Particle Physics and G_2-manifolds

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$\partial_\mu F^{\mu\nu} = j^\nu \ d\varphi = d * \varphi = 0$

and

ICTP Trieste $\bar{\Psi}(1 - \gamma_5)\Psi$

THANK YOU Simons Foundation!

- For supporting this collaboration on Special Holonomy!
- All of the staff and consultants in the Simons Foundation who have put a lot of time and effort into getting this collaboration going
- The staff at the Simons Center for all of their work in helping to make sure this Collaboration kickoff meeting runs smoothly and successfully

The Rich Physics-Mathematics Interface

- **ALL** the known physics in our entire Universe is *extremely well* modelled by the Standard Model of Particle physics plus General Relativity
- This **mathematical** model is based on these equations:
  - Maxwell equations $\leftrightarrow$ electricity and magnetism
  - Yang-Mills equations $\leftrightarrow$ strong and weak nuclear force
  - Dirac Equation $\leftrightarrow$ quarks and leptons
  - Klein-Gordon Equation $\leftrightarrow$ the Higgs boson
  - Einstein Equations $\leftrightarrow$ cosmology (galaxies, dark matter, dark energy)
- The impact of these equations upon mathematics, geometry in particular, cannot by emphasized enough
  - Index theory
  - Moduli spaces (Monopoles, Instantons, Higgs bundles, flat connections etc)
  - Donaldson Theory, Seiberg-Witten equations
  - Einstein manifolds
  - Calabi-Yau manifolds, mirror symmetry and Gromov-Witten invariants
  - This is just a selection of some 20th century highlights

String Theory

- Superstring theory is based on equations which describe 2d surfaces minimally embedded in space-time
- A remarkable fact is that the equations which describe the low energy harmonics of such strings include the:
  - Maxwells equations
  - Yang-Mills equations
  - Dirac Equation
  - Klein-Gordon Equation
  - Einstein Equations
- This is why Superstring theory is considered our best candidate for a unified description of the forces of nature
- One of the main goals is to understand solutions of string theory which look identical to our Universe

Extra Dimensions and Calabi-Yau Manifolds

- Superstring theories predict six extra dimensions of space.
- Calabi-Yau manifolds are used to model these dimensions (Candelas, Horowitz, Strominger and Witten '84)
- This area has been immensely fruitful in both mathematics and physics over the past three decades
- Mirror symmetry: Calabi-Yau manifolds come in topologically distinct pairs with remarkable mathematical relations between them
- Gromov-Witten theory of holomorphic curves
- Donaldson-Thomas invariants: gauge theory on Calabi-Yau manifolds
- Classifying the singularities of Calabi-Yau manifolds

M theory, G_2-manifolds and Particle Physics

- In the mid '90s Sen, Hull-Townsend and E.Witten discovered superstring dualities :– a web of connections between the superstring theories
- Culminated in M theory :– a unifying theory with seven extra dimensions
- G_2-holonomy manifolds are used to model these extra dimensions
- A beautiful picture has emerged from this, in which

all of physics has a completely geometric origin in M theory on a singular G_2-manifold

M theory, G_2-manifolds and Particle Physics

- In this picture, the particles and fields which are the basic ingredients of the Standard Model of Particles arise from geometric structures inside G_2-manifolds
- Singularites $\quad\leftrightarrow\quad$ Particles
  - Yang-Mills fields arise from (orbifold) singularities on 3d subspaces
  - Quarks and Leptons arise from *conical* singularities at points

M theory, G_2-manifolds and Particle Physics

- In the Calabi-Yau case, Yau's theorem allows the construction of many, many explicit examples
- G_2-manifolds are much more difficult to study explicitly
-
- Some of the key questions as far as physics is concerned are:
  - Constructing local models of conical singularities relevant for particle physics
  - (Hard) Constructing compact holonomy G_2-spaces with conical singularities
  - Understanding and enumerating special subspaces of G_2-manifolds known as associative submanifolds

