = \mathrm{SU}(2) = \mathrm{Spin}(3) \subset \mathrm{SO}(4)$ acts on $K(\mathfrak{su}(2)) = S^2_0(\mathbb{R}^3)$ preserving the symmetric functions of the eigenvalues of $R \in S^2_0(\mathbb{R}^3)$. Specifying a relation between $\sigma_2(R)$ and $\sigma_3(R)$ defines such an invariant subset $A \subset S^2_0(\mathbb{R}^3)$.

$$\sigma_3(R)^2 + \tfrac{4}{27}\sigma_2(R)^3 \leq 0.$$

Cases of interest in special holonomy

1. $\mathsf{SU}(n) \subset \mathsf{SO}(2n)$

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3. $\mathrm{Spin}(7) \subset \mathsf{SO}(8)$

$$K\big(\mathfrak{so}(7)\big) \simeq V^{0,2,0}(\mathfrak{so}(7)) \simeq \mathbb{R}^{168}.$$

The nearly Kähler structure equations

(M^6, ω, Υ) a nearly Kähler structure: An SU(3)-structure on M defined by $\omega \in \Omega^{1,1}(M)$ and $\Upsilon \in \Omega^{3,0}(M)$ satisfy

$$\mathrm{d}\omega = 3c\,\mathrm{Im}(\Upsilon) \qquad \text{and} \qquad \mathrm{d}\Upsilon = 2c\,\omega^2.$$

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The structure equations on the SU(3)-bundle $B \to M$ with coframing

$$(\eta, \theta) : TB \to \mathbb{C}^3 \oplus \mathfrak{su}(3)$$

are

$$\begin{aligned} \mathrm{d}\eta &= -\theta \wedge \eta + c\,\bar{\eta} \times \bar{\eta} \\ \mathrm{d}\theta &= -\theta \wedge \theta + R\,(\eta \wedge \eta) + c^2 R_0\,(\eta \times \eta) \end{aligned}$$

where $R : B \to K\big(\mathfrak{su}(3)\big)$ and $R_0 \in \mathfrak{su}(3) \otimes (\Lambda^2(\mathbb{C}^3))^*$ is the curvature of the G_2-invariant nearly-Kähler structure on the 6-sphere.

Tools for the general problem

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Example (Lie groups): If $A = \{a_0\}$ is a point, then $J(a_0) = 0$ is necessary and sufficient, where

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$$0 = \mathrm{d}(\mathrm{d}\omega) = \mathrm{d}\big(a_0\,(\omega \wedge \omega)\big) = J(a_0)\big(\omega \wedge \omega \wedge \omega\big).$$

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Example (Holonomy): Let $V = \mathbb{R}^n \oplus \mathfrak{h}$ and $\omega = (\eta, \theta) : TB \to V$. Then the structure equations

$$\mathrm{d}\eta = -\theta \wedge \eta \qquad \text{and} \qquad \mathrm{d}\theta = -\theta \wedge \theta + R(\eta \wedge \eta)$$

are of this form, with $A \subset V \otimes \Lambda^2(V^*)$ an affine space isomorphic to $K(\mathfrak{h})$, and (essentially) $a = R$.

A general structure equation problem

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In general, $0 = \mathrm{d}(\mathrm{d}\omega) = \mathrm{d}a{\wedge}(\omega{\wedge}\omega) + J(a)(\omega{\wedge}\omega{\wedge}\omega)$, so one must have

$$J(a_0) \in \sigma(T_{a_0}A \otimes V^*) \qquad \text{for all } a_0 \in a(X) \subset A,$$

where $\sigma : V \otimes \Lambda^2 V^* \otimes V^* \to V \otimes \Lambda^3 V^*$ is skew-symmetrization.

Involutivity: Let $T \subset V \otimes \Lambda^2 V^*$ be a linear subspace. Let $T^{(1)}$ be defined so that this sequence is exact:

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Let $F : (0) = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a flag of subspaces, and let $\rho_i : V \otimes \Lambda^2 V^* \to V \otimes \Lambda^2 V_i^*$ be restriction. Define, for $1 \leq i \leq n$,

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Cartan's Inequality: $\dim T^{(1)} \leq s_1^F(T) + 2s_2^F(T) + \cdots + ns_n^F(T).$

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When equality holds, we say T is *involutive*, F is said to be *T-regular*, and the sequence of numbers $s_i = s_i^F(T)$ is the *character sequence* of T.

A submanifold $A \subset V \otimes \Lambda^2 V^*$ is involutive if each $T_a A$ is involutive for $a \in A$ and, moreover, they all have the same character sequence.

Theorem A: Let $A \subset V \otimes \Lambda^2 V^*$ be a real-analytic involutive submanifold that satisfies, for all $a \in A$, that $J(a) \in \sigma(T_a A \otimes V^*)$.
Then (local) coframings of type A exist and, up to diffeomorphism, depend on s_q functions of q variables, where s_q is the last nonzero character.