M theory, G_2-manifolds and Particle Physics

- Solutions of these and the other problems in our programme would lead to tremendous progress in this geometric approach to particle physics
- Would provide a NEW framework for important physics questions such as:
  - the fermion mass problem (why the quark/lepton masses vary over many orders).
  - what is the nature of the dark matter?
- The fermion mass problem would be addressed by a G_2 analogue of Gromov-Witten theory of associative submanifolds in G_2-spaces with singularities
- One would address the dark matter problem by understanding which singularities are more typical and which geometric and topological features are not allowed

What is String Theory?

- String theory is based on equations which describe 2d surfaces minimally embedded in space-time
- A remarkable fact is that the equations which describe the low energy harmonics of such strings include:
  - Maxwell equations
  - Yang-Mills equations
  - Dirac Equation
  - Klein-Gordon Equation (Higgs equation)
  - Einstein Equations
- This is why String theory is considered our best candidate for a unified description of nature
- One of the main goals is to understand solutions of string theory which look identical to our Universe

Superstring theories

- The symmetries of string theory strongly restrict the number of consistent quantum string theories. Superstring" theories are the best understood.
- There are just five superstring theories:
  - Type IIA
  - Type IIB
  - Type I
  - $E_8 \times E_8$ Heterotic
  - $Spin(32)/\mathbf{Z_2}$ Heterotic
- All are defined on ten dimensional spacetime i.e. Lorentzian 10-manifolds
- They differ from each other by gauge symmetries, particle spectrum etc
- They all have supersymmetry –implies relations amongst equations e.g. Yang-Mills $\leftrightarrow$ Dirac

Superstring theories

- Superstring theories are defined (at weak coupling and in smooth regions of space) in ten Lorentzian dimensions ($M^{9,1}$
- We have physically observed only **three** dimensions of space so the **six extra dimensions** must be hidden
- Simplest solution is: $M^{9,1} = Z^6 \times R^{3,1}$ with product metric, Z^6 compact and $R^{3,1}$ approx flat.
- Simplest solution requires $g(Z)$ to be *Ricci flat* i.e. $Ric(g(Z)) = 0$
- Then we have, assuming (Cheeger-Gromoll splitting) Z is simply connected, that
  - $Hol(g(Z)) = O(6)$, $SO(6)$, or $SU(3)$
  - Only *known* examples of this type have $Hol(g(Z)) = SU(3)$
  - Yau's solution of Calabi conjecture generates examples
  - Calabi-Yau manifolds are supersymmetric: admit parallel spinor, θ, $\nabla_{g(Z)}\theta = 0$

- Question: are there compact, simply connected, Ricci flat manifolds with generic holonomy?
- Less simple solutions: "generalised Calabi-Yau", solutions of "Strominger system" (Hitchin)
- These are less well understood; very few known examples

Calabi-Yau manifolds and Mirror symmetry

- In the mid 80's Candelas, de la Ossa, Green and Parke realised
- Type IIA on a CY Z^6 has identical physics to Type IIB on a different CY $\hat{Z}^6$
- Remarkably, "classical" physics on $\hat{Z}^6 =$ "quantum" physics on Z^6.
  - "Classical" physics here are the period integrals of holomorphic 3-form as functions of complex structure moduli
  - "Quantum" physics here are "instantons" i.e. holomorphic maps from Riemann surfaces to Z^6
  - Numbers of holomorphic curves (Gromov-Witten invariants) related to period integrals
- Later, Strominger-Yau-Zaslow picture of CY's as T^3-fibrations in a limit lead to generalisations such as
- Coherent Sheaves on $Z \leftrightarrow$ special Lagrangians in $\hat{Z}$ plus flat connection (Fukaya)