More explicitly, the space of diffeomorphism classes of k-jets of coframings of type A has dimension

$$s + \binom{k}{1} s_1 + \binom{k+1}{2} s_2 + \cdots + \binom{k+n-1}{n} s_n\,,$$

where $s = \dim A$ and $(s_1, s_2, \ldots, s_n)$ is the sequence of Cartan characters of the subspaces $T_a A \subset V \otimes \Lambda^2 V^*$.

Proof idea: Set up the problem so that the Cartan-Kähler Theorem can be applied, generalizing (slightly) the arguments in Cartan's "infinite groups" papers from 1904–1910.

Example: The structure equations for SU(2)-holonomy

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We have $A \simeq K(\mathfrak{su}(2)) \simeq \mathbb{R}^5$ with

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (0, 3, 2, 0, 0, 0, 0).$$

and $\dim A^{(1)} = \dim K(\mathfrak{su}(2))^{(1)} = 12 = 2s_2 + 3s_3$, so it's involutive.

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Example: The SU(2) structure equations in which $R : B \to S^2_0(\mathbb{R}^3)$ has a double eigenvalue everywhere are not involutive:

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Applying $\mathrm{d}^2 = 0$ to the equations

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with $r \neq 0$ implies that there exist u_0, u_1, u_1, u_3 for which

$$\begin{aligned} \mathrm{d}r &= 4r\,(\quad u_0\,\eta_0 + u_1\,\eta_1 + u_2\,\eta_2 + u_3\,\eta_3) \\ \theta_2 &= 2\ \ (-u_2\,\eta_0 - u_3\,\eta_1 + u_0\,\eta_2 + u_1\,\eta_3) \\ \theta_3 &= 2\ \ (-u_3\,\eta_0 + u_2\,\eta_1 - u_1\,\eta_2 + u_0\,\eta_3) \end{aligned}$$

Applying $\mathrm{d}^2 = 0$ to the equations

$$\begin{pmatrix} \mathrm{d}\eta_0 \\ \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & 0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = - \begin{pmatrix} 2\,\theta_2 \wedge \theta_3 \\ 2\,\theta_3 \wedge \theta_1 \\ 2\,\theta_1 \wedge \theta_2 \end{pmatrix} + \begin{pmatrix} -2r^3 & 0 & 0 \\ 0 & r^3 & 0 \\ 0 & 0 & r^3 \end{pmatrix} \begin{pmatrix} \eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\ \eta_0 \wedge \eta_2 - \eta_3 \wedge \eta_1 \\ \eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2 \end{pmatrix},$$

with $r \neq 0$ implies that there exist u_0, u_1, u_1, u_3 for which

$$\begin{aligned} \mathrm{d}r &= 4r\,(\quad u_0\,\eta_0 + u_1\,\eta_1 + u_2\,\eta_2 + u_3\,\eta_3) \\ \theta_2 &= 2\ \ (-u_2\,\eta_0 - u_3\,\eta_1 + u_0\,\eta_2 + u_1\,\eta_3) \\ \theta_3 &= 2\ \ (-u_3\,\eta_0 + u_2\,\eta_1 - u_1\,\eta_2 + u_0\,\eta_3) \end{aligned}$$

These are structure equations for a coframing $(\eta_0, \eta_1, \eta_2, \eta_3, \theta_1)$ with coefficients (r, u_0, u_1, u_2, u_3) that still does not satisfy the involutivity hypothesis of Theorem A.

Differentiating the structure equations again yields relations of the form

$$\mathrm{d}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = U(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

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$$\mathrm{d}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = V(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

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Differentiating these last relations yields no more relations. Coupled with

$$\mathrm{d}r = 4r\,(u_0\,\eta_0 + u_1\,\eta_1 + u_2\,\eta_2 + u_3\,\eta_3)$$

This gives 8 'independent' coefficients in the structure equations for which $\mathrm{d}^2 = 0$ is an identity.

A generalization of Cartan's theorem

Theorem B: Consider a coframing η satisfying structure equations

$$\mathrm{d}\eta^i = -\tfrac{1}{2}\,C^i_{jk}(h)\,\eta^j\wedge\eta^k \qquad \mathrm{d}h^a = \big(F^a_i(h) + A^a_{i\alpha}(h)p^\alpha\big)\,\eta^i.$$

C^i_{jk}, F^a_i, and $A^a_{i\alpha}$ (where $1\le i,j,k\le n$, $1\le a\le s$, and $1\le\alpha\le r$) are specified functions on a domain $X\subset\mathbb{R}^s$.

Assume:

1. The functions C, F, and A are real analytic.
2. The tableau $A(h)=\big(A^a_{i\alpha}(h)\big)$ is rank r and *involutive*, with Cartan characters $s_1\ge s_2\ge\cdots\ge s_q > s_{q+1}=0$ for all $h\in\mathbb{R}^s$.
3. $\mathrm{d}^2=0$ reduces to equations of the form

$$0 = A^a_{i\alpha}(h)\big(\mathrm{d}p^\alpha + B^\alpha_j(h,p)\,\eta^j\big)\wedge\eta^i$$

   for some functions B^α_j. (*Torsion absorbable hypothesis*)

Then: Modulo diffeomorphism, the general real-analytic solution depends on s_q functions of q variables. Moreover, one can specify h and p arbitrarily at a point.

Theorem C: If $A \subset V \otimes \Lambda^2(V^*)$ is a real-analytic submanifold then any real-analytic coframing $\omega : TX \to V$ with mapping $a : X \to A$ satisfying the structure equation

$$\mathrm{d}\omega = a\,(\omega \wedge \omega)$$

can be found as a solution to a system of structure equations to which Theorem B applies.

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Example: The case of an SU(2)-holonomy metric for which R has a double eigenvalue everywhere prolongs to a system $\eta : TX \to \mathbb{R}^5$ and $h : X \to A \simeq \mathbb{R}^8$ where

$$\mathrm{d}\eta = C(h)(\eta \wedge \eta) \qquad \text{and} \qquad \mathrm{d}h = F(h)\eta$$

and to which Theorem B applies (i.e., it is involutive).

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3. $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$: $s_7 = 12$ is last nonzero character.

Curvature restrictions in the $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ case

The SU(2)-invariants on $K(\mathfrak{su}(2)) \simeq S^2_0(\mathbb{R}^3) \simeq \mathbb{R}^5$ are generated by $\sigma_2, \sigma_3 : S^2_0(\mathbb{R}^3) \to \mathbb{R}$, satisfying

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Not involutive. Prolong, apply Theorems B and C, yields that solutions only exist in the trivial case $c_2 = c_3 = 0$.

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3. $(\sigma_3(R))^2 + \frac{4}{27}\,(\sigma_2(R))^3 = 0$.
This is the 'double eigenvalue case', with nontrivial stabilizer $S^1 \subset \mathrm{SU}(2)$.
Not involutive. Prolong, apply Theorems B and C, yields a 2-parameter family of solutions, not all of which are complete, but some are.

The case $H = \mathrm{SO}(3)$ ($\dim M = 3$)

The SO(3)-invariants on $K(\mathfrak{so}(3)) \simeq S^2(\mathbb{R}^3) \simeq \mathbb{R}^6$ are generated by $\sigma_1, \sigma_2, \sigma_3 : S^2_0(\mathbb{R}^3) \to \mathbb{R}$.

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When eigenvalues c_i are distinct, Jacobi manifold is involutive with last nonzero character $s_2 = 3$. Otherwise, not involutive.

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Not involutive. Prolong, apply Theorems B and C, becomes involutive with last nonzero character $s_1 = 2$.

3. Fixed eigenvalues: $\big(\lambda_1(R), \lambda_2(R), \lambda_3(R)\big) \equiv (c, c, c)$
Involutive, but solutions depend on one constant. (Constant sectional curvature.)

Curvatures in $K(\mathfrak{h})$ with nontrivial H-stabilizers

Classifying the general H-invariant $A \subset K(\mathfrak{h})$ for which the corresponding H-structures have nontrivial solutions is probably intractable.

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The most promising candidate to date is the subset $S \subset K(\mathfrak{h})$ that consists of the curvatures that have nontrivial H-stabilizers. This is not a smooth manifold, but it can be 'stratified' into smooth pieces according to the stabilizer type, and these can be analyzed.

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This is the project that I have been engaged in.

Table: Stablized curvatures for subgroups of SU(3)

G	$\dim\left(K(\mathfrak{su}(3))\right)^G$	G-splitting of $\mathbb{C}^3$
U(2)	1	$\mathbb{C} \oplus \mathbb{C}^2$
SU(2)	1	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$
SO(3)	1	$\mathbb{R}^3 \oplus \mathbb{R}^3$
$\mathbb{T}^2$	3	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(p/q)^\dagger$	3	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(0)$	5	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(1)$	7	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

† $p/q \neq 0, 1$, where $S^1(p/q)$ is the circle of diagonal matrices $\mathrm{diag}(e^{ipt}, e^{iqt}, e^{-i(p+q)t})$.

Table: Stabilized curvatures of subgroups of G_2

G	$\dim\big(K(\mathfrak{g}_2)\big)^G$	G-splitting of $\mathbb{R}^7$
SU(3)	0	$\mathbb{R}^1 \oplus \mathbb{C}^3$
SO(4)	1	$\mathbb{R}^3 \oplus \mathbb{R}^4$
$U(2)_1$	2	$\mathbb{R}^3 \oplus \mathbb{R}^4$
$U(2)_2$	2	$\mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^4$
$\mathbb{T}^2$	5	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
$SU(2)_1$	3	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$
$SU(2)_2$	6	$\mathbb{R}^3 \oplus \mathbb{R}^4$
$SO(3)_1$	1	$\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$
$SO(3)_2$	1	$\mathbb{R}^7$
$S^1(p/q)^\dagger$	5	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(0)$	13	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(1/2)$	7	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
$S^1(1)$	9	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

† $p/q \neq 0, \frac{1}{2}, 1$