Superstring dualities and M theory

- In the early 90's was realised that the five superstring theories are inter-related (Sen, Hull-Townsend, Witten)
- A "new" theory emerged which is required in order to "interpolate" between the string theories
- This, so-called, M theory resides in eleven Lorentzian dimensions (so *seven extra dimensions*).
- The relevant fields of M theory on $M^{10,1}$ are a metric g, a 3-form C, and a gravitino Ψ (a vector with values in the spin bundle.)
  - E.g. M theory on $S^1 \times M^{9,1} \longrightarrow$ Type IIA on $M^{9,1}$ in small S^1 limit
  - E.g. M theory on $I \times M^{9,1} \longrightarrow E_8$ Heterotic on $M^{9,1}$ in small I limit (I is an interval)

Superstring dualities and M theory

- "Simplest" solutions of M theory with 7 compact dimensions have $g(M^{10,1}) = g(X^7) + g(R^{3,1})$ with X compact and Ricci flat
- Only *known* examples of this kind with $\pi_1(X) = 0$ have holonomy G_2. (Joyce, Kovalev, Corti-Haskins-Nordstrom-Pacini)
- Holonomy G_2-manifolds are supersymmetric i.e. have a parallel spinor (equiv to existence of a parallel 3-form)

G_2-manifolds in M theory

- G_2-manifolds (X) are models for the extra dimensions in M theory
- Smooth G_2-manifolds are not so relevant for particle physics: no non-Abelian gauge symmetries and no chiral fermions – both key ingredients of the Standard Model of Particle physics
- Codimension four orbifold singularities (along associative Q) give rise to Yang-Mills fields
- Codimension seven conical singularities (on Q) give rise to quarks and leptons
- Associative submanifolds play a key role: gauge couplings, Yukawa couplings (associatives which intersect triples of codimension seven singularities)

Some physics references for G_2-manifolds

- A review on the particle physics/dark matter: BSA, G. Kane and P. Kumar arXiv:1204.2795
- Codimension four singularities: BSA, arXiv/hep-th/9812205 and 0011089
- Codimension seven singularities and $U(1)$ quotients: M. Atiyah and E. Witten and BSA, E.Witten
- Physics Report on Physics of Singular G_2 and $Spin(7)$ manifolds, BSA and S. Gukov.

ADE-singularities, McKay correspondence and Particle Physics

- $\mathbf{R^4}/\Gamma_{ADE}$ has $3rk(ADE)$ moduli corresponding to the $rk(ADE)$ S^2's which desingularise the singularity preserving a Ricci flat metric.
- Think of this as a local model for $M^{10,1} = K3 \times R^{6,1}$
- i.e. $11 = 4 + 7$ instead of $11 = 7 + 4$!
- M theory has a 3-form potential C_3 which obeys a Maxwell equation,
- $d * dC = j_8$ whose source is an 8-form current supported on a 3d-submanifold.
- Electrically charged particles are extended surfaces (two space one time).
- These objects are called $M2$-branes

G_2-manifolds and Particle Physics: singularities

- $\tilde{\mathbf{R}}^{\mathbf{4}}/\Gamma_{ADE}$ with HK metric has $rk(ADE)$ harmonic 2-forms
- $H^2(\tilde{\mathbf{R}}^{\mathbf{4}}/\Gamma_{ADE}), \mathbf{Z}) \equiv \Gamma(ADE)$ the root lattice of the ADE Lie Algebra.
- If $C_3 = \alpha \wedge A_1$ with α harmonic and A_1 a 1-form on $\mathbf{R}^{\mathbf{6},\mathbf{1}}$
- A_1 obeys standard Maxwell equations, $d *_7 dA_1 = \cdots$
- Now the electric current is actually supported by point particles in $\mathbf{R}^{\mathbf{6},\mathbf{1}}$
- These are M2-branes which are of the form $\Sigma \times \mathbf{R}$ with
- $\Sigma \subset H_2(\tilde{\mathbf{R}}^{\mathbf{4}}/\Gamma_{ADE}) \equiv H^2(\tilde{\mathbf{R}}^{\mathbf{4}}/\Gamma_{ADE})$

G_2-manifolds and Particle Physics: singularities

- Since A_1 obeys Maxwell equations we effectively have a $U(1)$ gauge theory on $\mathbf{R}^{6,1}$
- This $U(1)$ naturally can be thought of as residing in the maximal torus of ADE
- In fact, we get a different $U(1)$ for each independent harmonic 2-form, hence the full maximal torus.
- The independent wrapped $M2$-branes look like particles, whose charges are the roots of the ADE Lie algebra
- The masses of these charged particles are given by the volumes of the corresponding two-cycles (supersymmetry)
- At singularities of the moduli space of HK metrics, some of the 2-cycles go to zero volume and the corresponding particles become massless

G_2-manifolds and Particle Physics: singularities

- At singularities of the moduli space of HK metrics, some of the 2-cycles go to zero volume and the corresponding particles become massless
- This is exactly described by the Higgs mechanism!
- In this case, the Higgs fields are the moduli of the hyperKahler metric!
- The dynamics is precisely described by 7d supersymmetric gauge theory with ADE gauge group

G_2-manifolds and Particle Physics: singularities

- When embedded into compact G_2 manifolds, these codimension four orbifold singularities are responsible for the gauge fields observed in nature
- What about quarks and leptons?
- These arise from additional singularities of codimension seven.
- The codim four singularities are supported along a 3-cycle $Q \subset X$
- The codim seven singularities will be at points on Q at which the fibers of $N(Q)$ become more singular
- Rank increases by one
- e.g. $\mathbf{R^4}/\Gamma_{A_N} \to \mathbf{R^4}/\Gamma_{A_{N+1}}$

G_2-manifolds and Particle Physics: singularities

- $\mathbf{R^4}/\Gamma_{ADE}$ has $3rk(ADE)$ moduli corresponding to the $rk(ADE)$ S^2's which desingularise the singularity preserving a Ricci flat metric.
- Pick one such S^2, fixing the remaining $3rk(ADE) - 3$ moduli 'at the origin'.
- This gives a 7d family of ALE spaces fibered over $\mathbf{R^3}$ with a 'more singular' fiber at the origin.
- Conjecture: this 7-manifold admits a G_2-holonomy metric
- For the A_1 case this is the Bryant-Salamon metric on $\mathbf{R}^+ \times \mathbf{CP^3}$ (Twistor)
- For A_{N+1} with generic fiber A_N there should be a G_2-holonomy metric on $\mathbf{R}^+ \times \mathbf{WCP^3_{N,N,1,1}}$

More on the Local Models

- So there ought to be G_2-holonomy metrics on "ALE-fibrations over 3-manifolds"
- In the examples above there are also $U(1)$ fibrations:
- $\mathbf{R}^+ \times \mathbf{CP^3}$ admits a $U(1)$ action whose quotient is $\mathbf{R}^+ \times \mathbf{S^5} = \mathbf{R^6}$
- The fixed points are a pair of special Lagrangian 3-planes in $\mathbf{R^6}$ which meet at $SU(3)$ angles.
- $\mathbf{R}^+ \times \mathbf{WCP^3_{PPQQ}}$ admits a $U(1)$ action whose quotient is $\mathbf{R}^+ \times \mathbf{S^5} = \mathbf{R^6}$
- The fixed point set is the same, but now the '1st Chern classes' of the $U(1)$ are P and Q
- Locally modelled on $U(1) \rightarrow \mathbf{R^4} \rightarrow \mathbf{R^3}$.

Collapsed Circle Limits

- G_2 manifolds could admit S^1-fibrations $S^1 \to X \to Z$
- This is natural from the point of view of the duality between M theory and Type IIA superstring theory
- The circles will collapse at codimension four i.e. codim three in Z.
- In string theory Z begins life as a quotient of a Calabi-Yau threefold by an antiholomorphic involution σ
- Near the fixed points of σ, X could be modelled by Atiyah-Hitchin fibered over Z_σ. Orientifolds
- cf Johannes Nordstrom and Mark Haskins talks

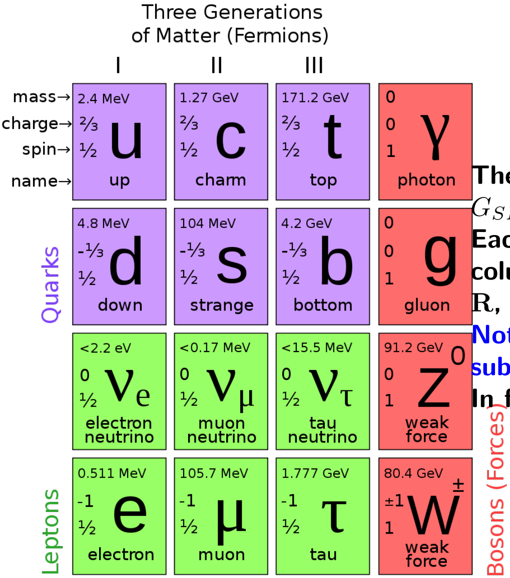
G_2-manifolds and Particle Physics: interactions

- The basic interactions in the Standard Model involve three particles
- Feynman diagrams have tri-valent vertices!
- There are interactions between the Higgs boson and pairs of fermions
- Each of these are codim 7 singularities
- In M theory these correspond to an associative submanifold which passes through the three corresponding codim 7 singularities
- M theory analogues of "world-sheet instantons" aka hol. curves in a CY
- Is there a mathematical theory which counts such associative submanifolds?

Compact G_2 manifolds

- The bigger picture leads to compact G_2-manifolds with codim 4 and codim 7 singluarities
- K3-fibrations over S^3 emerge as natural candidates
- The generic fiber would have e.g. an $A_4 \times A_6 \times A_7$ ADE singularity
- Then at isolated points on the base the A_4 enhances to A_5 and D_5.
- Associative submanifolds which intersect the codim 7 singularities arise from the "network" of 2-spheres in the fibers as they vary over the S^3

The Standard Model of Particle Physics

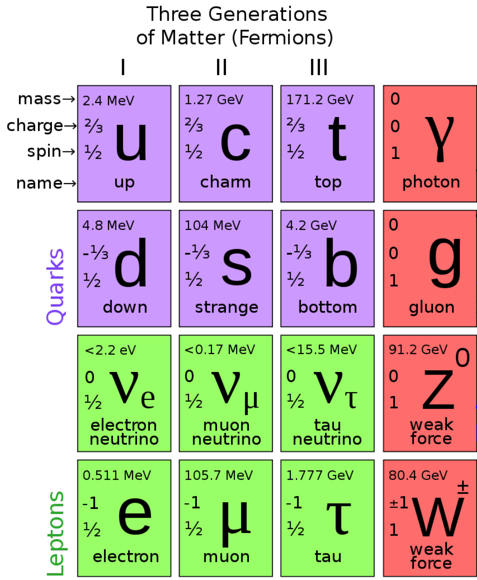


The gauge symmetry is $G_{SM} = SU(3) \times SU(2) \times U(1)$ **Each family i.e. 1st three columns is in a 15-dim rep,** $\mathbf{R}$**, of** G_{SM}.

Note: G_{SM} **is a maximal subgroup of** $SU(5)$

In fact $\mathbf{R} \equiv \wedge^1 \oplus \wedge^3$ **in** $SU(5)$

The Standard Model of Particle Physics



Using $SU(5)$ language:
The generic fiber has at least an $SU(5) = A_4$ singularity.
1st three columns are each $\wedge^1 \oplus \wedge^3$ of $SU(5)$
So three $\wedge^1$'s and three $\wedge^3$'s = 3 A_5 fibers and three D_5's.
Membrane instantons = associative submanifolds give hierarchy in masses

Geometric Origin of Particle Physics

- Understanding these G_2-manifolds better would lead to a completely geometric picture of all of particle physics!
- Moreover, it will likely lead to clues about the nature of dark matter

Other Issues which overlap with physics

- Would like to understand (much!) better the moduli space metric
- n.b. Kahler potential $K = -3ln(Vol(X))$, for complexified G_2-form: $\int_{H_3(X)}(\varphi + iC_3)$
- C_3 is the M theory 3-form 'potential'
- What are the geometric properties of the moduli space?
- **Important**: Can $Vol(X)$ be computed in practice as a function of the G_2 moduli space?
- Does the idea of an "ample" associative submanifold make sense, in analogy with "ample divisors"

G_2-manifolds in string theory

- In string theory, G_2-manifolds also play a role
- D-branes in string theory are volume minimising submanifolds with gauge bundles on them
- D-branes wrapped on associative and co-associative submanifolds as well as along X itself are described by "topological field theory" equations:
- e.g. Casson invariants, Donaldson invariants, G_2-instantons, G_2-monopoles
- Key point: supersymmetric configurations (e.g. calibrations) $\leftrightarrow$ interesting equations

Cosmology, Dark Matter and G_2-manifolds

- We have come to appreciate over the past ten years that the moduli of G_2-manifolds will, in typical models, dominate the energy density of the Universe during its first 1/100 th of a second!
- This is because they couple very weakly to ordinary matter, so dont thermalise, but their potential energy dilutes much more slowly than radiation as the Universe expands
- This means that DARK MATTER is NON-THERMALLY PRODUCED
- Which has significant implications for the properties and nature of Dark Matter
- Moreover, in this picture, Dark Matter must emerge from the cohomology of the G2-manifold and/or its singularities.

BACKUP

M theory, G_2-manifolds History

- '78 Cremmer, Julia, Scherk: discover 11d supergravity
- '83 Witten proves no-go theorem for 11d supergravity and quarks and leptons
- '84 Green-Schwarz discover anomaly cancellation, quickly followed by heterotic superstring
- '85 Candelas, Horowitz, Strominger, Witten: first Calabi-Yau solutions with quarks and leptons
- Late 80's: first evidence of mirror symmetry (Candelas, de la Ossa, Greene, Parkes)
- Bryant-Salamon '89: first local examples of holonomy G_2-manifolds
- Early 90's: Sen, Hull-Townsend, Witten: superstring dualities and M -theory in 11d
- Mid '90's Joyce constructs first compact G_2-manifolds
- Mid-90's First papers on M theory and G_2-manifolds (Papadopoulos-Townsend, Acharya)
- 1999-2000 Yang-Mills fields from orbifold singularities (Acharya)
- 2001 Chiral fermions (quarks/leptons) from conical singularities (Acharya-Witten, Atiyah-Witten), circumventing Witten's no-go theorem with singularities
- 2008-2012 Development of particle phenomenology models from G_2-manifolds (Acharya-Kane et al)
- 2010 Kovalev: compact examples from twisted connect sums
- 2012 Corti-Haskins-Nordstrom-Pacini – Large class of examples from generalised twisted connect sums
- 2017? First compact examples with conical singularities????

Some physics references for G_2-manifolds

- A recent review on the particle physics/dark matter: BSA, G. Kane and P. Kumar arXiv:1204.2795
- The M theory interpretation of the Fermi data: BSA, G. Kane, P. Kumar, R. Lu, B, Zheng arXiv:1205.5789
- Codimension four singularities: BSA, arXiv/hep-th/9812205 and 0011089
- Codimension seven singularities and $U(1)$ quotients: M. Atiyah and E. Witten and BSA, E.Witten

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